NOVEL SIGN OF SUPER EDGE-MAGIC GRAPH

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Abstract: In this paper, we introduce a new concept of super edge-magic sequence (SEMS) of a super edge-magic graph (SEMG) with p vertices and q edges. The super edge-magic sequence of natural numbers is denoted by < xᵢ >, 1 ≤ i ≤ q. This sequence need not to be monotonic. In this track, we also drive some families of super edge-magic graphs from fabrication of new super edge-magic sequences by considering additional parameter. We complete this paper by discussing the special case like monotonic sequences related to the super edge-magic sequence.

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1. Introduction

1.1. Background of Edge-Magic and Super Edge-Magic Graphs

Kotzig and Rosa introduced the concepts of magic valuation [11]. Ringel and Llado [15] called this type of valuation as edge-magic labeling. Enomoto et. al.[4] restricted the notion of edge-magic labeling of a graph to obtain the definition of super edge-magic labeling. A (p,q) graph G is called edge-magic if there exists a bijective function f : V(G) ∪ E(G) → {1, 2, 3, . . . , p + q} such
that $f(u) + f(v) + f(uv) = k$ is a magic constant for any edge $uv \in E(G)$. Moreover, $G$ is said to be super edge-magic if $f(V(G)) \rightarrow \{1, 2, 3, \ldots, p\}$. The following Lemma from [13] provides a necessary and sufficient condition for a graph to be super edge-magic.

**Lemma 1.1.** A graph $G$ with $p$ vertices and $q$ edges is super edge-magic if and only if there exists a bijective function $f : V(G) \rightarrow \{1, 2, 3, \ldots, p\}$ such that the set $S = \{f(x) + f(y) / xy \in E(G)\}$ consists of $q$ consecutive integers. In such a case, $f$ extends to a super edge-magic total labeling of $G$ with the magic constant $c = p + q + \min(S)$.

**Lemma 1.2.** If a graph $G$ with $p$ vertices and $q$ edges is super edge-magic then $q \leq 2p - 3$.

**Lemma 1.3.** Let $G$ be a triangle free super edge-magic graph with $p \ (\geq 4)$ vertices and $q$ edges. Then, $q \leq 2p - 5$.

### 1.2. Road Map of the Paper

The rest of the paper is organized as follows: In Section 2, we introduce the concept of super edge-magic sequence and construction of SEMG from SEMS. Also we give the limitations and upshots of super edge-magic sequence. Section 3, includes fabrication of new super edge-magic sequences and we drive some families of super edge-magic graphs. The last section covers a special case of sequence like monotonic sequence with their behavior in SEMS.

### 2. Proposed Work

#### 2.1. Definition and Construction

Now we define the concept of super edge-magic sequence and transmit it to the graph. We describe super edge-magic sequence analogously for graceful sequence [1], [2] and [3].

**Definition 2.1.** (Super Edge-Magic Sequence) Let $G$ be a super edge magic graph with $p$ vertices and $q$ edges. Here we introduce a new term i.e. a constant of super edge-magic sequence and is denoted by $\alpha^*$. A Sequence $< x_i >$ is said to be super edge- magic sequence if

$$\max_{1 \leq i \leq q} \{2x_i + i\} < \alpha^* + q \leq p + \min_{1 \leq i \leq q} \{x_i + i\} \ldots (2.1.a)$$
Where \( x_i \) is always the lower end vertex of the edge label \( p + i \), \( 1 \leq i \leq q \) i.e., \( x_i = \min \{ f(x), f(y) / xy \in E(G) \} \) and \( \alpha^* = \min(S) \), Where \( S = \{ f(x) + f(y) / xy \in E(G) \} \). We illustrate a super edge-magic sequence of Figure 1. SEMS of a

![Figure 1](image1)

![Figure 2](image2)

**SEM Graph**: Suppose \((4,4,2,1,2,3,1,3,1)\) is a given sequence. For \(1 \leq i \leq q\), \( \max \{2x_i + i\} = 14, \min \{x_i + i\} = 5, \alpha^* = 6, p = 10, q = 9 \). Then above data satisfies the condition (2.1.a).

