International Journal of Pure and Applied Mathematics
Volume 85 No. 1 2013, 149-154
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v85i1.12
ijpam.eu

AN IMPROVED BOUND ON THE POISSON-NEGATIVE BINOMIAL ERROR

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Abstract: The Stein-Chen method is used to obtain a new non-uniform bound on the error of the negative binomial cumulative distribution function with parameters n and p and the Poisson cumulative distribution function with mean $n(1-p)$. The bound obtained in this study is sharper than those reported in [8].

AMS Subject Classification: 62E17, 60F05
Key Words: cumulative distribution function, negative binomial distribution, Poisson distribution, non-uniform bound, Stein-Chen method

1. Introduction

The negative binomial distribution with parameters $n > 0$ and $p \in (0, 1)$ is an important discrete distribution. It is widely used in many areas of probability and statistics. For $n = 1$, it is referred to as the geometric distribution with parameter p. Let X be the negative binomial random variable with parameters

Received: February 6, 2013

n and $p = 1 - q$, then its probability distribution function is of the form

$$p_X(x) = \frac{\Gamma(n+x)}{\Gamma(n)x!} q^x p^n, \; x = 0, 1, ..., \tag{1.1}$$

and $E(X) = \frac{nq}{p}$ and $Var(X) = \frac{nq}{p^2}$, respectively. For $\lambda = \frac{nq}{p}$ and $p = \frac{n}{n+\lambda}$, it becomes

$$p_X(x) = \frac{\lambda^x}{x!} \frac{\Gamma(n+x)}{\Gamma(n)(n+\lambda)^x} \left(\frac{1}{1+\frac{\lambda}{n}} \right)^n, \; x = 0, 1, ... \tag{1.2}$$

From (1.2), if $n \to \infty$ and $q \to 0$ while λ remains fixed, then $p_X(x) \to \frac{e^{-\lambda}\lambda^x}{x!}$ for every $x \in \mathbb{N} \cup \{0\}$, that is, the negative binomial distribution with parameters n and p converges to the Poisson distribution with mean λ. In this case, some authors have tried to derive uniform bounds for the total variation distance between the negative binomial and Poisson distributions, which can be found in [3−7] and [10]. Let us consider the probability distribution function (1.1), by setting $\lambda = nq$ and $p = \frac{n-\lambda}{n}$, it can be expressed as

$$p_X(x) = \frac{\lambda^x}{x!} \frac{\Gamma(n+x)}{\Gamma(n)n^x} \left(1 - \frac{\lambda}{n} \right)^n, \; x = 0, 1, ... \tag{1.3}$$

Observe that if $n \to \infty$ and $q \to 0$ while λ remains fixed, then $p_X(x) \to \frac{e^{-\lambda}\lambda^x}{x!}$ for every $x \in \mathbb{N} \cup \{0\}$. Therefore, the negative binomial distribution with parameters n and p also converges to the Poisson distribution with mean $\lambda = nq$ when n is large and q is small. In the case of $n \in \mathbb{N}$, Teerapabolarn [8] gave a non-uniform bound on this convergence as follows:

$$|\mathbb{NB}_{n,p}(x_0) - \mathbb{P}_\lambda(x_0)| \leq (e^\lambda - 1) \min\left\{1, \frac{1}{p(x_0+1)}\right\} q, \tag{1.4}$$

where $\mathbb{NB}_{n,p}(x_0) = \sum_{j=0}^{x_0} \frac{\Gamma(n+j)}{\Gamma(n)j!} q^j p^n$ and $\mathbb{P}_\lambda(x_0) = \sum_{j=0}^{x_0} \frac{e^{-\lambda}\lambda^j}{j!}$ are the negative binomial and Poisson cumulative distribution functions at $x_0 \in \mathbb{N} \cup \{0\}$. In this study, we focus on improving the bound in (1.4) to be more sharper for any positive real number n.

The Stein-Chen method is important tool for giving the main result as mentioned in Section 2. In Section 3, we use this method to obtain a non-uniform bound for this error. Concluding remarks are presented in the last section.

2. Method

Stein's method was first introduced by Stein [7]. The version appropriate for the Poisson case was first developed by Chen [1], which is referred to as the Stein-Chen method. Following [9], Stein's equation of the Poisson cumulative distribution function with parameter $\lambda > 0$ is of the form

$$h_{x_0}(x) - \mathbb{P}_\lambda(x_0) = \lambda f_{x_0}(x+1) - x f_{x_0}(x), \tag{2.1}$$

where $x_0, x \in \mathbb{N} \cup \{0\}$ and function $h_{x_0} : \mathbb{N} \cup \{0\} \to \mathbb{R}$ is defined by

$$h_{x_0}(x) = \begin{cases} 1 & \text{if } \ x \leq x_0, \\ 0 & \text{if } \ x > x_0 \end{cases}$$

and

$$f_{x_0}(x) = \begin{cases} (x-1)!\lambda^{-x}e^\lambda[\mathbb{P}_\lambda(x-1)[1-\mathbb{P}_\lambda(x_0)]] & \text{if } x \leq x_0, \\ (x-1)!\lambda^{-x}e^\lambda[\mathbb{P}_\lambda(x_0)[1-\mathbb{P}_\lambda(x-1)]] & \text{if } x > x_0, \\ 0 & \text{if } x = 0. \end{cases} \tag{2.2}$$

Note that $f_{x_0}(x) \geq 0$ for every $x \in \mathbb{N} \cup \{0\}$.

