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ON SUBSPACE-TRANSITIVE OPERATORS

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Abstract: The purpose of the present paper is to treat a notion, which can be viewed as a localization of the recent notion of subspace-transitivity. We conclude this paper to answer in affirmative one question asked by Madore and Martinez-Avandano with an additional assumption.

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1. Introduction

Let X be a Banach space. In what follows, the symbol T stands for a bounded linear operator acting on T and M will be a nonzero closed subspace of X. Consider any subset C of X. The symbol $Orb(T,C)$ denotes the orbit of C under T, i. e. $Orb(T,C) = \{T^n x : x \in C, n = 0,1,2,...\}$. If $C = \{x\}$ is a singleton and the orbit $Orb(T,x)$ is dense in X, then the operator T is called hypercyclic and the vector x is a hypercyclic vector for T. Observe that in case the operator is hypercyclic the underlying Banach space X should be separable. Then it is well known and easy to show that an operator T is hypercyclic if and only if T is topologically transitive, to be precise, for every pair of nonempty open sets U, V of X there exists a non-negative integer n such that $T^n(U) \cap V \neq \emptyset$. The study of hypercyclicity goes back a long way, and has

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been investigated in more general settings, for example in topological vector spaces. A nice source of examples and properties of hypercyclic operators is the survey article [5], and see also recent books [1], [4].

Recently, B. F. Madore and R. A. Martinez-Avendano in [7] introduced the concept of subspace-hypercyclicity. An operator T is subspace-hypercyclic (or M-hypercyclic) for a subspace M of X if there exists a vector $x \in X$ such that the intersection of its orbit and M is dense in M. Also authors introduced the notion of subspace-transitivity (or M-transitivity) and show that M-transitivity implies M-hypercyclicity, and C. M. Le in [6] construct an operator T such that it is M-hypercyclic but it is not M-transitive. The authors in [7] prove several results analogous to hypercyclicity case. Other sources of examples and some properties of notions relating subspace-hypercyclicity are [8], [9].

The purpose of this paper is twofold. Firstly, we somehow *localize* the notion of subspace-transitivity by introducing certain set, which we called M-extended limit set of x under T, $J(T, M, x)$, for an operator T and a given vector x.

It is worthwhile to mention that the notion of $J-$*class operators* was introduced by G. Costakis and A. Manoussos in [3], [2], and with it, the authors localized the notion of hypercyclicity.

In [7] authors raised five questions relating subspace-hypercyclic-ity. We are interested in the first one. The second purpose of this paper is an application of the localization notion of subspace-transitivity in order to answer in the affirmative question:" let T be an invertible operator. If T is subspace-transitive for some M, is T^{-1} subspace-hypercyclic for some space? If so, for which space?"

2. Preliminaries and Some Results

Definition 1. Let $T \in B(X)$. We say that T is *M-hypersyclic* if there exists $x \in X$ such that $Orb(T, x) \cap M$ is dense in M. Such a vector x is called an *M-hypersyclic* vector for T.

Definition 2. Let $T \in B(X)$. We say that T is *M-transitive* if for any nonempty open subsets U, V of M there exists a non-negative integer n such that $T^{-n} \cap V$ contains a relatively open nonempty subset of M.

The proof of the following proposition can be found in [7].

Proposition 3. *Let $T \in B(X)$. Then the following conditions are equivalent:*

(i) The operator T is M-transitive.

(ii) For any nonempty sets U and V, both relatively open, there exists $n \geq 0$ such that $T^{-n} \cap V$ is a relatively open nonempty subset of M.

(iii) For any nonempty sets U and V, both relatively open, there exists $n \geq 0$ such that $T^{-n} \cap V$ is nonempty and $T^n(M) \subseteq M$.

In [6], [7] the authors prove that M-transitivity implies M-hypercyclicity and the convers is not always correct.