**Note 2.1.** In this entire paper sequence means super edge-magic sequence.

### 2.2. Edifice of a Super Edge-Magic Graph from the Sequence

Let \((x_1, x_2, x_3, \ldots, x_q)\) be the sequence having 'q' terms. Compute \( \alpha^* \) and \( p \) using the condition (2.1.a) as follows:

\[
\text{Let} \quad m = \max_{1 \leq i \leq q} \{2x_i + i\} \quad \text{and} \quad n = \min_{1 \leq i \leq q} \{x_i + i\}
\]

Then by (2.1.a), \( m < \alpha^* + q \leq p + n \). In this construction, \( \alpha^* \) is independent for \( \alpha^* \geq m - q + 1 \). Based on particular \( \alpha^* \), and by (2.1.a) choose \( p \geq \alpha^* + q - n \). For every \( \alpha^* \), there exist many \( p \) values such that all they must give super edge-magic graphs. Here we note that each \( p \) in this domain, the super edge-magic graph is unique. Identify the edges of super edge-magic graph corresponding a particular \( \alpha^* \) and 'p' by the following way:

1.
The super edge-magic graph can be drawn by identifying all the edges like above.

**Concrete Example.** Suppose the given sequence is \((3,3,3,1,2,2,2,1)\), \(q=8\), \(m = 11\) and \(n = 4\). On simplification of (2.1.a), we obtain \(3 < \alpha^* \leq p - 4\). The possibilities of \(\alpha^*\) and \(p\) is respectively: \(\alpha^* = \{4,5,6,\ldots\}\) and \(p = \{8,9,10,\ldots\}\). In this domain of \(\alpha^*\) and \(p\), given sequence is super edge-magic sequence. For \(\alpha^* = 4, p= \{8,9,10,\ldots\}\). If \(\alpha^* = 4\), and we select appropriate \(p = 8\). We identify the edges by the following way:

Then the super edge-magic graph is shown in Figure 2

### 2.3. Limitations and Upshots of Super Edge-Magic Sequence

**Lemma 2.3.1.** Let \((x_1, x_2, \ldots, x_q)\) be any super edge-magic sequence. \(x_i\) denotes lower end vertex of the edge label \(p + i, 1 \leq i \leq q\). Then lower end vertex is always strictly less than the upper end vertex, i.e., \(x_i < \alpha^* + q - i - x_i\) and upper end vertex is less than or equal to \(p\).

**Proof.** Let \((x_1, x_2, \ldots, x_q)\) be any super edge-magic sequence. Then by definition (2.1), this sequence satisfies the condition (2.1.a).

From LHS of (2.1.a): \(x_i < \alpha^* + q - i - x_i\) for all \(i, 1 \leq i \leq q\) ......... (2.3.a)

From RHS of (2.1.a): \(\alpha^* + q - i - x_i \leq p\) for all \(i, 1 \leq i \leq q\) ......... (2.3.b)

Also from (2.3.a) and (2.3.b), \(x_i < \alpha^* + q - i - x_i \leq p\) This completes the proof.

**Proposition 2.3.1.** Let \(< x_i >, 1 \leq i \leq q\) be any super edge-magic sequence. The lower end vertex is at most \(p-1\). i.e., \(x_i \leq p - 1\).

