NOVEL SIGN OF SUPER EDGE-MAGIC GRAPH

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Abstract: In this paper, we introduce a new concept of super edge-magic sequence (SEMS) of a super edge-magic graph (SEMG) with p vertices and q edges. The super edge-magic sequence of natural numbers is denoted by < xᵢ >, 1 ≤ i ≤ q. This sequence need not to be monotonic. In this track, we also drive some families of super edge-magic graphs from fabrication of new super edge-magic sequences by considering additional parameter. We complete this paper by discussing the special case like monotonic sequences related to the super edge-magic sequence.

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1. Introduction

1.1. Background of Edge-Magic and Super Edge-Magic Graphs

Kotzig and Rosa introduced the concepts of magic valuation [11]. Ringel and Llado [15] called this type of valuation as edge-magic labeling. Enomoto et. al.[4] restricted the notion of edge -magic labeling of a graph to obtain the definition of super edge-magic labeling. A (p,q) graph G is called edge-magic if there exists a bijective function f : V(G) ∪ E(G) → {1, 2, 3, . . . , p + q} such
that \( f(u) + f(v) + f(uv) = k \) is a magic constant for any edge \( uv \in E(G) \). Moreover, \( G \) is said to be super edge-magic if \( f(V(G)) \rightarrow \{1, 2, 3, \ldots, p\} \). The following Lemma from [13] provides a necessary and sufficient condition for a graph to be super edge-magic.

**Lemma 1.1.** A graph \( G \) with \( p \) vertices and \( q \) edges is super edge-magic if and only if there exists a bijective function \( f : V(G) \rightarrow \{1, 2, 3, \ldots, p\} \) such that the set \( S = \{f(x) + f(y) / xy \in E(G)\} \) consists of \( q \) consecutive integers. In such a case, \( f \) extends to a super edge-magic total labeling of \( G \) with the magic constant \( c = p + q + \min(S) \).

**Lemma 1.2.** If a graph \( G \) with \( p \) vertices and \( q \) edges is super edge-magic then \( q \leq 2p - 3 \).

**Lemma 1.3.** Let \( G \) be a triangle free super edge-magic graph with \( p \) (\( \geq 4 \)) vertices and \( q \) edges. Then, \( q \leq 2p - 5 \).

### 1.2. Road Map of the Paper

The rest of the paper is organized as follows: In Section 2, we introduce the concept of super edge-magic sequence and construction of SEMG from SEMS. Also we give the limitations and upshots of super edge-magic sequence. Section 3, includes fabrication of new super edge-magic sequences and we drive some families of super edge-magic graphs. The last section covers a special case of sequence like monotonic sequence with their behavior in SEMS.

### 2. Proposed Work

#### 2.1. Definition and Construction

Now we define the concept of super edge-magic sequence and transmit it to the graph. We describe super edge-magic sequence analogously for graceful sequence [1], [2] and [3].

**Definition 2.1.** (Super Edge-Magic Sequence) Let \( G \) be a super edge magic graph with \( p \) vertices and \( q \) edges. Here we introduce a new term i.e. a constant of super edge-magic sequence and is denoted by \( \alpha^* \). A Sequence \( \langle x_i \rangle \) is said to be super edge-magic sequence if

\[
\max_{1 \leq i \leq q} \{2x_i + i\} < \alpha^* + q \leq p + \min_{1 \leq i \leq q} \{x_i + i\} \ldots \ldots (2.1.a)
\]
Where \( x_i \) is always the lower end vertex of the edge label \( p + i \), \( 1 \leq i \leq q \) i.e., \( x_i = \min\{f(x), f(y)/xy \in E(G)\} \) and \( \alpha^* = \min(S) \), Where \( S = \{f(x) + f(y)/xy \in E(G)\} \). We illustrate a super edge-magic sequence of Figure 1. SEMS of a

\[ \text{Figure 1} \]

SEM Graph: Suppose \((4,4,2,1,2,3,1,3,1)\) is a given sequence. For \( 1 \leq i \leq q \), \( \max\{2x_i + i\} = 14, \min\{x_i + i\} = 5, \alpha^* = 6, p = 10, q = 9 \). Then above data satisfies the condition (2.1.a).

