Abstract: Recently an algebra based on proportional calculi was introduced by Tamilarasi and Mekalai in the year 2010 known as $TM$–algebras [3]. Kanadaraj and Chandramouleeswaran [5] introduced the notion of derivation on $d$–algebras. In [6], we introduced the notion of derivations on TM-algebras. In this paper, we introduce the notion of $t$–derivation on TM-algebras. We study the properties of regular $t$–derivations on a TM-algebra and prove that the set of all $t$–derivations on a TM-algebra forms a semigroup under a suitable binary composition.

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Key Words: BCK/BCI algebras, TM-algebras, derivations, $t$–derivations

1. Introduction

It is well known that BCK and BCI-algebras are two classes of algebras of logic. They were introduced by Imai and Iseki [1] and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper sub class of the BCI-algebras. J.Neggers and H.S.Kim [2] introduced the notion of $d$–algebras which is another generalization of BCK–algebras.
Recently another algebra based on proportional calculi was introduced by Tamilarasi and Mekalai in the year 2010 known as $TM$–algebras. In their paper [3] they claimed that $TM$–algebra was the generalization of BCK and BCI algebras. But this was proved wrong in [4], by giving counter examples.

Motivated by the notion of derivations on rings and near-rings Jun and Xin [7] studied the notion of derivation on BCI-algebras. In [5], the authors introduced the notion of derivation on $d$–algebras, another generalisation of BCK-algebras.

In our paper [6], we introduced the notion of derivation on $TM$–algebras. In this paper, we introduce the notion of $t$–derivation on TM-algebras. We study the properties of regular $t$–derivations on $X$ and prove that the set of all $t$–derivations on a TM-algebra $X$ forms a semigroup under a suitable binary composition.

2. Preliminaries

In this section, we recall some basic definitions and results that are needed for our work.

**Definition 2.1.** A $TM$–algebra $(X, *, 0)$ is a non-empty set $X$ with a constant 0 and a binary operation $*$ satisfying the following axioms:

1. $x * 0 = x$

2. $(x * y) * (x * z) = z * y \forall x, y, z \in X$.

**Lemma 2.2.** The following properties hold in a $TM$–algebra $X$.

1. $x * x = 0$.

2. $(x * y) * x = 0 * y$.

3. $x * (x * y) = y$.

4. $(x * y) * z = (x * z) * y$.

5. $x * 0 = 0 \Rightarrow x = 0$. In other words $x \leq 0 \Rightarrow x = 0$.

6. $0 * (x * y) = y * x = (0 * x) * (0 * y)$.

7. $(x * z) * (y * z) = (x * y)$.
Remark 2.3. In a $TM$–algebra $X$, by definition, $x \land y = y \ast (y \ast x)$. However, by property (3) above, we have $x = y \ast (y \ast x)$. Hence, in a $TM$-algebra we have $x \land y = x \ \forall \ x, y \in X$.

Definition 2.4. In any $TM$–algebra $X$, we define a partial order $\leq$ by putting $x \leq y$ if and only if $x \ast y = 0$.

Definition 2.5. A non-empty subset $S$ of a $TM$–algebra $(X, \ast, 0)$ is said to be a subalgebra of $X$ if $x \ast y \in S$ whenever $x, y \in S$.

Definition 2.6. Let $(X, \ast, 0)$ be a $TM$–algebra. A self map $d : X \to X$ is said to be a $(l, r)$–derivation on $X$, if $d(x \ast y) = (d(x) \ast y) \land (x \ast d(y))$. $d$ is said to be a $(r, l)$–derivation on $X$, if $d(x \ast y) = (x \ast d(y)) \land (d(x) \ast y)$. It is said to be a derivation on $X$ if $d$ is both a $(l, r)$–derivation and a $(r, l)$–derivation on $X$.

3. $t$–Derivations on $TM$–Algebra

Definition 3.1. A $TM$–algebra $X$ is said to be associative if $(x \ast y) \ast z = x \ast (y \ast z)$ for all $x, y, z \in X$.

Example 3.2. Let $(X, \ast, 0)$ be a $TM$–algebra with the Cayley table.

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Then $X$ is an associative $TM$–algebra.

Definition 3.3. Let $X$ be a $TM$–algebra. Then for any $t \in X$, we define a self map $d_t : X \to X$ by $d_t(x) = x \ast t$ for all $x \in X$.

