International Journal of Pure and Applied Mathematics
Volume 85 No. 1 2013, 13-22
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v85i1.2

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ON SYSTEMATIC GENERATION OF BIHARMONIC FUNCTIONS

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Abstract: We present some results for systematic generation of biharmonic functions that are not readily obtainable by a direct application of the separation of variable technique to the biharmonic equation. Almansi's theorem and the Kelvin transformation were adapted to obtain the results, and they are presented as theorems followed by simple proofs. The results are not only labour-saving, but also have important implications for the construction of solutions to boundary value problems involving composite media with curved geometry.

AMS Subject Classification: 31B30
Key Words: systematic generation, harmonic, biharmonic functions, Almansi's theorem, Kelvin transformation

1. Introduction

The biharmonic equation

$$\nabla^4\Phi = 0, \tag{1}$$

where ∇ is the del operator, has applications in many areas of continuum mechanics. For example, in solid mechanics, it is used to model elasto-static deformation in the absence of body forces and its solution Φ may represent

Received: May 11, 2012

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the Airy stress function for a two-dimensional, isotropic, linear elastic solid [1] or the deflection of a clamped thin plate [2]. In fluid mechanics, it can be used to describe the motion of an incompressible viscous fluid at low Reynolds number and its solution would, in this case, represent the stream function for Stokes flow [3]. For boundary value problems involving composite media, if the solution for the corresponding homogeneous problem is known and if the perturbation solution due to the presence of inhomogeneities can be obtained in terms of the known homogeneous solution, then one can easily construct the solution for the composite media, see for example, ([4], [5], [6]). For the class of problems which are governed by the biharmonic equation, the pertinent question is as follows: *Given a biharmonic function Φ which characterizes the field in a homogeneous medium, what are the additional biharmonic functions that may be superimposed on Φ in order to obtain the complete solution for the corresponding heterogeneous medium?* Such knowledge is fundamental to the establishment of a representation theorem for the solution for composite media in terms of the solution for the corresponding homogeneous medium and this simplifies the analysis, particularly when the solution to the homogeneous problem is readily obtainable by standard techniques.

The purpose of this paper is to show how biharmonic functions can be generated, in a systematic way, from known harmonic and biharmonic functions without solving the biharmonic equation directly. Specifically, we shall address the generation of biharmonic functions in polar coordinates. The method discussed here can be used to find those solutions of a biharmonic equation which are not readily obtainable by the separation of variable procedure, but may be required for the construction of complete solutions to some biharmonic boundary value problems of continuum mechanics.

2. Fundamental Solutions

Let (r, θ) be the polar coordinates which are related to the Cartesian coordinates (x, y) through the equations

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

For axially symmetric problems, the biharmonic equation (1) takes the form:

$$\left(\frac{d^4}{dr^4} + \frac{2}{r}\frac{d^3}{dr^3} - \frac{1}{r^2}\frac{d^2}{dr^2} + \frac{1}{r^3}\frac{d}{dr}\right)\Phi = 0, \tag{2}$$

where Φ is a function of r only and does not depend on θ. Equation (2) is of Euler-Cauchy type and it can be solved by introducing a new variable $t = \ln r$ to transform it to a differential equation with constant coefficients. This procedure yields the basic solutions:

$$r^2, \quad \ln r, \quad r^2 \ln r.$$

If Φ is a function of both r and θ, equation (1) takes the form:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\Phi = 0. \tag{3}$$

Equation (3) can be solved by the separation of variable technique to obtain the basic solutions:

$$f(r)\cos(n\theta), \quad f(r)\sin(n\theta), \quad r\theta\cos\theta, \quad r\theta\sin\theta, \quad r^2\theta,$$

where $f(r)$ could be a polynomial or a power series in r, and n is an integer. The general solution of equation (1) which provides a means of solving problems involving circular and radial boundaries, whether or not they are axially symmetric, is a series solution involving Fourier series in θ and power series in r. The detailed expressions can be found in the literature ([2],[7]) .

3. Systematic Generation of Additional Biharmonic Functions

Our concern here is to develop a systematic way of generating additional solutions of the biharmonic equation (3) by a means other than the separation of variable technique. Noting that the biharmonic operator ∇^4 is obtained by repeated application of Laplace's operator ∇^2, where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}, \tag{4}$$

it is reasonable to expect that a close relationship exists between harmonic and biharmonic functions. Indeed, one can combine two harmonic functions in a specific manner to obtain a biharmonic function [8]. A harmonic (or biharmonic) function defined in a region $\mathcal{D}$ can also be transformed into an image harmonic (or biharmonic) function defined in a region $\tilde{\mathcal{D}}$ ([4], [9]). These two ideas form the foundation for the present work.

