International Journal of Pure and Applied Mathematics
Volume 85 No. 2 2013, 339-347
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v85i2.11

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A NEW RAMANUJAN CONTINUED FRACTION AND THEIR EXPLICITR VALUES

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Abstract: In this paper, we study new Ramanujan continued fraction and establish some new modular identities of their and some explicit values.

AMS Subject Classification: 33D10, 40A15, 11A55, 30B70
Key Words: continued fraction, Theta functions

1. Introduction

In Chapter 16 of his second notebook [2], Ramanujan develops the theory of theta-function and is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \mid ab \mid < 1, \tag{1}$$

$$= (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty$$

where $(a;q)_0 = 1$ and $(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots$.

Received: December 14, 2012

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Following Ramanujan, we defined

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{3}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty \tag{4}$$

and

$$\chi(q) := (-q;q^2)_\infty . \tag{5}$$

Ramanujan recorded many q-continued fractions and some of their explicit values in his second notebook [7] and in his lost notebook [8]. The following beautiful continued fraction identity was recorded by Ramanujan in his second notebook and can be found in [1, p. 11, Entry 11]:

$$\frac{(-a)_\infty(b)_\infty - (a)_\infty(-b)_\infty}{(-a)_\infty(b)_\infty + (a)_\infty(-b)_\infty} = \frac{a-b}{1-q\ +}\ \frac{(a-bq)(aq-b)}{1-q^3\quad +}\ \frac{q(a-bq^2)(aq^2-b)}{1-q^5\quad +\cdots} \tag{6}$$

where either q, a, and b are complex numbers with mod $q < 1$, or q, a, and b are complex numbers with $a = bq^m$ for some integer m. Several elegant q-continued fractions can be expressed in terms of Ramanujan's theta-functions. The most famous of them is the celebrated Rogers-Ramanujan continued fraction $R(q)$ is defined as

$$R(q) := \frac{q^{1/5} f(-q,-q^4)}{f(-q^2,-q^3)} = \frac{q^{1/5}}{1\ +}\ \frac{q}{1\ +}\ \frac{q^2}{1\ +}\ \frac{q^3}{1\ +}\ \cdots, \quad |q| < 1, \tag{7}$$

On page 365 of his Lost Notebook [8], Ramanujan recorded five identities showing the relationships between $R(q)$ and five continued fractions $R(-q)$, $R(q^2)$, $R(q^3)$, $R(q^4)$, and $R(q^5)$. He also recorded these identities at the scattered places of his Notebooks [7]. L. J. Rogers [9] established the modular equations relating $R(q)$ and $R(q^n)$ for n=2,3,5, and 11. The last of these equations cannot be found in Ramanujan's works.

The Ramanujan's cubic continued fraction $G(q)$ is defined as

$$G(q) := \frac{q^{1/3} f(-q,-q^5)}{f(-q^3,-q^3)} = \frac{q^{1/3}}{1\ +}\ \frac{q+q^2}{1\ +}\ \frac{q^2+q^4}{1\ +}\ \cdots, \quad |q| < 1, \tag{8}$$

The continued fraction (8) was first introduced by Ramanujan in his second letter to G. H. Hardy [5]. He also recorded the continued fraction (8) on page 365 of his Lost Notebook [8] and claimed that there are many results for $G(q)$ similar the results obtained for the famous Rogers-Ramanujan continued fraction (7).

Motivated by the above cited works on the continued fractions, if $a = q$ and $b = -q$ in (7), we obtain

$$\frac{(-q,q)^2_\infty - (q,q)^2_\infty}{(-q,q)^2_\infty + (q,q)^2_\infty} = \frac{2q}{1-q\ +}\ \frac{q^2(1+q)^2}{1-q^3\ +}\ \frac{q^3(1+q^2)^2}{1-q^5\ +\cdots} \tag{9}$$

In this paper, we study the Ramanujan's continued fraction $U(q)$ defined by

$$U(q) = \frac{2q}{1-q\ +}\ \frac{q^2(1+q)^2}{1-q^3\ +}\ \frac{q^3(1+q^2)^2}{1-q^5\ +\cdots} \tag{10}$$

which is the right hand side of the equality (9).