Definition 3.4. Let $X$ be a $TM$–algebra. Then for any $t \in X$, a self map $d_t : X \to X$ is called a $(l, r) – t$–derivation of $X$ if it satisfies the condition $d_t(x \ast y) = (d_t(x) \ast y) \land (x \ast d_t(y))$ for all $x, y \in X$.

Example 3.5. Let $(X, \ast, 0)$ be a $TM$–algebra with the following Cayley table.

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Define, when \( t = 0 \), \( d_t(x) = x \ \forall \ x \in X \).

when \( t = 1 \), \( d_t(0) = 2, \ d_t(1) = 0, \ d_t(2) = 1 \).
when \( t = 2 \), \( d_t(0) = 1, \ d_t(1) = 2, \ d_t(2) = 0 \).
For each \( t \in X \), \( d_t \) is a \((l, r) - t\) derivation of \( X \).

**Remark 3.6.** In a \( TM \) algebra, \( x \wedge y = y \ast (y \ast x) = x \ \forall \ x, y \in X \). We can observe that by using the above property we take \( d_t \) is a \((l, r) - t\) derivation of \( X \) then \( d_t(x \ast y) = d_t(x) \ast y \).

**Definition 3.7.** Let \( X \) be a \( TM \) algebra. Then for any \( t \in X \) a self map \( d_t : X \rightarrow X \) is called a \((r, l) - t\) derivation of \( X \) if it satisfies the condition \( d_t(x \ast y) = (x \ast d_t(y)) \wedge (d_t(x) \ast y) \) for all \( x, y \in X \).

**Remark 3.8.** We can observe that, if \( d_t \) is a \((r, l) - t\) derivation of \( X \), then \( d_t(x \ast y) = x \ast d_t(y) \) for all \( x, y \in X \).

**Definition 3.9.** Let \( X \) be a \( TM \) algebra. Then for any \( t \in X \), a self map \( d_t : X \rightarrow X \) is called a \( t \) derivation on \( X \) if \( d_t \) is both a \((l, r) - t\) derivation and a \((r, l) - t\) derivation on \( X \).

**Example 3.10.** Consider the \( TM \) algebra \((X, \ast, 0)\) in 3.2. Define the mapping \( d_t \) as follows:

When \( t = 0 \), \( d_t(x) = x \ \forall \ x \in X \).
When \( t = 1 \), \( d_t(0) = 1, \ d_t(1) = 0, \ d_t(2) = 3, \ d_t(3) = 2 \).
When \( t = 2 \), \( d_t(0) = 2, \ d_t(1) = 3, \ d_t(2) = 0, \ d_t(3) = 1 \).
When \( t = 3 \), \( d_t(0) = 3, \ d_t(1) = 2, \ d_t(2) = 1, d_t(3) = 0 \).
For each \( t \in X \), \( d_t \) is a \( t \) derivation of \( X \).

**Remark 3.11.** Any self map \( d_t \) of a \( TM \) algebra \( X \) is a \((l, r) - t\) derivation on \( X \).

**Proposition 3.12.** Let \( d_t \) be a self map of an associative \( TM \) algebra \( X \). Then \( d_t \) is a \((r, l) - t\) derivation of \( X \).

**Proof.** Let \( X \) be an associative \( TM \) algebra. Then we have

\[
\begin{align*}
 d_t(x \ast y) &= (x \ast y) \ast t \\
 &= (x \ast t) \ast y \quad (\therefore (x \ast y) \ast z = (x \ast z) \ast y) \\
 &= ((x \ast t) \ast y) \ast 0 \\
 &= (((x \ast t) \ast y) \ast ((x \ast t) \ast y) \ast ((x \ast t) \ast y)) \\
 &\quad (\therefore x \ast x = 0) \\
 &= (((x \ast t) \ast y) \ast ((x \ast t) \ast y) \ast ((x \ast y) \ast t)) \\
 &\quad (\therefore (x \ast y) \ast z = (x \ast z) \ast y)
\end{align*}
\]
\[
\begin{align*}
&= ((x * t) * y) * (((x * t) * y) * (x * (y * t))) \\
&= (x * (y * t)) \land ((x * t) * y) \\
&= (x * d_t(y)) \land (d_t(x) * y)
\end{align*}
\]
\[
\therefore d_t \text{ is a } (r, l) - t\text{-derivation of } X.
\]

By combining the remark 3.11 and proposition 3.12, we get the following theorem.