3.1. Almansi's Theorem

Almansi's theorem [8] gives a relationship between harmonic and biharmonic functions . In order to arrive at some deductions which are crucial for further discussion, we shall state and prove this theorem for a two-dimensional region thus:

Theorem 1 (Almansi). *Let $u_1(r,\theta)$ and $u_2(r,\theta)$ be two harmonic functions in a two dimensional region $\mathcal{D}$, then*

$$u(r,\theta) = (r^2 - a^2)u_1(r,\theta) + u_2(r,\theta) \tag{5}$$

is biharmonic, where $r^2 = x^2 + y^2$ and a is an arbitrary constant. Conversely, if $u(r,\theta)$ is a given biharmonic function in a two-dimensional region $\mathcal{D}$ whose boundary is intersected in at most one point by each radius vector with an origin inside $\mathcal{D}$, then two harmonic functions, $u_1(r,\theta)$ and $u_2(r,\theta)$, can be determined such that equation (5) holds.

Proof. Let $u_1(r,\theta)$ and $u_2(r,\theta)$ be harmonic functions, then $\nabla^2 u_1 = 0$ and $\nabla^2 u_2 = 0$. By direct differentiation of equation (5) we obtain

$$\nabla^2 u = 4u_1 + 4r\frac{\partial u_1}{\partial r}. \tag{6}$$

In order to get the result (6), we have made use of the identity

$$\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2\nabla f.\nabla g, \tag{7}$$

with $f = r^2 - a^2$ and $g = u_1(r,\theta)$. A repeated application of the Laplace operator to the result (6) gives $\nabla^4 u = 0$. This completes the proof of the first part.

Conversely, let u be a biharmonic function and suppose that there exist two functions u_1 and u_2 such that equation (5) holds. Then, u_2 is harmonic if and only if

$$\nabla^2 u_2 \equiv \nabla^2 \left[u - (r^2 - a^2)u_1\right] = 0. \tag{8}$$

By equation (8), we obtain the linear differential equation,

$$\frac{\partial u_1}{\partial r} + \frac{u_1}{r} = \frac{1}{4}\left(\frac{1}{r}\right)\nabla^2 u,$$

whose solution is

$$u_1 = \frac{1}{4}\left(\frac{1}{r}\right)\int \nabla^2 u\, dr. \tag{9}$$

Since u is biharmonic,

$$\nabla^4 u = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]\nabla^2 u = 0,$$

and

$$\frac{\partial^2}{\partial\theta^2}\nabla^2 u = -r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\nabla^2 u. \tag{10}$$

By differentiating equation (9) and using (10), we obtain

$$4\nabla^2 u_1 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\left[\frac{1}{r}\int \nabla^2 u\, dr\right] - \frac{1}{r^3}\int r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\nabla^2 u\, dr. \tag{11}$$

It is readily verified that the right side of equation (11) vanishes, showing that u_1 as determined by equation (9) is harmonic. This completes the proof. □

Two important results that follow from Almansi's theorem are presented in Theorem 2.

Theorem 2. *(i) If $f(r,\theta)$ is a harmonic function in a two-dimensional region $\mathcal{D}$, then $(r^2 - a^2)f(r,\theta)$ is a biharmonic function in $\mathcal{D}$;*

(ii) If $F(r,\theta)$ is a biharmonic function in a two-dimensional region $\mathcal{D}$, then $\frac{1}{r}\int \nabla^2 F(r,\theta)\, dr$ is a harmonic function in $\mathcal{D}$.

Proof. The proof of the first part of Theorem 2 readily follows by direct differentiation and the use of identity (7) with $g = r^2 - a^2$ and $f = f(r,\theta)$.

For the second part, let $F(r,\theta)$ be a biharmonic function. We know that

$$4\nabla^2\left[\frac{1}{r}\int \nabla^2 F(r,\theta)\, dr\right] = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\int \frac{1}{r}\nabla^2 F(r,\theta)\, dr + \frac{1}{r^3}\int \frac{\partial^2}{\partial\theta^2}\nabla^2 F(r,\theta)\, dr. \tag{12}$$

By direct differentiation of the first term on the right side of equation (12), we find that it is equal to

$$\frac{1}{r}\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\nabla^2 F(r,\theta) + \frac{1}{r^3}\int \nabla^2 F(r,\theta)\, dr. \tag{13}$$