In Section 3, we establish some modular relations connecting $U(q)$ and $U(q^n)$. In Section 4, we find the some explicit evaluations of $U(q)$.

2. Preliminary Results

In this section, we collect the necessary results required to prove our main results.

Lemma 1.

$$\varphi(q) + \varphi(-q) = 2\varphi(q^4) \tag{11}$$

$$\varphi(q)\varphi(-q) = \varphi^2(-q^2) \tag{12}$$

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2) \tag{13}$$

For the proofs of (11),(12) and (13), see [2, Entry 25(i),(iii) and (vi), p.40].

Lemma 2.

$$\varphi(q) = \sqrt{z} \tag{14}$$

and

$$\varphi(-q) = \sqrt{z}(1-t)^{1/4}. \tag{15}$$

For the proofs of (14) and (15), see [2, Entry 10(i),(ii), p.122].

Lemma 3. *If β is of degree 2 over α, then*

$$(1-\sqrt{1-\alpha})(1-\sqrt{\beta}) = 2\sqrt{\beta(1-\alpha)}. \tag{16}$$

For a proof of (16), see [1, Entry 17.3.1, p.385].

Lemma 4. *If β has degree 3 over α, then*

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} = 1. \tag{17}$$

For a proof of (17), see [2, Entry 5(ii), p.230].

3. Relation between $U(q)$ and $U(q^n)$

Theorem 5. *We have*

$$U(q) := \frac{1-\varphi^2(-q)}{1+\varphi^2(-q)} \tag{18}$$

Proof. From the equations (9) and (10), weget

$$U(q) = \frac{\dfrac{(-q,q)^2_\infty}{(q,q)^2_\infty} - 1}{\dfrac{(-q,q)^2_\infty}{(q,q)^2_\infty} + 1} \tag{19}$$

Employing the equation (2) in the above equation (19), we obtain

$$U(q) := \frac{\dfrac{1}{\varphi^2(-q)} - 1}{\dfrac{1}{\varphi^2(-q)} + 1} \tag{20}$$

the above equation can written as (18). □

Corollary 6. *We have*

$$\varphi^2(-q) := \frac{1-U(q)}{1+U(q)} \tag{21}$$

Proof. Easily from the equations (18). □

Theorem 7. *If* $x := U(q)$, $y := U(-q)$, *and* $z := U(-q^4)$, *then*

$$\begin{aligned}(6z - 5 - 12y - 4z^2y - 8y^2 + 8y^2z - 5z^2 + 16zy)x^2 + (+16y^2z\\ - 4 - 4y^2z^2 - 12y^2 + 16z - 14y - 12z^2 + 36zy - 14z^2y)x\\ - 5y^2z^2 - 4y - 8z^2 - 12z^2y + 16zy + 8z - 5y^2 + 6y^2z = 0.\end{aligned} \tag{22}$$

Proof. Squaring the equation (11), weget

$$2\varphi(q)\varphi(-q) = (4\varphi^2(q^4) - \varphi^2(q) - \varphi^2(-q))^2. \tag{23}$$

Again squaring the above equation (23), we obtain

$$\begin{aligned} &2\varphi^2(q)\varphi^2(-q) - 16\varphi^4(q^4) + 8\varphi^2(q)\varphi^2(q^4) \\ &+ 8\varphi^2(-q)\varphi^2(q^4) - \varphi^4(q) - \varphi^4(-q) = 0. \end{aligned} \tag{24}$$