**Theorem 3.13.** Let \( X \) be an associative \( TM \)-algebra. For ant \( t \in X \), a self map \( d_t \) is a \( t \)-derivation on \( X \).

**Definition 3.14.** A self map \( d_t \) of a \( TM \)-algebra \( X \) is said to be \( t \)-regular if \( d_t(0) = 0 \).

**Example 3.15.** In example 3.10 \( d_t \) is a regular \( t \)-derivation on \( X \) when \( t = 0 \). However, \( t = 1 \) or \( t = 2 \) or \( t = 3 \), \( d_t \) is not a regular \( t \)-derivation of \( X \).

**Proposition 3.16.** For any self map \( d_t \) of a \( TM \)-algebra \( X \), the following holds:

1. If \( d_t \) is a \( (l, r) \)-\( t \)-derivation of \( X \), \( d_t(x) = d_t(x) \land x \forall x \in X \).

2. If \( d_t \) is a \( (r, l) \)-\( t \)-derivation of \( X \), \( d_t(x) = x \land d_t(x) \) for all \( x \in X \) if and only if \( d_t \) is \( t \)-regular.

**Proof.**

1. Let \( d_t \) be a \( (l, r) \)-\( t \)-derivation of \( X \). Then we have

\[
\begin{align*}
  d_t(x) & = d_t(x * 0) \\
        & = (d_t(x) * 0) \land (x * d_t(0)) \\
        & = d_t(x) \land (x * d_t(0)) \\
        & = (x * d_t(0)) * ((x * d_t(0)) * d_t(x)) \\
        & = (x * d_t(0)) * ((x * d_t(x)) * d_t(0) \quad (\because (x * y) * z = x * (z * y)) \\
        & = x * (x * d_t(x)) \quad (\because (x * z) * (y * z) = x * y) \\
        & = d_t(x) \land x
\end{align*}
\]

\[
\therefore d_t \text{ is a } (l, r) - t\text{-derivation of } X.
\]

2. Let \( d_t \) be a \( (r, l) \)-\( t \)-derivation of \( X \) and \( d_t(x) = x \land d_t(x) \quad \cdots \cdots (1) \).
Put $x = 0$ in (1), we have

\[
\begin{align*}
d_t(0) &= 0 \land d_t(0) \\
&= d_t(0) \ast (d_t(0) \ast 0) \\
&= d_t(0) \ast d_t(0) \\
&= 0
\end{align*}
\]

\[
\therefore \ d_t \text{ is } t-\text{regular.}
\]

Conversely, suppose that $d_t$ is $t-$regular $(r, l) - t-$derivation of $X$. Then

\[
\begin{align*}
d_t(x) &= d_t(x \ast 0) \\
&= (x \ast d_t(0)) \land (d_t(x) \ast 0) \\
&= (x \ast 0) \land d_t(x) \quad (\because d_t(0) = 0) \\
&= x \land d_t(x)
\end{align*}
\]

Hence the proof.

**Theorem 3.17.** Let $d_t$ be a $(l, r) - t-$derivation of a $TM-$algebra. Then the following hold.

1. $d_t(0) = d_t(x) \ast x \ \forall \ x \in X$.

2. $d_t$ is one-one.

3. $d_t$ is $t-$regular then it is the identity map.

4. If there is an element $x \in X$ such that $d_t(x) = x$, then $d_t$ is the identity map.

5. If $x \leq y$ then $d_t(x) \leq d_t(y)$ for all $x, y \in X$.

**Proof.**

1. Let $d_t$ be a $(l, r) - t-$derivation of a $TM-$algebra $X$.

Then we have $d_t(0) = d_t(x \ast x) = d_t(x) \ast x \quad (\because d_t \text{ is a } (l, r) - t-\text{derivation})$

2. Let $d_t(x) = d_t(y)$ for all $x, y \in X$.

Then $x \ast t = y \ast t$ and by applying the right cancellation law we have, $x = y$. 
3. Let $d_t$ be a $t$–regular and $x \in X$. Now,

$$x \ast x = 0 = d_t(0) = d_t(x \ast x) = d_t(x) \ast x$$

Hence by right cancellation law, $d_t(x) = x \forall x \in X$, showing that $d_t$ is the identity map.