Also, by integration of the second term on the right side of equation (12), taking into account the relationship (10), we find that it is an additive inverse of the first term. Consequently,

$$\nabla^2\left[\frac{1}{r}\int \nabla^2 F(r,\theta)\, dr\right] = 0.$$

□

3.2. Kelvin's Inversion Theorem

Kelvin's inversion theorem provides a means of transforming a harmonic (or biharmonic) function defined in a region $\mathcal{D}$ into a harmonic (or biharmonic) function defined in the Kelvin image region $\tilde{\mathcal{D}}$. The Kelvin image $\tilde{\mathcal{D}}$ of a region $\mathcal{D}$ in a plane (or in space) consists of all points $\tilde{r}$ generated by the transformation:

$$r \mapsto \tilde{r} = \frac{a^2}{r}, \qquad r\tilde{r} \neq 0, \tag{14}$$

which maps every point at a radial distance r from a given point O to its inverse point with respect to a circle (or a sphere) of radius $a > 0$ with centre at O. The inverse transformation, obtained by interchanging r and $\tilde{r}$ in equation (14), is also a Kelvin transformation.

Kelvin's inversion theorem for harmonic functions [4] and its extension to biharmonic functions [9] may be stated with respect to a spherical coordinate system (r, θ, ϕ) as follows.

Theorem 3 (Kelvin)**.** *(i) If $u(r, \theta, \phi)$ is a harmonic function, then so is $v = \frac{a^2}{r} u\left(\frac{a^2}{r}, \theta, \phi\right)$.*

(ii) If $U(r, \theta, \phi)$ is a biharmonic function, then so is $V = rU\left(\frac{a^2}{r}, \theta, \phi\right)$.

The details of the proof of Theorem 3 by direct differentiation have been provided in the cited references and will not be repeated here. However, it is noted that the fundamental singularity of Laplace's operator is $\frac{1}{r}$ while r is the fundamental singularity of the biharmonic operator. Thus, the transformed function must be multiplied by the fundamental singularity of the relevant operator for harmonicity and biharmonicity to be preserved under the Kelvin transformation.

In the case of two dimensions, the application of Kelvin's transformation (14) to Laplace's operator and the biharmonic operator in plane polar coordinates (r, θ) gives

$$\nabla^2 f(r, \theta) = \frac{\tilde{r}^4}{a^4} \tilde{\nabla}^2 f(\tilde{r}, \theta), \qquad \nabla^4 F(r, \theta) = \frac{\tilde{r}^6}{a^6} \tilde{\nabla}^4 \left[\frac{\tilde{r}^2}{a^2} F(\tilde{r}, \theta)\right], \tag{15}$$

where the quantities ∇, $f(r, \theta)$ and $F(r, \theta)$ are defined in a two-dimensional region $\mathcal{D}$ while $\tilde{\nabla}$, $f(\tilde{r}, \theta)$ and $F(\tilde{r}, \theta)$, respectively, are the corresponding quantities in the Kelvin image $\tilde{\mathcal{D}}$ of $\mathcal{D}$. The results (15) are obtained by making use of the fact that the chain rule for differentiation gives,

$$\frac{\partial}{\partial r} = -\left(\frac{\tilde{r}^2}{a^2}\right) \frac{\partial}{\partial \tilde{r}},$$

$$\frac{\partial^2}{\partial r^2} = 2\left(\frac{\tilde{r}^3}{a^4}\right)\frac{\partial}{\partial \tilde{r}} + \left(\frac{\tilde{r}^4}{a^4}\right)\frac{\partial^2}{\partial \tilde{r}^2} \tag{16}$$

where $\tilde{r} = a^2/r$. Equation (15) leads to two results which are presented below in Theorem 4.

Theorem 4. *Let $\tilde{\mathcal{D}}$ be the Kelvin image of a two-dimensional region $\mathcal{D}$.*
(i) If $f(r,\theta)$ is harmonic in $\mathcal{D}$, then $f\left(\frac{a^2}{r},\theta\right)$ is harmonic in $\tilde{\mathcal{D}}$.
(ii) If $F(r,\theta)$ is biharmonic in $\mathcal{D}$, then $r^2F\left(\frac{a^2}{r},\theta\right)$ is biharmonic in $\tilde{\mathcal{D}}$.

