MAJORIZATION FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS USING HURWITZ LERCH ZETA FUNCTION

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Abstract: In the present paper, we investigate the majorization problems for functions belonging to the class $J_{s,b}^{p,n} f(z)$ is considered. Moreover we point out some new or known consequences of our main result.

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1. Introduction  

Let $A_p(n)$ be the class of functions which are analytic and $p$-valent in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ of the form  

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{p+k}z^{p+k} \quad (p, n \in \mathbb{N} = \{1, 2, \ldots\}) \quad (1)$$

For $g(z) \in A_p(n)$ given by  

$$g(z) = z^p + \sum_{k=n}^{\infty} b_{k+p}z^{k+p} \quad (2)$$

the Hadamard product of $f(z)$ and $g(z)$ is denoted by

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\[(f \ast g)(z) = z^p + \sum_{k=n}^{\infty} a_{k+p}b_{k+p}z^{k+p} = (g \ast f)(z) \quad (3)\]

The following we recall a general Hurwitz-Lerch Zeta function \(\Phi(z, s, a)\) defined by (see [16])
\[
\Phi(z, s, a) := \sum_{k=0}^{\infty} \frac{z^k}{(k + a)^s} \quad (4)
\]

\((a \in \mathbb{C} \{\mathbb{Z}_{0}^-\}; s \in \mathbb{C}, \Re(s) > 1 and |z| = 1)\) where, as usual, \(\mathbb{Z}_{0}^- := \mathbb{Z} \setminus \{\mathbb{N}\} \quad (\mathbb{Z} := \{0, \pm1, \pm2, \pm3,\ldots\}; \mathbb{N} := \{1, 2, 3,\ldots\})\). Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function \(\Phi(z, s, a)\) can be found in the recent investigations by Choi and Srivastava [3], Ferreira and Lopez [4], Garg et al. [5], Lin and Srivastava [7], Lin et al. [8], and others.

For the class of analytic functions denote by \(A\) consisting of functions of the form \(f(z) = z + \sum_{k=2}^{\infty} a_kz^k\) Srivastava and Attiya [15] (see also Raducanu and Srivastava [13]) introduced and investigated the linear operator:
\[
\mathcal{J}_{s, b} : A \rightarrow A
\]
defined in terms of the Hadamard product (or convolution) by
\[
\mathcal{J}_{s, b}f(z) = G_{s, b} \ast f(z) \quad (5)
\]
\((z \in U; b \in \mathbb{C} \{\mathbb{Z}_{0}^-\}; \mu \in \mathbb{C}; f \in A)\), where, for convenience,
\[
G_{s, b}(z) := (1 + b)^s[\Phi(z, s, b) - b^{-s}] \quad (z \in U). \quad (6)
\]

It is easy to observe from (given earlier by [13]) (1), (5) and (6) that
\[
\mathcal{J}_{s, b}f(z) = z + \sum_{k=2}^{\infty} \left(1 + \frac{b}{k + b}\right)^s a_kz^k. \quad (7)
\]

Motivated essentially by the above-mentioned Srivastava-Attiya operator, we define the operator
\[
\mathcal{J}_{s, b}^{p, n}(f) : A_p(n) \rightarrow A_p(n)
\]
which is defined as
\[
\mathcal{J}_{s, b}^{p, n}f(z) = z^p + \sum_{k=1}^{\infty} C^s_b(p, n)a_{p+k}z^{p+k} \quad (z \in U; f(z) \in A_p) \quad (8)
\]
where
\[
C_s^b(k,p) = \left( \frac{p + b}{k + p + b} \right)^s
\]  
and (throughout this paper unless otherwise mentioned) the parameters \( s, b, \) are constrained as
\[ b \in \mathbb{C} \setminus \{ \mathbb{Z}_0^- \}; s \in \mathbb{C} \text{ and } p, \in \mathbb{N}. \]