**Note 2.1.** In this entire paper sequence means super edge-magic sequence.

**2.2. Edifice of a Super Edge-Magic Graph from the Sequence**

Let \((x_1, x_2, x_3, \ldots, x_q)\) be the sequence having ’q’ terms. Compute \( \alpha^* \) and \( p \) using the condition (2.1.a) as follows:

\[ Letm = \max_{1 \leq i \leq q} \{2x_i + i\} \text{ and } n = \min_{1 \leq i \leq q} \{x_i + i\} \]

Then by (2.1.a), \( m < \alpha^* + q \leq p + n \). In this construction, \( \alpha^* \) is independent for \( \alpha^* \geq m - q + 1 \). Based on particular \( \alpha^* \), and by (2.1.a) choose \( p \geq \alpha^* + q - n \). For every \( \alpha^* \), there exist many \( p \) values such that all they must give super edge-magic graphs. Here we note that each \( p \) in this domain, the super edge-magic graph is unique. Identify the edges of super edge-magic graph corresponding a particular \( \alpha^* \) and ’p’ by the following way:

1.
The super edge-magic graph can be drawn by identifying all the edges like above.

**Concrete Example.** Suppose the given sequence is \((3,3,3,1,2,2,2,1)\), \(q=8\), \(m=11\) and \(n=4\). On simplification of (2.1.a), we obtain \(3 < \alpha^* \leq p - 4\). The possibilities of \(\alpha^*\) and \(p\) is respectively: \(\alpha^* = \{4,5,6,\ldots\}\) and \(p = \{8,9,10,\ldots\}\). In this domain of \(\alpha^*\) and \(p\), given sequence is super edge-magic sequence. For \(\alpha^* = 4\), \(p = \{8,9,10,\ldots\}\). If \(\alpha^* = 4\), and we select appropriate \(p = 8\). We identify the edges by the following way:

Then the super edge-magic graph is shown in Figure 2

### 2.3. Limitations and Upshots of Super Edge-Magic Sequence

**Lemma 2.3.1.** Let \((x_1, x_2, \ldots, x_q)\) be any super edge-magic sequence. \(x_i\) denotes lower end vertex of the edge label \(p + i\), \(1 \leq i \leq q\) Then lower end vertex is always strictly less than the upper end vertex, i.e., \(x_i < \alpha^* + q - i - x_i\) and upper end vertex is less than or equal to \(p\).

**Proof.** Let \((x_1, x_2, \ldots, x_q)\) be any super edge-magic sequence. Then by definition (2.1), this sequence satisfies the condition (2.1.a).

From LHS of (2.1.a): \(x_i < \alpha^* + q - i - x_i\) for all \(i, 1 \leq i \leq q\) .... (2.3.a)

From RHS of (2.1.a): \(\alpha^* + q - i - x_i \leq p\) for all \(i, 1 \leq i \leq q\) .... (2.3.b)

Also from (2.3.a) and (2.3.b), \(x_i < \alpha^* + q - i - x_i \leq p\) This completes the proof.

**Proposition 2.3.1.** Let \(< x_i >, 1 \leq i \leq q\) be any super edge-magic sequence. The lower end vertex is at most \(p-1\). i.e., \(x_i \leq p-1\).

**Theorem 2.3.1.** A graph is a super edge-magic graph if and only if \(G\) has super edge-magic sequence.

**Proof.** Necessary Part. Let \((x_1, x_2, \ldots, x_q)\) be super edge-magic sequence. Then by definition (2.1), it has one super edge-magic graph.
Sufficient Part. Let us assume that $G$ is super edge-magic. Here $\alpha^* = \min\{f(u) + f(v)/uv \in E(G)\}$ and $S = \{\alpha^*, \alpha^* + 1, \alpha^* + 2, \ldots, \alpha^* + q - 1\}$ has $"q"$ consecutive integers. i.e., $S = \{\alpha^* + q - i/1 \leq i \leq q\}$ For each edge $e=uv \in E(G)$, $x_i = \min\{f(u), f(v)/f(u) + f(v) = \alpha^* + q - i/1 \leq i \leq q\}$ The other end is $\alpha^* + q - i - x_i, \forall i, 1 \leq i \leq q$. By the second part of the lemma (2.3.1),