Employing the equations (21) in the above equation (24), we obtain (22). □

Theorem 8. *If* $x := U(q)$, $y := U(-q)$, *and* $z := U(q^2)$, *then*

$$2z - x - z^2x - y - z^2y + 2zyx = 0. \tag{25}$$

Proof. Squaring the equation (12) and using equation (21), we obtain (25). □

Theorem 9. *If* $x := U(q)$, $y := U(-q)$, *and* $z := U(-q^2)$, *then*

$$x + y - 2z - zy - zx + 2yx = 0. \tag{26}$$

Proof. Using the equations (13) and (21), we obtain (26). □

Theorem 10. *If* $x := U(-q)$, $y := U(-q^2)$, *and* $z := U(-q^4)$, *then*

$$\begin{aligned} &(2y^2z - 2y - 2y^2 - 1 + 4zy - 2z^2y - 3z^2)x^2 \\ &+ (12zy - 4z^2 - 4y^2 - 2z^2y - 2y + 4y^2z + 4z)x \\ &- 2y - 2z^2 + 4zy - 2z^2y - y^2z^2 - 3y^2 + 2z = 0. \end{aligned} \tag{27}$$

Proof. Using the equations (11) and (13) eliminating $\varphi(-q)$, we obtain

$$2\varphi(q)\varphi(q^4) = \varphi^2(q) + 2\varphi^2(q^4) - \varphi^2(q^2). \tag{28}$$

Squaring the above equation (28), we obtain

$$2\varphi^2(q)\varphi^2(q^2) + 4\varphi^2(q^2)\varphi^2(q^4) - \varphi^4(q) - \varphi^4(q^2) - 4\varphi^4(q^4) = 0. \tag{29}$$

Employing the equations (21) in the above equation (29), we get (27). □

Theorem 11. *If* $x := U(q)$, $y := U(-q)$, $z := U(q^2)$, *and* $w := U(-q^2)$, *then*

$$\begin{aligned}
&(w^2z^2 + 4wy^2z^2 - 4w^2y^2z - 2wz^2 + 1 + 2w^2z + w^2 + z^2 - 4wz \\
&- 2w + 4wy^2 - 4y^2z + 2z)x^2 + (8wz - 2w^2z^2 + 4z - 4wz^2 - 2z^2 \\
&- 2w^2 + 4w^2z - 4w - 2)xy2wy^2z^2 - 4z + 2w^2y^2z - 4wy^2z + 2y^2z \\
&+ y^2z^2 + 4wz^2 - 4w^2z + 4w + w^2y^2z^2 + y^2 - 2wy^2 + w^2y^2 = 0.
\end{aligned} \tag{30}$$

Proof. From the equations (14) and (15), weget

$$\frac{\varphi(-q)}{\varphi(q)} = (1-t)^{1/4}, \quad 0 < t < 1. \tag{31}$$

The equation (16) can be written as

$$\beta = \left[\frac{1-\sqrt{1-\alpha}}{1+\sqrt{1-\alpha}}\right]^2. \tag{32}$$

Employing the equations (21) and (31) in the above equation (32), we obtain (30). □

Theorem 12. *If* $x := U(q)$, $y := U(-q)$, $z := U(q^3)$, *and* $w := U(-q^3)$, *then*