It is easily verified from (8)
\[
z(J_{s+1,b}^{p,n} f)'(z) = (p + b)J_{s,b}^{p,n} f(z) - bJ_{s+1,b}^{p,n} f(z)
\]  
For two analytic functions \( f, g \in A_p \) we say that \( f \) is subordinate to \( g \) written \( f(z) \prec g(z) \) if there exists a schwarz function \( \omega(z) \) which (by definition) is analytic in \( U \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \) for all \( z \in U \), such that \( f(z) = g(\omega(z)), z \in U. \) Furthermore, if the function \( g(z) \) is univalent in \( U \), then we have the following equivalence
\[
f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(U) \subset g(U)
\]  
If \( f(z) \) and \( g(z) \) are analytic functions in \( U \), then due to MacGregor [9] we may say that \( f(z) \) is majorized by \( g(z) \) in \( U \) and written as
\[
f(z) \ll g(z) \quad (z \in U)
\]  
if there exists a function \( \phi(z) \), analytic in \( U \), such that
\[
|\phi(z)| < 1 \text{ and } f(z) = \phi(z)g(z) \quad (z \in U)
\]  
It is noted that the notation of majorization (12) is closely related to the concept of quasi-subordination between analytic functions in \( U \) which was considered earlier by [1, 2], on the other hand, investigated several majorization problems involving a number of subclasses of analytic functions in \( U \). In present sequel to the work of Altintas et al [2] we propose to investigate the corresponding majorization problem associated with the class of multivalent functions based on Srivastava-Attitya operator as defined below.

**Definition 1.** A function \( f(z) \in A_p(n) \) and suppose that \( g(z) \in J_{s,b}^{p,n}(\gamma) \) of \( p \)-valent functions of complex order \( \gamma \neq 0 \) in \( U \) if and only if
\[
\text{Re}(1 + \frac{1}{\gamma} (zJ_{s+1,b}^{p,n} f(q+1)(z) - p + q)) > 0
\]  
\[
(z \in U; p \in \mathbb{N}; q \in \mathbb{N}_0; \gamma \in \mathbb{C} - \{0\}; |2\gamma - (p + q)| \leq (p + q))
\]
Motivated by earlier works of [6, 10, 12, 14] in this paper we investigate majorization problems for the function class $J_{p,n}^{s,b}(\gamma)$ of $p$-valently stalike functions of complex order $\gamma \neq 0$ in open unit disc $U$.

2. Main Result

**Theorem 1.** A function $f(z) \in A_p(n)$ and suppose that $g(z) \in J_{s,b}^{p,n}(\gamma)$ if $[J_{s+1,b}^p f(z)](q)$ is majorized by $[J_{s+1,b}^p g(z)](q)$ in $U$ then

$$| [J_{s,b}^p f(z)](q) | \leq [J_{s,b}^p g(z)](q) \quad (|z| \leq r_0) \tag{15}$$

where $r_0$ is given by

$$r_0 = r_0(p, b, \gamma) = \frac{L - \sqrt{L^2 - 4(p + q)|2\gamma - (p + q)|}}{2|2\gamma - (p + q)|} \tag{16}$$

where $L = (p + b) + |2\gamma - (p + b)|; p \in N; \gamma \in C - (0)$

*Proof.* since $g(z) \in J_{s,b}^{p,n}(\gamma)$ we find from ,if

$$h(z) = 1 + \frac{1}{\gamma} \left( \frac{[J_{s+1,b}^p g(z)](q+1)}{[J_{s+1,b}^p g(z)](q)} - p + q \right) \tag{17}$$

then $\Re\{h(z)\} > 0(z \in U)$ and

$$h(z) = \frac{1 + \omega(z)}{1 - \omega(z)} (\omega \in A) \tag{18}$$

where

$$\omega(z) = c_1z + c_2z^2 + \ldots \tag{19}$$

and $A$ denotes the well known class of bounded analytic functions in $U$ and satisfies the conditions

$$\omega(0) = 0, and |\omega(z)| \leq |z|(z \in U) \tag{20}$$

making use of (17)and(18) we get

$$\frac{[J_{s+1,b}^p g(z)](q+1)}{[J_{s+1,b}^p g(z)](q)} = \frac{p + q + (2\gamma - p + q)\omega(z)}{1 - \omega(z)} \tag{21}$$
In view of equation (10)

\[ |[J_{s+1,b}^p g(z)](q)| \leq \frac{(1 + |z|)(p + b)}{p + b - (2\gamma - p + b)|z|} |[J_{s,b}^p g(z)](q)| \]  