$$\alpha^* + q \leq p + \min_{1 \leq i \leq q} \{x_i + i\}....(2.3.c)$$

And the first part of the lemma (2.3.1),

$$\max_{1 \leq i \leq q} \{2x_i + i\} < \alpha^* + q....(2.3.d)$$

From (2.3.c) and (2.3.d), $(x_1, x_2, \ldots, x_q)$ is super edge-magic sequence. This completes the proof.

3. Fabrication of New Super Edge-magic Sequences

In this section, we fabricate new SEMS of SEMG by means of extension. We will do this construction as explained analogously in [1], [2].

Theorem 3.1. If $(x_1, x_2, \ldots, x_q)$ represents a super edge-magic sequence of a graph $G$ on $"q"$ edges with

$$\alpha^* (\langle x_i \rangle) = \max_{1 \leq i \leq q} \{2x_i + i\} + 1 - q......(3.a)$$

$$p(\langle x_i \rangle) = \max_{1 \leq i \leq q} \{2x_i + i\} - \min_{1 \leq i \leq q} \{x_i + i\} + 1......(3.b)$$

Then the sequence $(x^* + 1 - x_1, x^* + 1 - x_2, \ldots, x^* + 1 - x_{q-1}, x^* + 1 - x_q, x_1, x_2, \ldots, x_q)$ represents super edge-magic sequence on $2q$ edges with same $\alpha^*$, for $x^* = \max\{x_i/1 \leq i \leq q\}$.
Proof. Let \( y_i = x^* + 1 - x_i, \ 1 \leq i \leq q \) \( y_q + i = x_i, \ 1 \leq i \leq q \) Then the sequence becomes: \( (y_1, y_2, y_3 \ldots y_q, y_{q+1}, \ldots, y_{2q}) \)

\[
\max_{1 \leq i \leq q} \{2y_i + i\} \leq 2x^* + q . . . . . (3.c)
\]

Suppose \( x^* \) is occurring in the position of the least label \('r'\) in the sequence \( < x_i > \)

\[
2x^* + q < 2x^* + q + r \leq \max_{1 \leq i \leq q} \{2y_{q+i} + q + i\}
\]

\[
= \max_{1 \leq i \leq q} \{2x_i + i\} + q
\]

\[
= \alpha * (< x_i >) + 2q - 1 by (3.a)
\]

\[
2x^* + q < \alpha * (< x_i >) + 2q - 1 . . . . . (3.d)
\]

\[
\max_{q+1 \leq i \leq 2q} \{2y_i + i\} = \max_{1 \leq i \leq q} \{2x_i + i\} + q
\]

\[
\max_{q+1 \leq i \leq 2q} \{2y_i + i\} = \alpha * (< x_i >) + 2q - 1 . . . . . (3.e)
\]

using (3.c) and (3.e)

\[
\max_{1 \leq i \leq 2q} \{2y_i + i\} = \alpha * (< x_i >) + 2q - 1 . . . . . (3.f)
\]

By applying (3.a), \( \alpha * (< y_i >) = \max_{1 \leq i \leq 2q} \{2y_i + i\} + 1 - 2q \)

\[
= \alpha * (< x_i >) (by 3.f)
\]

\( \alpha * (< y_i >) = \alpha * (< x_i >). \)

This completes the proof.