$$\begin{aligned}
&12yw^2z - 12y^2wz + 24y^2w^2z^2 - 12ywz^2 - 12y^3wz^2 + 6y^4w^2z^2 + 12y^3w^2z \\
&- 4y^4wz + 6y^2w^4z^2 - 12y^2w^3z + 6y^2w^2z^4 + y^4w^4z^4 - 12y^2z^3w - 16y^3z^4w \\
&+ 12yz^4w - 12yz^2w^3 - 12yzw^4 - 12y^3z^2w^3 + 16y^3zw^4 + (1 - 12y^2wz \\
&+ 12yw^2z + 24y^2w^2z^2 - 12ywz^2 - 12y^3wz^2 + 6y^4w^2z^2 + 12y^3w^2z + 6y^2w^4z^2 \\
&- 4y^4w^3z - 12y^2w^3z + 6y^2w^2z^4 - 12y^2z^3w + 12y^3z^4w - 16yz^4w - 4y^4z^3w \\
&- 12yz^2w^3 + 16yzw^4 - 12y^3z^2w^3 - 12y^3zw^4 - 12yz - 12y^2z^3w^3 - 4wz \\
&+ 12yw + 6y^2w^2 + 6y^2z^2 + 6w^2z^2 - 16y^3w + 16y^3z + y^4w^4 + y^4z^4 + w^4z^4 \\
&+ 16yz^3 - 12y^3z^3 - 16yw^3 + 12y^3w^3 + 12yz^3w^2 + 12y^3z^3w^2 - 12yz^3w^4 \\
&+ 12yz^4w^3 - 4z^3w^3 + 16y^3z^3w^4 - 16y^3z^4w^3)x^4 + (24y^3wz - 4y + 12y^4wz^2 \\
&+ 40ywz + 72y^2wz^2 - 72yw^2z^2 - 12y^4w^2z - 72y^3w^2z^2 + 12y^4w^3z^2 + 12y^4w^4z \\
&- 72y^2w^2z + 72y^2w^3z^2 - 12y^4w^2z^3 - 72y^2w^2z^3 + 12y^2z^4w - 12y^2w^4z \\
&- 12y^2w^4z^3 - 16y^4w^4z^3 + 40y^3z^3w + 24yz^3w - 12y^4z^4w - 12yz^2w^4 + 24yzw^3 \\
&+ 40y^3zw^3 - 12y^3z^2w^4 + 12y^2z^4w^3 + 12z - 12w - 12yw^2 + 12y^2w - 12yz^2
\end{aligned}$$

$$
\begin{aligned}
&+12wz^2-12y^2z-12w^2z-12y^3w^2+16y^4w-12y^3z^2-16y^4z+16w^3\\
&+12y^2w^3+12w^3z^2-16w^4z-12y^4w^3-12y^2z^3-12w^2z^3+12y^4z^3-16z^3\\
&+12w^4z^3-4y^3z^4+16z^4w-4y^3w^4-12z^4w^3+16y^4z^4w^3+40yz^3w^3\\
&-12yz^4w^2-4yz^4w^4+24y^3z^3w^3-12y^3z^4w^2)x^3+(72yw^2z-72y^2wz\\
&+168y^2w^2z^2-72ywz^2-72y^3wz^2+24y^4w^2z^2+72y^3w^2z-12y^4wz\\
&+6y^4w^4z^2-12y^4w^3z+24y^2w^4z^2-72y^2w^3z+6y^4w^2z^4+24y^2w^2z^4\\
&+6y^2w^4z^4-72y^2z^3w-12y^3z^4w-12yz^4w-12y^4z^3w-72yz^2w^3\\
&+12yzw^4-72y^3z^2w^3+12y^3zw^4-72y^2z^3w^3-12y^4z^3w^3+6y^2+6w^2\\
&-12yw+24y^2w^2+6z^2+12yz-12wz+24y^2z^2+24w^2z^2-12y^3w\\
&+12y^3z+6y^4z^2+6y^2w^4-12w^3z+6w^4z^2+6y^2z^4+6w^2z^4+12yz^3\\
&+12y^3z^3-12yw^3-12z^3w-12y^3w^3-12z^3w^3+72yz^3w^2+12yz^3w^4\\
&-12yz^4w^3+72y^3z^3w^2+12y^3z^3w^4-12y^3z^4w^3+6y^4w^2)x^2+(40y^3wz\\
&+24ywz+72y^2wz^2-72yw^2z^2-12y^4w^2z+12y^4wz^2-72y^3w^2z^2\\
&+12y^4w^3z^2-16y^4w^4z-72y^2w^2z+72y^2w^3z^2-12y^4w^2z^3+12y^2z^4w\\
&-72y^2w^2z^3-12y^2w^4z-12y^2w^4z^3+12y^4w^4z^3+24y^3z^3w+40yz^3w\\
&+16y^4z^4w-12yz^2w^4+40yzw^3+24y^3zw^3-12y^3z^2w^4+12y^2z^4w^3\\
&-16z+16w-4y^3-12yw^2+12y^2w-12yz^2+12wz^2-12y^2z-12w^2z\\
&-12y^3w^2-12y^4w-12y^3z^2+12y^4z-12w^3+12y^2w^3+12w^3z^2\\
&+12w^4z+16y^4w^3-12y^2z^3-12w^2z^3-16y^4z^3+12z^3-16w^4z^3\\
&-4yz^4-4yw^4-12z^4w+16z^4w^3-12y^4z^4w^3+24yz^3w^3-12yz^4w^2\\
&+40y^3z^3w^3-12y^3z^4w^2-4y^3z^4w^4)x-12y^2z^3w^3-4y^4z^3w^3\\
&+y^4-16yw+6y^2w^2+16yz+6y^2z^2+6w^2z^2+12y^3w-12y^3z+w^4\\
&+z^4-4w^3z-12yz^3+16y^3z^3+12yw^3-4z^3w-16y^3w^3+12yz^3w^2\\
&+16yz^3w^4-16yz^4w^3+12y^3z^3w^2-12y^3z^3w^4+12y^3z^4w^3=0. \qquad (33)
\end{aligned}
$$