Proof. By equation (15),

$$r^2\nabla^2 f(r,\theta) = \tilde{r}^2\tilde{\nabla}^2 f(\tilde{r},\theta)$$

and

$$r^3\nabla^4 F(r,\theta) = \tilde{r}^3\tilde{\nabla}^4\left(\frac{\tilde{r}^2}{a^2}F(\tilde{r},\theta)\right).$$

By hypothesis, $\nabla^2 f(r,\theta) = 0$ and $\nabla^4 F(r,\theta) = 0$. Therefore, the right hand sides of the above equations vanish, and the proof is complete. □

Theorem 4 shows that, in two dimensions, the Kelvin transformation preserves harmonicity while its action on a biharmonic function multiplies the transformed function by the fundamental singularity, r^2, of the biharmonic operator. There is, thus, a slight difference between the action of the Kelvin transformation on harmonic and biharmonic functions in two- and three- dimensional spaces.

4. Further Results

Further results obtained by a systematic application of Theorems 2 and 4 are presented below in Theorem 5.

Theorem 5. *If $F(r,\theta)$ is a biharmonic function , then*
(i) $\left(r - \frac{a^2}{r}\right)\int \nabla^2 F(r,\theta)\,dr$ is biharmonic;
(ii) $r\int r^2\nabla^2 F\left(\frac{a^2}{r},\theta\right)\,dr$ is harmonic;
(iii) $\left(r^3 - a^2 r\right)\int r^2\nabla^2 F\left(\frac{a^2}{r},\theta\right)\,dr$ is biharmonic .

Proof.

(i) Let $\phi = \left(r - \frac{a^2}{r}\right) \int \nabla^2 F(r,\theta)\, dr$. Then, $\phi = fg$, where $f = r^2 - a^2$ and $g = \frac{1}{r}\int \nabla^2 F(r,\theta)\, dr$. Using the identity (7) and simplifying the resulting expressions, we get

$$\nabla^2 \left[\left(r - \frac{a^2}{r}\right) \int \nabla^2 F(r,\theta)\, dr\right] = 4\nabla^2 F(r,\theta). \tag{17}$$

The biharmonicity of ϕ follows immediately by a repeated application of the Laplace operator to equation (17) and the fact that $F(r,\theta)$ is biharmonic, by hypothesis.

(ii) By Kelvin's transformation (14),

$$r \int r^2 \nabla^2 F\left(\frac{a^2}{r}, \theta\right) dr \qquad \equiv \qquad -\frac{a^4}{\tilde{r}} \int \tilde{\nabla}^2 F(\tilde{r},\theta)\, d\tilde{r},$$

and by the first part of equation (15)

$$\nabla^2 \left[r \int r^2 \nabla^2 F\left(\frac{a^2}{r}, \theta\right) dr\right] = -\tilde{r}^4 \tilde{\nabla}^2 \left[\frac{1}{\tilde{r}} \int \tilde{\nabla}^2 F(\tilde{r},\theta)\, d\tilde{r}\right]. \tag{18}$$

The harmonicity of $\frac{1}{\tilde{r}}\int \tilde{\nabla}^2 F(\tilde{r},\theta)\, d\tilde{r}$ follows from Theorem 2(ii) when $\tilde{r}$ is written in place of r and $\tilde{\nabla}$ is written in place of ∇. Therefore, the right side of equation (18) vanishes and the proof is complete.

(iii) Let

$$\psi = r \int r^2 \nabla^2 F\left(\frac{a^2}{r}, \theta\right) dr,$$

then

$$\nabla^4 \left[\left(r^3 - a^2 r\right) \int r^2 \nabla^2 F\left(\frac{a^2}{r}, \theta\right) dr\right] = \nabla^4 \left[\left(r^2 - a^2\right) \psi\right].$$

On using the identity (7) repeatedly, we get

$$\nabla^4 \left[\left(r^2 - a^2\right) \psi\right] = 4\nabla^2 \left(\psi + r\frac{\partial \psi}{\partial r}\right) = 4\left(2 + r\frac{\partial}{\partial r}\right) \nabla^2 \psi = 0, \tag{19}$$

since ψ is a harmonic function, by Theorem 5(ii). □

5. Concluding Remarks

In this paper, we have presented a systematic way of generating biharmonic functions without solving the biharmonic equation directly. The results stated in Theorems 2, 4 and 5 provide a means of finding additional biharmonic functions which may be required for the construction of complete solutions to biharmonic boundary value problems in continuum mechanics. The availability of such results could facilitate the application of the perturbative method of solution to two-dimensional biharmonic boundary-value problems involving composite media with curved geometry. Analogous results for three-dimensional biharmonic functions have been used to construct solutions for the Stokes's stream function induced by the presence of an arbitrary axisymmetric singularity in a space occupied by three spherical-layered immiscible incompressible viscous fluids [10].

Acknowledgments

The author gratefully acknowledges some helpful suggestions by Professor K. Aderogba, whose publications inspired this work.

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