Corollary 3.1. If \( (x^* + 1 - x_1, x^* + 1 - x_2, \ldots, x^* + 1 - x_{q-1}, x^* + 1 - x_q, x_1, x_2, \ldots, x_q) \) represents super edge-magic sequence on \( 2q \) edges satisfying the conditions in theorem 3.1 then the number of vertices is given by

\[ \alpha * (< x_i >) + 2q - \min_{1 \leq i \leq 2q} \{y_i + i\}. \]

Proof. Using the Theorem 3.1:

\[
\min_{1 \leq i \leq 2q} \{y_i + i\} = \min_{1 \leq i \leq q} \{y_i + i\}
\]

\[
= \min_{1 \leq i \leq q} \{(x^* + 1 - x_i) + i\}\]
(by 3.2) and we have
\[ p(< y_i >) = \alpha * (< x_i >) + 2q - \min_{1 \leq i \leq q} \{ x^* - x_i + i + 1 \} \]

\[ p(< y_i >) = \alpha * (< x_i >) + 2q - \min_{1 \leq i \leq 2q} \{ y_i + i \} \]

This completes the proof.

Let us illustrate the theorem and corollary of 3.1, by the following:

Suppose \(< x_i > = (3, 2, 2, 1, 1)\), by the theorem 3.1, \(\alpha* = 3\), \(p(G_1) = 4\), \(x^* = 3\), \(< y_i > = (1, 2, 2, 3, 3, 3, 2, 2, 1, 1)\), and by corollary 3.1, \(p(G_2) = 11\). Then the graph as shown in Figure 3:

\[ \text{Figure 3} \]

\[ \text{Figure 4} \]

**Theorem 3.2.** If \((x_1, x_2, \ldots, x_q)\) represents a super edge-magic sequence of a graph \(G\) on \(q\) edges with
\[ \alpha * (< x_i >) = \max_{1 \leq i \leq q} \{ 2x_i + i \} + 1 - q \ldots \ldots (3.g) \]
\[ p(< x_i >) = \max_{1 \leq i \leq q} \{ 2x_i + i \} - \min_{1 \leq i \leq q} \{ x_i + i \} + 1 \ldots \ldots (3.h) \]
Then \((x_{*} + 1 - x_{q}, x_{*} + 1 - x_{q-1}, \ldots, x_{*} + 1 - x_{2}, x_{*} + 1 - x_{1}, x_{1}, x_{2}, \ldots, x_{q})\) represents super edge-magic sequence on \(2q\) edges with same \(\alpha^*\), for \(x^* = \max\{x_{i}/1 \leq i \leq q\}\).

Proof. Let \(y_{i} = x^* + 1 - x_{q} - i + 1, 1 \leq i \leq q y_{q+i} = x_{i} , 1 \leq i \leq q\) The given sequence becomes: \((y_{1}, y_{2}, \ldots, y_{q}, y_{q+1}, \ldots, y_{2q})\)

\[
\max_{1 \leq i \leq q} \{2y_{i} + i\} = \max_{1 \leq i \leq q} \{2(x^* + 1 - x_{q-i+1} + i)\}
\]

\[
= 2x^* + 2 - \min_{1 \leq i \leq q} \{2x_{q-i+1} + i\}
\]

\[
\max_{1 \leq i \leq q} \{2y_{i} + i\} < 2x^* + q \ldots (3.i)
\] Using theorem 3.1, \(2x^* + q < \alpha^* (< x_{i}>) + 2q - 1\) and By (3.e)

\[
\max_{q+1 \leq i \leq 2q} \{2y_{i} + i\} \leq \alpha^* (< x_{i}>) + 2q - 1 \ldots (3.j)
\]

By (3.i) and (3.j),

\[
\max_{1 \leq i \leq 2q} \{2y_{i} + i\} = \alpha^* (< x_{i}>) + 2q - 1
\]

\[
\alpha^* (< y_{i}>) = \max_{1 \leq i \leq 2q} \{2y_{i} + i\} + 1 - 2q
\]

By applying (3.g) \(\alpha^* (< y_{i}>) = \alpha^* (< x_{i}>)\)

This completes the proof.