4. Let $d_t(x) = x$ for some $x \in X$.

$$0 = x \ast x = d_t(x) \ast x = d_t(x \ast x) = d_t(0).$$

showing that $d_t$ is $t$–regular. Hence by (3) $d_t$ is the identity map

5. Since $x \leq y$,

$$d_t(x) \ast d_t(y) = (x \ast t) \ast (y \ast t) = (x \ast y) = 0$$

thus proving $d_t(x) \leq d_t(y)$.

**Theorem 3.18.** Let $X$ be a $TM$–algebra and $d_t$ be a $t$–derivation on $X$. If $x \leq y$ and $d_t(x \ast y) = d_t(x) \ast d_t(y)$ for all $x, y \in X$. Then $d_t(x) = d_t(y)$.

**Proof.**

$$d_t(x) = d_t(x \ast 0)$$
$$= d_t(x \ast (x \ast y)) \quad (\because x \leq y)$$
$$= d_t(x) \ast d_t(x \ast y) \quad (\because d_t(x \ast y) = d_t(x) \ast d_t(y))$$
$$= d_t(x) \ast (d_t(x) \ast d_t(y))$$
$$= d_t(y) \quad (\because x \ast (x \ast y) = y)$$

**Theorem 3.19.** Let $d_t$ be a $t$–regular $(r, l)$$t$–derivation of a $TM$–algebra $X$. Then the following hold.

1. $d_t(x) = x$.

2. $d_t(x) \ast y = x \ast d_t(y)$ for all $x, y \in X$.

3. $d_t(x \ast y) = d_t(x) \ast y = d_t(x) \ast d_t(y) = x \ast d_t(y)$.

4. $Ker(d_t) = \{x \in X : d_t(x) = 0\}$ is a sub algebra of $X$.

**Proof.**

1. Since $d_t$ is $t$–regular $(r, l)$–$t$–derivation of $X$, for any $x \in X$, we have

$$d_t(x) = d_t(x \ast 0) = x \ast d_t(0) = x \ast 0 = x.$$
2. If $d_t$ is $t$-regular $(r, l) - t$-derivation of $X$ then by (1), $d_t(x) = x$ for all $x \in X$.
   Thus, $d_t(x) * y = x * y = x * d_t(y)$.

3. If $d_t$ is $t$-regular $(r, l) - t$-derivation of $X$ then by (1), $d_t(x) = x \forall x \in X \cdots \cdots (1)$.
   For $x, y \in X$, $d_t(x * y) = x * y = d_t(x) * d_t(y)$ \quad \text{(By (1))}
   If $d_t$ is a $(r, l) - t$-derivation of $X$ then $d_t(x * y) = x * d_t(y)$.
   $d_t(x * y) = x * y = d_t(x) * y$. \quad \text{($\therefore x = d_t(x)$)}
   Hence $d_t(x * y) = d_t(x) * y = x * d_t(y) = x * y$.

4. Since $d_t$ is $t$-regular, $d_t(0) = 0$. Then $0 \in Ker(d_t)$ showing that $Ker(d_t)$ is a non-empty set.
   Let $x, y \in Ker(d_t)$, then $d_t(x) = 0, d_t(y) = 0$. Now
   
   $d_t(x * y) = x * y = d_t(x) * d_t(y) = 0 * 0 = 0$.
   
   Therefore $(x * y) \in Ker(d_t)$, proving that $Ker(d_t)$ is a sub-algebra of $X$.

**Proposition 3.20.** Let $X$ be a $TM$-algebra. Then $Ker(d_t) = \{0\}$ if and only if $d_t$ is $t$-regular.

**Proof.** Obviously when $Ker(d_t) = \{0\}$ $d_t(0) = 0$, showing that $d_t$ is $t$-regular.
   On the other hand, if $x \in Ker(d_t)$, $d_t$ is $t$-regular shows that,
   
   $0 = d_t(0) = d_t(x * x) = d_t(x) * x = 0 * x$.
   
   Thus, $x = 0$, showing that $Ker(d_t) = \{0\}$.

**Definition 3.21.** Let $X$ be a $TM$-algebra and let $d_t, d'_t$ be two self maps of $X$. Then we define $d_t \circ d'_t : X \rightarrow X$ by $(d_t \circ d'_t)(x) = d_t(d'_t(x))$ for all $x \in X$.