(22)

Since \([J_{s+1,b}^p f(z)](q)\) is majorized by \([J_{s,b}^p g(z)](q)\) in \(U\) then we have

\[ [J_{s+1,b}^p f(z)](q) = \phi(z)[J_{s+1,b}^p g(z)](q) \]  

(23)

Differentiating with respect to \(z\) and then multiplying \(z\) we get

\[ z([J_{s+1,b}^p f(z)](q+1) = z\phi'(z)[J_{s+1,b}^p g(z)](q) + z\phi(z)[J_{s+1,b}^p g(z)](q+1) \]  

(24)

Using (10) in the above equation we get

\[ (p + b)[J_{s,b}^p f(z)](q) = (\frac{z}{p + b})(\phi'(z))[J_{s+1,b}^p g(z)](q) + \varphi(z)[J_{s,b}^p g(z)](q) \]  

(25)

Noting that the Schwarz function \(\phi(z)\) satisfies

\[ |\phi'(z)| \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2} \]  

(26)

and using (22) and (26) in (25) we have

\[ |[J_{s,b}^p f(z)](q)| \leq \{ |\phi(z)| + \frac{|z|(1 - |\phi(z)|^2)}{(1 - |z|)} \frac{1}{(p + b) - (2\gamma - (p + b))|z|} \} |[J_{s,b}^p g(z)](q)| \]

setting \(|z| = r\) and \(|\phi(z)| = \rho\) \((0 \leq \rho \leq 1)\)

\[ |[J_{s,b}^p f(z)](q)| \leq \frac{\psi(\rho)}{(1 - r)(p + b) - (2\gamma - (p + b))|r|} |[J_{s,b}^p g(z)](q)| \]  

(27)

where

\[ \psi(\rho) = -\rho^2 r + \rho(1 - r)(p + b) - (2\gamma - (p + b))r \]  

takes its maximum value at \(\rho = 1\) with \(r_0 = r_0(p, b, \gamma)\) given by Furthermore, if \(0 \leq \sigma \leq r_0 = r_0(p, b, \gamma)\), the function \(\varphi(\rho)\) defined by

\[ \varphi(\rho) = -\rho^2 r + \rho(1 - r)(p + b) - (2\gamma - (p + b))r + r \]  

(28)

is an increasing function on \((0 \leq \rho \leq 1)\) so that

\[ \varphi(\rho) \leq \varphi(1) = (1 - r)(p + b) - (2\gamma - (p + b))r \]  

(29)

\((0 \leq \rho \leq 1)\), \(0 \leq \sigma \leq r_0 = r_0(p, b, \gamma)\) then setting \(\rho = 1\) in we conclude that holds true for \(|z| \leq r_0(p, b, \gamma)\). This completes the proof of Theorem 1. \(\square\)
Corollary 2. Let the function \( f(z) \in A_p \) and \( g(z) \in J_{s,b}^{1,n}(\gamma) \) If \( [J_{s+1,b}^{1,n} f(z)] \) is majorized by \( [J_{s+1,b}^{1,n} g(z)] \) in \( U \), then
\[
||[J_{s,b}^{1,n} f(z)]|| \leq ||[J_{s,b}^{1,n} g(z)]|| (|z| \leq r_1),
\]
where
\[
r_1 = \frac{Q \pm \sqrt{Q^2 - 4(1 + q)|2\gamma - (1 + q)|}}{2|2\gamma - (1 + q)|}
\]
where
\[
Q = (1 + b) + |2\gamma - (1 + b)|
\]
Putting \( q = 0 \) and
\[
\gamma = (1 - \frac{\alpha}{p})\cos \lambda e^{-i\lambda} (|\lambda| < \frac{\pi}{2}; 0 \leq \alpha \leq p)
\]
in Theorem 1 we have the following corollary