Proof. The equation (16) can be written as

$$\alpha\beta=[1-\{(1-\alpha)(1-\beta)\}^{1/4}]^4. \qquad (34)$$

Employing the equations (21) and (31) in the above equation (34), we obtain (33). □

4. Explicit Evaluation of $U(q)$

Theorem 13. *We have*

$$(i)\ U(e^{-n\pi}) = \frac{1-\varphi^2(-e^{-n\pi})}{1+\varphi^2(-e^{-n\pi})} \quad and \quad (ii)\ U(-e^{-n\pi}) = \frac{1-\varphi^2(e^{-n\pi})}{1+\varphi^2(e^{-n\pi})} \tag{35}$$

Proof. Put $q := e^{-n\pi}$ and $q := -e^{-n\pi}$ in the equation (18), we obtain (35). □

In his first notebook, Ramanujan's second notebook [7] recorded many elementary values of $\varphi(q)$. In particularly, he recorded $\varphi(e^{-n\pi})$ and $\varphi(-e^{-n\pi})$ for $n = 1, 2, 4, 8, 1/2, and 1/4$. Ramanujan also recoded values of $\varphi(e^{-n\pi})$ for $n = 3, 5, 7, 9, and 45$. Proof of these can found in [3]. Yi [10] also evaluated $\varphi(e^{-n\pi})$ for $n = 1, 2, 3, 4, 5, and 6$ and $\varphi(-e^{-n\pi})$ for $n = 1, 2, 4, 6, 8, 10, and 12$. Noting from [3, Entry 1(i) and (ii), p.325], we have

$$\varphi(e^{-\pi}) = \frac{\pi^{1/4}}{\Gamma(3/4)} \quad and \quad \varphi(-e^{-\pi}) = 2^{-1/4}\frac{\pi^{1/4}}{\Gamma(3/4)}. \tag{36}$$

Employing (36) in the equation (35), we get

$$U(e^{-\pi}) = \frac{\sqrt{2} - \dfrac{\sqrt{\pi}}{\Gamma^2(3/4)}}{\sqrt{2} + \dfrac{\sqrt{\pi}}{\Gamma^2(3/4)}} \quad and \quad U(-e^{-\pi}) = \frac{1 - \dfrac{\sqrt{\pi}}{\Gamma^2(3/4)}}{1 + \dfrac{\sqrt{\pi}}{\Gamma^2(3/4)}}.$$

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