**Example 3.22.** Consider the TM-algebra given in example 3.2. The self-maps $d_t, d'_t : X \rightarrow X$ given by
   
   $d_t(0) = 1, d_t(1) = 0, d_t(2) = 3, d_t(3) = 2$
   $d'_t(0) = 2, d'_t(1) = 3, d'_t(2) = 0, d'_t(3) = 1$ are $t$-derivations on $X$.
   Now define a self map $(d_t \circ d'_t) : X \rightarrow X$ by
   
   $(d_t \circ d'_t)(0) = 3, (d_t \circ d'_t)(1) = 2, (d_t \circ d'_t)(2) = 1, (d_t \circ d'_t)(3) = 0$.
   Then it is easily checked that $(d_t \circ d'_t)(x) = d_t(d'_t(x))$ for all $x \in X$ is also a $t$-derivation of $X$. 
Proposition 3.23. Let \( X \) be a \( TM \)-algebra and let \( d_t, d'_t \) be a \((l, r) - t\) -derivation of \( X \). Then \( (d_t \circ d'_t) \) is also a \((l, r) - t\) -derivation of \( X \).

Proof. Let \( X \) be a \( TM \)-algebra and let \( d_t, d'_t \) be \((l, r) - t\) -derivations of \( X \). Then for all \( x, y \in X \). We have

\[
(d_t \circ d'_t)(x \ast y) = d_t(d'_t(x \ast y)) \\
= d_t(d'_t(x) \ast y) \quad (\because d'_t \text{ is a } (l, r) - t \text{-derivation of } X) \\
= (d_t(d'_t(x)) \ast y) \quad (\because d_t \text{ is a } (l, r) - t \text{-derivation of } X) \\
= (d_t \circ d'_t)(x) \ast y
\]

\( \therefore (d_t \circ d'_t) \) is a \((l, r) - t\) -derivation of \( X \).

Proposition 3.24. Let \( X \) be a \( TM \)-algebra and let \( d_t, d'_t \) be \((r, l) - t\) -derivations of \( X \). Then \( (d_t \circ d'_t) \) is also a \((r, l) - t\) -derivation of \( X \).

Proof. Let \( X \) be a \( TM \)-algebra and let \( d_t, d'_t \) be \((r, l) - t\) -derivations of \( X \).

Then for all \( x, y \in X \). We have

\[
(d_t \circ d'_t)(x \ast y) = d_t(d'_t(x \ast y)) \\
= d_t(x \ast d'_t(y)) \quad (\because d'_t \text{ is a } (r, l) - t \text{-derivation of } X) \\
= x \ast d'_t(d_t(y)) \quad (\because d_t \text{ is a } (r, l) - t \text{-derivation of } X) \\
= x \ast (d_t \circ d'_t)(y)
\]

\( \therefore (d_t \circ d'_t) \) is a \((r, l) - t\) -derivation of \( X \).

Combining the above two propositions we get the following theorem.

Theorem 3.25. Let \( X \) be a \( TM \)-algebra and let \( d_t, d'_t \) be \( t \)-derivation of \( X \). Then \( (d_t \circ d'_t) \) is also a \( t \)-derivation of \( X \).

Theorem 3.26. Let \( X \) be a \( TM \)-algebra. Let \( d_t \) be a \((r, l) - t\) -derivation of \( X \) and \( d'_t \) be a \((l, r) - t\) -derivation of \( X \). Then \( d_t \circ d'_t = d'_t \circ d_t \).

Proof. Let \( d'_t \) be a \((l, r) - t\) -derivation of \( X \). Then we have \( d'_t(x \ast y) = d'_t(x) \ast y \).

Now \( (d_t \circ d'_t)(x \ast y) = d_t(d'_t(x \ast y)) = d_t(d'_t(x)) \ast y = d'_t(x) \ast d_t(y) \quad \cdots \cdots (1) \quad (\because d_t \text{ is a } (r, l) - t \text{-derivation of } X) \)

Again \( (d_t \circ d'_t)(x \ast y) = d'_t(d_t(x \ast y)) \)
\[
\dot{d}_t(x \ast d_t(y)) \\
= \dot{d}_t(x \ast d_t(y)) \\
(: \cdot \dot{d}_t \text{ is a } (r, l) - t \text{- derivation of } X) \\
= \dot{d}_t(x) \ast d_t(y) \quad \cdots \cdots \quad (2) \\
(: \cdot \dot{d}_t \text{ is a } (l, r) - t \text{- derivation of } X)
\]

From (1) and (2), \((d_t \circ \dot{d}_t)(x \ast y) = (\dot{d}_t \circ d_t)(x \ast y)\).