Lemma 2.1. *For $x_0 \in \mathbb{N}$, then the following inequality holds:*

$$\sup_{x \geq 2} f_{x_0}(x) \leq \frac{2\lambda^{-2}(e^\lambda - \lambda - 1)}{x_0 + 1}. \tag{2.3}$$

Proof. Following [9], we have $f_{x_0}(x) \leq f_{x_0}(x_0+1)$ for every $x \in \mathbb{N}$. Thus,

$$\begin{aligned} f_{x_0}(x_0+1) &= x_0!\lambda^{-(x_0+1)}e^\lambda \mathbb{P}_\lambda(x_0)(1-\mathbb{P}_\lambda(x_0)) \\ &\leq \frac{1}{x_0+1} + \frac{\lambda}{(x_0+1)(x_0+2)} + \frac{\lambda^2}{(x_0+1)(x_0+2)(x_0+3)} + \cdots \\ &\leq \frac{1}{x_0+1}\left\{1 + \frac{\lambda}{3} + \frac{\lambda^2}{12} + \cdots\right\} \\ &= \frac{2\lambda^{-2}(e^\lambda - \lambda - 1)}{x_0+1}. \end{aligned}$$

Hence, the inequality (2.3) holds. □

Lemma 2.2. *Let $x_0 \in \mathbb{N}$ and $\lambda = nq$, then we have the following:*

$$\mathbb{P}_\lambda(x_0) - \mathbb{NB}_{n,p}(x_0) \leq 1 - e^\lambda p^n. \tag{2.4}$$

Proof. $\mathbb{P}_\lambda(x_0) - \mathbb{NB}_{n,p}(x_0) = e^{-\lambda}\sum_{k=0}^{x_0}\frac{\lambda^k}{k!} - p^n\sum_{k=0}^{x_0}\frac{\lambda^k}{k!}\left(\frac{n+k-1}{n}\cdots\frac{n}{n}\right) \leq e^{-\lambda}\sum_{k=0}^{x_0}\frac{\lambda^k}{k!} - p^n\sum_{k=0}^{x_0}\frac{\lambda^k}{k!} \leq 1 - e^\lambda p^n$. □

3. Result

The following theorem shows a new non-uniform bound on $|\mathbb{NB}_{n,p}(x_0) - \mathbb{P}_\lambda(x_0)|$.

Theorem 3.1. *For $x_0 \in \mathbb{N}\cup\{0\}$, if $\lambda = nq$ then we have the following:*

$$|\mathbb{NB}_{n,p}(x_0) - \mathbb{P}_\lambda(x_0)| \begin{cases} = e^{-\lambda} - p^n & \text{if } x_0 = 0, \\ \leq \min\left\{1 - e^\lambda p^n, \frac{2(e^\lambda - \lambda - 1)}{(x_0+1)np}\right\} & \text{if } x_0 > 0. \end{cases} \tag{3.1}$$

Proof. It is clear for $x_0 = 0$. For $x_0 > 0$, Teerapabolarn [9] showed that

$$\mathbb{NB}_{n,p}(x_0) - \mathbb{P}_\lambda(x_0) = -\sum_{x=1}^{\infty} xqp_X(x)f_{x_0}(x+1) < 0.$$

Therefore, we obtain

$$\begin{aligned} 0 &\leq \mathbb{P}_\lambda(x_0) - \mathbb{NB}_{n,p}(x_0) \\ &= \sum_{x=1}^{\infty} xqp_X(x)f_{x_0}(x+1) \\ &\leq \sup_{x\geq 2} f_{x_0}(x)\sum_{x=1}^{\infty} xqp_X(x) \\ &\leq \frac{2(e^\lambda - \lambda - 1)}{(x_0+1)np} \quad \text{(by (2.3))}, \end{aligned} \tag{3.2}$$

and follows from (2.4), we obtain

$$0 \leq \mathbb{P}_\lambda(x_0) - \mathbb{NB}_{n,p}(x_0) \leq 1 - e^\lambda p^n. \tag{3.3}$$

Hence, from (3.2) and (3.3), we have (3.1). □

Corollary 3.1. *For $x_0 \in \mathbb{N}$ and $\lambda = nq$, then the following inequality holds:*

$$\min\left\{1 - e^\lambda p^n, \frac{2(e^\lambda - \lambda - 1)}{(x_0+1)np}\right\} < (e^\lambda - 1)\min\left\{1, \frac{1}{p(x_0+1)}\right\} q. \tag{3.4}$$

Proof. It can be seen that $\frac{2(e^{\lambda}-\lambda-1)}{n} = \frac{2\left\{\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\cdots\right\}q}{\lambda} = \left\{\lambda + \frac{\lambda^2}{3} + \cdots\right\} q < \left\{\lambda + \frac{\lambda^2}{2!} + \cdots\right\} q = (e^{\lambda} - 1)q$. Hence the inequality (3.4) holds. □

4. Conclusion

The non-uniform bound on the error in Theorem 3.1, determined by the Stein-Chen method, is an estimate for the error of the negative binomial cumulative distribution function with parameters n and p and the Poisson cumulative distribution with mean $\lambda = nq = n(1-p)$. With this bound, it is indicated that the result gives a good approximation when q is sufficiently small. In addition, following Corollary 3.1, this bound is sharper than those reported in [8].

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