**Corollary 3.2.** If \((x_{*} + 1 - x_{q}, x_{*} + 1 - x_{q-1}, \ldots, x_{*} + 1 - x_{2}, x_{*} + 1 - x_{1}, x_{1}, x_{2}, \ldots, x_{q})\) represents super edge-magic sequence on \(2q\) edges satisfying the conditions in theorem 3.2 then the number of vertices is given by

\[
\alpha^* (< x_{i}>) + 2q - \min_{1 \leq i \leq 2q} \{y_{i} + i\}
\]

Proof. The sequence \(< y_{i} >\) and other details are followed by theorem 3.2.

\[
\min_{1 \leq i \leq 2q} \{y_{i} + i\} = \min_{1 \leq i \leq q} \{(x^* + 1 - x_{q-i+1})\}
\]

\[
= (x^* + 1) - \max_{1 \leq i \leq q} \{(x_{q-i+1})\}
\]

\[
p(< y_{i} >) = \max_{1 \leq i \leq 2q} \{2y_{i} + i\} - \min_{1 \leq i \leq 2q} \{y_{i} + i\} + 1
\]
By using (3.h)

\[ p(<y_i>) = \alpha * (<x_i>) + 2q - \min_{1 \leq i \leq 2q} \{y_i + i\} \]

This completes the proof.

\textbf{Concrete Example for the theorem and corollary of 3.2:} Suppose \( <x_i> = (3,2,2,1,1) \), by using theorem 3.2, \( \alpha* = 3 \), \( p(G_1) = 4 \), \( x^* = 3 \), \( <y_i> = (3,3,2,2,1,3,2,2,1,1) \), and by corollary 3.2, \( p(G_2) = 9 \). Then the graph as shown in Figure 4.

**Theorem 3.3.** If \((x_1,x_2,\ldots,x_q)\) represents a super edge-magic sequence with

\[ \alpha * (<x_i>) = \max_{1 \leq i \leq q} \{2x_i + i\} + 1 - q \]

and

\[ p(<x_i>) = \max_{1 \leq i \leq q} \{2x_i + i\} - \min_{1 \leq i \leq q} \{x_i + i\} + 1 \]

Then \((\alpha* + q - 1 - x_1, \alpha* + q - 2 - x_2, \ldots, \alpha* - x_q, \lambda, x_1, x_2, \ldots, x_q)\) represents a super edge-magic sequence on \(2q+1\) edges with same \(\alpha*\), choose \(\lambda\) such that \(1 \leq \lambda < q\).

**Proof.** This is an immediate consequence of definition and by applying theorem 3.1.

**Remark 3.1.** Based on the parameter \(<x_i>\), \(q\), \(\alpha*\), At least one \(\lambda\) such that \(1 \leq \lambda < q\) would give SEMG.

\textbf{Let us illustrate the theorem and corollary of 3.3 as follows:} \(<x_i> = (2,1,2,2,1), \alpha* = 4, p(G_1) = 6, <y_i> = (6,6,4,3,3,2,1,2,2,1), \lambda = 3, p(G_2) = 8\). Figure 5 shows the graph.
Remark 3.2. Theorems from 3.1 to 3.3 have the same property that the sequence \(< x_i >\) and \(< y_i >\) have same \(\alpha^*\). Each sequence gives one super edge-magic graph say \(G_1, G_2\) respectively then the graph \(G_2\) contains \(G_1\) always. \(G_1\) indicated by block color and \(G_2\) indicated by pink color.

Theorem 3.4. If \((x_1, x_2, \ldots, x_q)\) represents a super edge-magic sequence of a graph \(G\) on \(q\) edges with

\[
\alpha^*(<x_i>) = \max_{1 \leq i \leq q} \{2x_i + i\} + 1 - q
\]

\[
p(<x_i>) = \max_{1 \leq i \leq q} \{2x_i + i\} - \min_{1 \leq i \leq q} \{x_i + i\} + 1
\]

Then \((2x_1 + 1, 2x_1, 2x_2 + 1, 2x_2, \ldots, 2x_q + 1, 2x_q)\) represents super edge-magic sequence of a graph \(H\) such that \(\alpha^*(H) = 2\alpha^*(G)\) and \(p(H) = 2p(G)\).

Proof. This is an immediate consequence on modification of \(< x_i >\) by definition.