This is true for all \(x, y \in X\). In particular this true for all \(x\) and \(y = 0\).

Put \(y = 0\), \((d_t \circ \dot{d}_t)(x \ast 0) = (\dot{d}_t \circ d_t)(x \ast 0)\)

\((d_t \circ \dot{d}_t)(x) = (\dot{d}_t \circ d_t)(x)\) for all \(x \in X\).

Hence \(d_t \circ \dot{d}_t = \dot{d}_t \circ d_t\).

The following theorem can be easily obtained by above theorem 3.26.

**Theorem 3.27.** Let \(X\) be a TM-algebra and let \(d_t, \dot{d}_t\) be two \(t\)-derivations of \(X\), then \(d_t \circ \dot{d}_t = \dot{d}_t \circ d_t\).

**Definition 3.28.** Let \(X\) be a TM-algebra and let \(d_t, \dot{d}_t\) be two self maps of \(X\). Then we define \(d_t \ast \dot{d}_t : X \rightarrow X\) defined by \((d_t \ast \dot{d}_t)(x) = d_t(x) \ast \dot{d}_t(x)\) for all \(x \in X\).

**Example 3.29.** Consider the TM-algebra \((X, \ast, 0)\) given in 3.2. Define \(d_t : X \rightarrow X\) by

\[d_t(0) = 1, \; d_t(1) = 0, \; d_t(2) = 3, \; d_t(3) = 2\] be a \(t\)-derivation of \(X\).

Define \(\dot{d}_t : X \rightarrow X\) by \(\dot{d}_t(0) = 2, \; \dot{d}_t(1) = 3, \; \dot{d}_t(2) = 0, \; \dot{d}_t(3) = 1\) be a \(t\)-derivation of \(X\).

Now \((d_t \ast \dot{d}_t)(0) = 3 = d_t(0) \ast \dot{d}_t(0)\).

\((d_t \ast \dot{d}_t)(1) = 3 = d_t(1) \ast \dot{d}_t(1)\) \((d_t \ast \dot{d}_t)(2) = 3 = d_t(2) \ast \dot{d}_t(2)\) \((d_t \ast \dot{d}_t)(3) = 3 = d_t(3) \ast \dot{d}_t(3)\).

**Theorem 3.30.** Let \(X\) be a TM-algebra and let \(d_t, \dot{d}_t\) be two \(t\)-derivations of \(X\). Then \(d_t \ast \dot{d}_t = \dot{d}_t \ast d_t\).

**Proof.** Let \(X\) be a TM-algebra and let \(d_t, \dot{d}_t\) be \(t\)-derivation of \(X\). Then we have

\[(d_t \circ \dot{d}_t)(x \ast y) = d_t(\dot{d}_t(x \ast y))
= d_t(d_t(x) \ast y)
= d_t(d_t(x) \ast y) \quad (: \cdot \dot{d}_t \text{ is a } (l, r) - t \text{- derivation of } X) \\
= \dot{d}_t(x) \ast d_t(y) \quad \cdots \cdots \quad (1) \\
(: \cdot \dot{d}_t \text{ is a } (r, l) - t \text{- derivation of } X)\]

Again \((d_t \circ \dot{d}_t)(x \ast y) = d_t(\dot{d}_t(x \ast y))\)
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Now, from (1) and (2), \( d_t(y) \) is also a \( (r, l) - t \)-derivation of \( X \).

From (1) and (2), \( d_t(x) * d_t(y) = d_t(x) * d_t(y) \) \( \cdots \cdots \cdots (2) \)

Hence \( d_t * d_t = d_t * d_t \).

**Definition 3.31.** Let \( L_t Der(X) \) denote the set of all \( (l, r) - t \)-derivations of \( X \). Define the binary operation \( \wedge \) on \( L_t Der(X) \) as follows: For \( d_t, d_t' \in L_t Der(X) \), define \( (d_t \wedge d_t')(x) = d_t(x) \wedge d_t'(x) \) \( \forall x \in X \).

**Lemma 3.32.** If \( d_t \) and \( d_t' \) are \( (l, r) - t \)-derivations on \( X \). Then \( (d_t \wedge d_t') \) is also a \( (l, r) - t \)-derivation on \( X \).