Theorem 4. *Let T is M-transitive. Then for any nonempty open subset U of M, $\cap_{n=0}^{\infty} T^n(U) \cap M$ is dense in M.*

Proof. Let U be a nonempty subset of M,by previous proposition there exists some $k \geq 0$ such that

$$T^{-k}U \cap V \neq \emptyset \quad \text{and} \quad T^k(M) \subseteq M$$

hence

$$\emptyset \neq T^k(T^{-k}(V) \cap U) \subseteq V \cap T^k(U).$$

Therefore $\cap_{n=0}^{\infty} T^n(U) \cap M$ is dense in M. □

Definition 5. Let $T \in B(X)$. Then for any subsets $A, B \subseteq M$, the *return set from A to B* defined as

$$N_{(T,M)}(A, B) = \{n \geq 0 : T^{-n}(A) \cap B \;\; \textit{is nonempty open subset of} \;\; M\}$$

In this notation, T is M-transitive if and only if, for any nonempty sets $U \subseteq M$ and $V \subseteq M$, both relatively open, the return set $N_{(T,M)}(U, V)$ is nonempty.

Remark 6. When T is M-transitive, the proposition3 rearrange the return set $N_{(T,M)}(U, V)$ for any nonempty sets $U \subseteq M$ and $V \subseteq M$, both relatively open, as below:

$$N_{(T,M)}(U, V) = \{n \geq 0 : T^{-n}(U) \cap V \neq \emptyset \quad \textit{and} \quad T^n(M) \subseteq M\}.$$

Theorem 7. *Let T is an M-transitive operator. Then for any pair U, V, both nonempty relatively open subsets of M, the return set $N_{(T,M)}(U, V)$ is infinite.*

Proof. Since T is M-transitive, there exists $n \geq 0$ such that $W = T^{-n}(U) \cap V$ is nonempty relatively open subset of M. Consider two distinict points x, y in W and two relatively open subsets W_1 and W_2 of M such that

$$x \in W_1, \;\; y \in W_2, \;\; W_1 \subseteq W, \;\; W_2 \subseteq W, \;\; W_1 \cap W_2 = \emptyset,$$

consequently there exsists $k \geq 1$ such that

$$T^{-k}(W_1) \cap W_2 \neq \emptyset, \;\; T^k(M) \subseteq M$$

hence

$$\emptyset \neq T^{-k}(W_1) \cap W_2 \subseteq T^{-k}(W) \cap W \subseteq T^{-(k+n)}(U) \cap V. \tag{1}$$

Since $T^n(M) \subseteq M$, so $T^{(k+n)}(M) \subseteq M$. Therefore (1) implies that the intersection of $N_{(T,M)}(U, V)$ and Natural numbers is nonempty. Proceeding inductively we find infinite integers $n \in N_{(T,M)}(U, V)$. □

Remark 8. An equivalent definition of an M-transitive operator is the following: for any nonempty sets $U \subseteq M$ and $V \subseteq M$, both relatively open, and for any $N \geq 1$, there exists a positive integer $n > N$ such that $T^{-n}(U) \cap V$ is a relatively open nonempty subset of M.

Definition 9. Let T be an operator. For every $x \in M$ the set

$$\begin{aligned} J(T, M, x) = \{y \in M : & \; for\ every\ relatively\ open\ neighborhoods \\ & U, V \text{ of } x, y \text{ in } M \text{ respectively, and every positive integer } N, \\ & \text{there exists } n > N \text{ such that } T^n(U) \cap V \neq \emptyset \text{ and } T^n(M) \subseteq M\} \end{aligned}$$

denote the *M-extended limit set* of x under T.

Proposition 10. *An eqivalent definition of $J(T, M, x)$ is the following.*

$$\begin{aligned} J(T, M, x) = \{y \in M : & \text{ there exists a strictly increasing sequence} \\ & \text{of positive integers } \{k_n\} \text{ and a sequence } \{x_n\} \subset M \\ & \text{such that } x_n \longrightarrow x \text{ and } T^{k_n} x_n \longrightarrow y \text{ and for every} \\ & n, \; T^{k_n}(M) \subseteq M\}. \end{aligned}$$