Remark 3.3. \(\alpha^*\) of \(H\) is twice number of \(\alpha^*\) of \(G\), so that the structure of the graph \(G\) is embedded in \(H\) and also the labeling of \(G\) is exactly doubled.

\(\triangleright\)Let us illustrate the theorem 3.4 by the following: In the succeeding figures the color green indicates the graph \(G\) and blue indicates \(H\).

- If \(G\) has one connected component then the corresponding \(H\) as follows: \(< x_i > = (4,4,2,1,2,3,1,3,1)\), \(\alpha^*(G) = 6\), \(p(G) = 10\), \(q(G) = 9\). The sequence for \(H = (9,8,9,8,5,4,3,2,5,4,7,6,3,2,7,6,3,2)\), \(p(H) = 20\), \(q(H) = 18\), \(\alpha^*(H) = 12\). Then the graph as shown in Figure 6

\[\text{Figure 6}\]

\[\text{Figure 7}\]
• If G has more than one connected components the corresponding H as follows: \(< x_i > = (4, 5, 6, 6, 6, 6)\) \(\alpha^*(G) = 13\), \(p(G) = 14\), \(q(G) = 6\). The sequence for \(H = (9, 8, 11, 10, 13, 12, 13, 12, 13, 12, 13, 12)\), \(\alpha^*(H) = 26\), \(p(H) = 28\), \(q(H) = 12\). Then the graph as shown in Figure 7

**Proposition 3.1.** If any super edge-magic sequence does not contain the element ‘1’, then \((m-1)\) number of deficiency can be reduced by subtracting \((m-1)\) in each element of that sequence, where \(m\) is minimum element of the given super edge-magic sequence.

**Proof.** This is an immediate consequence of theorem 3.4.

4. Monotonic Sequences and their Behavior in Super Edge-Magic Graph

**Proposition 4.1.** If the sequence \(< x_i >, 1 \leq i \leq q\) defines a super edge-magic graph with \(x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_q (= 1)\) then the feasible range of \(\alpha^*\) is \(x_q + 2 \leq \alpha^* \leq x_1 + 2\).

The range of \(\alpha^*\) is \((x_q + 2, x_1 + 2)\) but all \(\alpha^*\) need not give super edge-magic graph. To find \(\alpha^*\) which will give super edge-magic graph from the following.

**Proposition 4.2.** Let \(< x_i >, 1 \leq i \leq q\) be super edge-magic sequence such that \(x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_q\) with \(x_q = 1\),

\[
\alpha = \max_{1 \leq i \leq q} \{2x_i + i\}\text{ and }\beta = \min_{1 \leq i \leq q} \{x_i + i\}
\]

. Then the number of super edge-magic graph is \((q + 1) - (\alpha - \beta)\) which is denoted by ‘r’. Moreover, the value of \(\alpha^*\) is \(\alpha - q + j\), \(1 \leq j \leq r\) and the corresponding value of \(p\) is at least \(\alpha^* - \beta + q\).

**Definition 4.1.** Let \(< x_i >, 1 \leq i \leq q\) be super edge-magic sequence such that \(x_1 \leq x_2 \leq x_3 \leq \ldots \leq x_q\) then number of digits appeared in the sequence \(< x_i >, 1 \leq i \leq q\) is said to be ”orderofthesequence” and denoted by ‘n’.

**Proposition 4.3.** Let \(< x_i >, 1 \leq i \leq q\) be super edge-magic sequence such that \(x_1 \leq x_2 \leq x_3 \leq \ldots \leq x_q\) with \(x_1 = 1\). Then \(\alpha^* = 2x_q + 1\), \(p = 2x_q + q - 1\) must be super edge-magic graph with deficiency[13] is \(n-1\).

5. Conclusion

The relevance of this paper is two fold. First, we introduced definition of SEMS and construction method for SEMG from SEMS. In a consequent step,
the limitations and upshots of SEMS were also discussed. Second, Fabrication of New SEMS and behavior of monotonic sequences of super edge-magic graph were discussed.

References