Corollary 3. Let the function \( f(z) \in A_p \) and \( g(z) \in J_{s,b}^{p,n}(\alpha) \) \((|\lambda| < \frac{\pi}{2})\). If
\[
|J_{s,b}^{p,n} f(z)| \leq |J_{s,b}^{p,n} g(z)| (|z| \leq r_2),
\]
r2 = \( r_2(p, b, \lambda) \) is given by
\[
r_2 = \frac{\delta \pm \sqrt{\delta^2 - 4(p + b)|2(1 - \frac{\alpha}{p})\cos \lambda e^{-i\lambda} - (b + p)|}}{2|2(1 - \frac{\alpha}{p})\cos \lambda e^{-i\lambda} - (b + p)|}
\]
where
\[
\delta = (p + b) + |2(1 - \frac{\alpha}{p})\cos \lambda e^{-i\lambda} - (p + b)|
\]
putting \( m = 0 \) in corollary 2 we have the following corollary:

The proof of our next result is essentially based upon the following lemma, for the class of starlike and convex functions of complex order \( \gamma \) considered and studied by Nasar[11].

Lemma 1. If \( f \in C(\gamma) \), the class of convex functions of order \( \gamma \) where \((\gamma \in C \{0\})\), then \( f \in S(\frac{1}{2}\gamma) \), that is \( C(\gamma) \subset S(\frac{1}{2}\gamma) \) the class of starlike functions of order \( \frac{\gamma}{2} \).

Theorem 4. Let the function \( f(z) \in A_p \) and \( g(z) \in C_{s,b}^{p,n} \) if \( J_{s+1,b}^{p,n} f(z) \) is majorized by \( J_{s+1,b}^{p,n} g(z) \) in \( U \) then
\[
||[J_{s,b}^{p,n} f(z)]^{(q)}|| \leq ||[J_{s,b}^{p,n} g(z)]^{(q)}|| (|z| \leq r_4)
\]
where $r_4$ is given by

$$r_4 = \frac{T - \sqrt{T^2 - 4(p + q)|\gamma - (p + q)|}}{2|\gamma - (p + q)|} \quad (36)$$

where

$$T = (p + b) + |\gamma - (p + b)| \quad (37)$$

The proof of our next result is essentially based upon the following lemmas due to Altintas[1].

**Lemma 2.** If the function $h(z)$ of the form $h(z) = 1 - \sum_{k=1}^{\infty} c_k z^k$ be in the class $R(\lambda, \gamma)$ if it satisfies the condition

$$\Re (h'(z) + \lambda zh'(z)) > \alpha$$

then

$$\sum_{k=1}^{\infty} c_k \leq \frac{\gamma}{1 + \Re(\lambda)} \quad (38)$$

**Lemma 3.** If the function $h(z)$ of the form $h(z) = 1 - \sum_{k=1}^{\infty} c_k z^k$ be in the class $R(\lambda, \gamma)$ if it satisfies the condition

$$\Re (h'(z) + \lambda zh'(z)) > \alpha$$

then

$$1 - \frac{|\gamma|}{1 + \Re(\lambda)} |z| \leq |h(z)| \leq 1 + \frac{|\gamma|}{1 + \Re(\lambda)} |z|, (z \in U) \quad (39)$$