**Proof.** Let \( d_t, d_t' \) be \( (l, r) - t \)-derivation on \( X \). Then we have
\[
(d_t \wedge d_t')(x \ast y) = d_t(x \ast y) \wedge d_t'(x \ast y) \quad \text{(By definition)}
\]
\[
= (d_t(x) \ast y) \wedge (d_t'(x) \ast y) \quad \text{\( \therefore \) \( d_t, d_t' \) are \( (l, r) - t \)-derivations}
\]
\[
= (d_t'(x) \ast y) \ast ((d_t(x) \ast y) \ast (d_t(x) \ast y))
\]
\[
= d_t(x) \ast y \quad \cdots \cdots \cdots (1).
\]

Again,
\[
(d_t \wedge d_t')(x \ast y) = (d_t(x) \wedge d_t'(x)) \ast y
\]
\[
= (d_t(x) \ast (d_t'(x) \ast d_t(x))) \ast y
\]
\[
= d_t(x) \ast y \quad \cdots \cdots \cdots (2).
\]

From (1) and (2), \( (d_t \wedge d_t')(x \ast y) = (d_t \wedge d_t')(x \ast y) \). Hence \( (d_t \wedge d_t') \) is a \( (l, r) - t \)-derivation of \( X \).

**Lemma 3.33.** The binary composition \( \wedge \) defined on \( L_t Der(X) \) is associative.

**Proof.** Let \( X \) be a TM-algebra. Let \( d_t, d_t', d_t'' \) be \( (l, r) - t \)-derivations on \( X \). Now,
\[
((d_t \wedge d_t') \wedge d_t'')(x \ast y) = (d_t \wedge d_t')(x \ast y) \wedge d_t''(x \ast y)
\]
\[
= (d_t(x) \ast y) \wedge (d_t'(x) \ast y) \quad \text{(by lemma 3.32)}
\]
Then it is a semi-group under the binary operation $\land$. K. Iseki, An BCI-algebras, Math. Seminar Notes. 

From (1) and (2), (\(d_t(x) \ast y \ast ((d_t(x) \ast y) \ast (d_t(x) \ast y))\))
\[= d_t(x) \ast y \quad \text{\cdots \cdots (1)} \quad (\because y \ast (y \ast x) = x)\]

Again
\[d_t(x) \ast y \ast ((d_t(x) \ast y) \ast (d_t(x) \ast y))\]
\[= (d_t(x) \ast y) \ast (d_t(x) \ast y) \quad (\text{By lemma 3.32})\]
\[= (d_t(x) \ast y) \ast (d_t(x) \ast y)\]
\[\because d_t \text{ is a } (l, r) - t - \text{derivation of } X\]
\[= (d_t(x) \ast y) \ast (d_t(x) \ast y)\]
\[= d_t(x) \ast y \quad \text{\cdots \cdots (2)}\]

From (1) and (2), (\((d_t \land d_t')(x \ast y) = (d_t \land d_t')(x \ast y)\))

Put $y = 0$, we get \((d_t \land (d_t \land d_t')(x) = (d_t \land (d_t \land d_t'))(x)\) for all $x \in X$.

Hence \((d_t \land d_t) \land d_t' = d_t \land (d_t \land d_t')\).

This prove that the binary operation $\land$ is associative.

Combining the above two lemmas we get the following theorem.

**Theorem 3.34.** $L_tDer(X)$ is a semi-group under the binary operation $\land$ defined by \((d_t \land d_t')(x) = d_t(x) \land d_t'(x)\) for all \(x \in X\), and \(d_t, d_t' \in L_tDer(X)\).

**Definition 3.35.** Let $R_tDer(X)$ denote the set of all \((r, l) - t - \text{derivations}\) on $X$. Define the binary operation $\land$ on $R_tDer(X)$ as follows: For \(d_t, d_t' \in R_tDer(X)\). Define \((d_t \land d_t')(x) = d_t(x) \land d_t'(x)\).

Analogously we prove the following theorem.

**Theorem 3.36.** $R_tDer(X)$ is a semi-group under the binary operation $\land$ defined by \((d_t \land d_t')(x) = d_t(x) \land d_t'(x)\) for all $x \in X$ and \(d_t, d_t' \in R_tDer(X)\).

Combining the above two theorems we get the following theorem

**Theorem 3.37.** If $tDer(X)$ denotes the set of all \(t - \text{derivations}\) on $X$ then it is a semi-group under the binary operation $\land$ defined by \((d_t \land d_t')(x) = d_t(x) \land d_t'(x)\) for all $x \in X$ and \(d_t, d_t' \in tDer(X)\),

**References**