Proof. Let us prove that

$$J(T, M, x) \subseteq \{y \in M : \text{there exists a strictly increasing sequence}$$

$$of\ positive\ integers\ \{k_n\}\ and\ a\ sequence\ \{x_n\} \subset M$$
$$such\ that\ x_n \longrightarrow x\ and\ T^{k_n}x_n \longrightarrow y\ and\ for\ every$$
$$n,\ T^{k_n}(M) \subseteq M\}.$$

since the converse inclusion is obvious. Let $y \in J(T, M, x)$ and consider the open balls

$$U_n = B(x, \frac{1}{n}) \cap M,\ V_n = B(y, \frac{1}{n}) \cap M,\ for\ n = 1, 2, ...$$

and $N = k_{n-1}$, $k_0 = 1$. Then there exists $k_n > N = k_{n-1}$ such that

$$T^{k_n}(U_n) \cap V_n \neq \emptyset \quad and \quad T^{k_n}(M) \subseteq M.$$

Hence there exists $x_n \in U_n$ such that $T^{k_n} \in V_n$ and $T^{k_n}(M) \subseteq M$. Therefore $\{k_n\}$ is an strictly increasing sequence of positive integers and $\{x_n\}$ is a sequence in X such that $x_n \longrightarrow x$ and $T^{k_n}x_n \longrightarrow y$ and for every n, $T^{k_n}(M) \subseteq M$. □

3. Main Results

The following characterization of M-transitive operators help us to answer in the affirmative question:" let T be an invertible operator. If T is an T-transitive for some M, is T^{-1} subspace-hypercyclic for some space? If so, for which space?".

Theorem 11. *Let T be an operator on X. Then the following conditions are equivalent:*
(i) T is an M-transitive.
(ii) For every $x \in M$, $J(T, M, x) = M$.

Proof. We first prove that *(i)* implies *(ii)*. Let $x \in U$, $y \in V$ and U, V be relatively open subsets of M and $N \geq 1$. There exists $n > N$ such that $U \cap T^{-n}(V)$ is nonempty and $T^n(M) \subseteq M$. Thus $y \in J(T, M, x)$, and consequently $J(T, M, x) = M$.

We will show that *(ii)* ⇒ *(i)*. Let $U \subseteq M$, $V \subseteq M$, both nonempty and relatively open. Consider $x_0 \in U$, $y_0 \in V$. Since $J(T, M, x_0) = M$, there exists $n \geq 1$ such that $T^n(V) \cap U \neq \emptyset$ and $T^n(M) \subseteq M$. Proposition3 implies that T is an M-transitive operator. □

The next example will show that subspace-hypercyclicity does not imply subspace-transitivity with respect to M.

Example 12. Let $\lambda \in \mathbf{C}$ be of modulus greater than 1 and let B be the backward shift on l^2. Let m be a positive integer and M be the subspace of l^2 consisting of all sequences with zero on the first m entries, that is:

$$M = \{\{a_n\}_{n=0}^{\infty} \in l^2 : \ a_n = 0 \quad for \quad n \leq m\}$$

then $T = \lambda B$ is M-hypercyclic, see [7]. Now consider

$$V = \{\{a_n\}_{n=0}^{\infty} \in l^2 : \ a_n = 0 \; for \; n \leq m \quad and \quad | \, a_n \, |> 0 \; for \; n > m\}$$

so V is relatively open subset of M. If $N = m + 1$, then for every $n > N$, $T^n(V) \cap M = \emptyset$. Thus for every $x \in M$, $J(T, M, x) \neq M$.

Theorem 13. *Let T be an invertible operator and M-transitive. Then T^{-1} is M-hypercyclic.*

Proof. Let $x, y \in M$. Since T is M-transitive, so $J(T, M, x) = M$. If U, V are relatively open subsets of M such that contain x, y respectively, then there exists $n > 1$,

$$T^n(U) \cap V \neq \emptyset \qquad and \qquad T^n(M) \subseteq M$$

thus invertibility of T implies that

$$T^{-n}(M) \subseteq M \qquad and \qquad U \cap T^{-n}(V) \neq \emptyset.$$

Hence for every $x \in M$, $x \in J(T^{-1}, M, y)$. This means for every $y \in M$,

$$M = J(T^{-1}, M, y)$$

or equivalently T^{-1} is M-transitive. □

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