Finally we prove

**Theorem 5.** Let the function $f(z) \in A_p$ and $g(z) \in R(\lambda, \gamma)$ be analytic in $U$ and suppose that the function $g(z)$ is so normalised that it also satisfies the following inclusion property:

$$[J_{p,n}^{s,b} g(z)](q) \quad (\text{where } J_{p,n}^{s,b} g(z) \in R(\lambda, \gamma))$$

if $[J_{s,b}^{p,n} f(z)](q)$ is majorized by $[J_{s,b}^{p,n} g(z)](q) (z \in U)$, then

$$|[J_{s,b}^{p,n} f(z)](q)| \leq |[J_{s,b}^{p,n} g(z)](q)| (|z| \leq r_5) \quad (40)$$
where \( r_5 \) is given by
\[
r_5 = r_5(\lambda, \gamma)
\] (41)
is the smallest root of the equation
\[
|\gamma|r^3 - (1 + \Re(\lambda)r^2) - [2 + |\gamma| + 2\Re(\lambda)]r + 1 + \Re(\lambda) = 0
\] (42)

Proof. For an appropriately normalised analytic function \( g(z) \) satisfying the inclusion property we find from the assertion of Lemma 3 that
\[
\left| \frac{J_{s,b}^{p,n}g(z)}{J_{s+1,b}^{p,n}g(z)}(q) \right| \geq 1 - \frac{|\gamma|}{1 + \Re(\lambda)}r (|z| = r; 0 < r < 1)
\] (43)
or equivalently that
\[
||J_{s+1,b}^{p,n}g(z)||^q \leq \frac{1 + \Re(\lambda)}{1 + \Re(\lambda) - |\gamma|r} \left| J_{s,b}^{p,n}g(z) \right|^q (|z| = r; 0 < r < 1)
\] (44)
Since
\[
[J_{s+1,b}^{p,n}f(z)](q) < [J_{s+1,b}^{p,n}g(z)](q) (z \in U)
\]
there exists an analytic function \( \omega \) such that
\[
[J_{s+1,b}^{p,n}f(z)](q) = \omega(z) [J_{s+1,b}^{p,n}g(z)](q)
\]
and \( |\omega(z)| < 1 \) Thus in view of and just as in the proof of Theorem 1, we have
\[
|\omega(z)| \leq \frac{1 - |\omega(z)|^2}{1 - |z|^2} (z \in U)
\]
and
\[
[J_{s,b}^{p,n}f(z)](q) \leq (|\omega(z)| + \frac{1 - |\omega(z)|^2}{1 - r^2} \frac{1 + \Re(\lambda)}{1 + \Re(\lambda) - |\gamma|r}) [J_{s,b}^{p,n}g(z)](q)
\] (45)
Where we have set \( |\omega(z)| = \rho \) and the function \( \theta(\rho) \) defined by
\[
\theta(\rho) = \{1 + \Re(\lambda)\} + (1 - r^2)1 + \Re(\lambda) - |\gamma|\rho - \{1 + \Re(\lambda)\}\rho^2 (0 \leq \rho \leq 1)
\] (46)
takes on its maximum value at \( \rho = 1 \) with \( r = r_5(\lambda, \gamma) \) given by Moreover if \( 0 \leq \eta \leq r_5(\lambda, \gamma) \) where \( r_5(\lambda, \gamma) \) is the root of the cubic equation such that
\[
0 < r_5(\lambda, \gamma) < 1 \text{ then the function } \vartheta(\rho) \text{ defined by}
\]
\[
\vartheta(\rho) = \{1 + \Re(\lambda)\} + (1 - \eta^2)1 + \Re(\lambda) - |\gamma|\eta\rho - \{1 + \Re(\lambda)\}\rho^2 (0 \leq \rho \leq 1)
\] (47)
is seen to be increasing function on the interval \( 0 \leq \rho \leq 1 \), so that
\[
\vartheta(\rho) = \theta(1) = (1 - \eta^2)1 + \Re(\lambda) - |\gamma|\eta(0 \leq \rho \leq 1; 0 \leq \eta \leq r_5(\lambda, \gamma)
\] (48)
consequently, upon setting \( \rho = 1 \) in , we complete the proof of Theorem 3. \( \square \)
References