**Theorem 2.3.1.** A graph is a super edge-magic graph if and only if \(G\) has super edge-magic sequence.

**Proof.** Necessary Part. Let \((x_1, x_2, \ldots, x_q)\) be super edge-magic sequence. Then by definition (2.1), it has one super edge-magic graph.
Sufficient Part. Let us assume that $G$ is super edge-magic. Here $\alpha^* = \min\{f(u) + f(v)/uv \in E(G)\}$ and $S = \{\alpha^*, \alpha^* + 1, \alpha^* + 2, \ldots, \alpha^* + q - 1\}$ has $"q"$ consecutive integers. i.e., $S = \{\alpha^* + q - i/1 \leq i \leq q\}$ For each edge $e=uv \in E(G)$, $x_i = \min\{f(u), f(v)/f(u) + f(v) = \alpha^* + q - i/1 \leq i \leq q\}$ The other end is $\alpha^* + q - i - x_i, \forall i, 1 \leq i \leq q$. By the second part of the lemma (2.3.1),
\[
\alpha^* + q \leq p + \min_{1 \leq i \leq q} \{x_i + i\} \ldots (2.3.c)
\]
And the first part of the lemma (2.3.1),
\[
\max_{1 \leq i \leq q} \{2x_i + i\} < \alpha^* + q \ldots (2.3.d)
\]
From (2.3.c) and (2.3.d), $(x_1, x_2, \ldots, x_q)$ is super edge-magic sequence. This completes the proof.

### 3. Fabrication of New Super Edge-magic Sequences

In this section, we fabricate new SEMS of SEMG by means of extension. We will do this construction as explained analogously in [1], [2].

**Theorem 3.1.** If $(x_1, x_2, \ldots, x_q)$ represents a super edge-magic sequence of a graph $G$ on "$q$" edges with
\[
\alpha^* (< x_i >) = \max_{1 \leq i \leq q} \{2x_i + i\} + 1 - q \ldots (3.a)
\]
\[
p(< x_i >) = \max_{1 \leq i \leq q} \{2x_i + i\} - \min_{1 \leq i \leq q} \{x_i + i\} + 1 \ldots (3.b)
\]
Then the sequence $(x^* + 1 - x_1, x^* + 1 - x_2, \ldots, x^* + 1 - x_{q-1}, x^* + 1 - x_q, x_1, x_2, \ldots, x_q)$ represents super edge-magic sequence on $2q$ edges with same $\alpha^*$, for $x^* = \max\{x_i/1 \leq i \leq q\}$. 

\[
\begin{array}{cccccccc}
3 & 3 & 3 & 1 & 2 & 2 & 2 & 1 \\
\downarrow & & & & & & & \\
8 & 7 & 6 & 7 & 5 & 4 & 3 & 3 \\
\end{array}
\]
Proof. Let \( y_i = x^* + 1 - x_i, \ 1 \leq i \leq q \) \( y_q + i = x_i, \ 1 \leq i \leq q \) Then the sequence becomes: \((y_1, y_2, y_3 \ldots y_q, y_{q+1}, \ldots, y_{2q})\)

\[
\max_{1 \leq i \leq q} \{2y_i + i\} \leq 2x^* + q ...... (3.c)
\]

Suppose \( x^* \) is occurring in the position of the least label \( 'r' \) in the sequence \(< x_i >\)

\[
2x^* + q < 2x^* + q + r \leq \max_{1 \leq i \leq q} \{2y_{q+i} + q + i\} = \max_{1 \leq i \leq q} \{2x_i + i\} + q = \alpha*(< x_i >) + 2q - 1by(3.a)
\]

\[
2x^* + q < \alpha*(< x_i >) + 2q - 1...... (3.d)
\]

\[
\max_{q+1 \leq i \leq 2q} \{2y_i + i\} = \max_{1 \leq i \leq q} \{2x_i + i\} + q
\]

\[
\max_{q+1 \leq i \leq 2q} \{2y_i + i\} = \alpha*(< x_i >) + 2q - 1...... (3.e)
\]

using (3.c) and (3.e)

\[
\max_{1 \leq i \leq 2q} \{2y_i + i\} = \alpha*(< x_i >) + 2q - 1...... (3.f)
\]

By applying (3.a), \( \alpha*(< y_i >) = \max_{1 \leq i \leq 2q} \{2y_i + i\} + 1 - 2q = \alpha*(< x_i >) (by 3.f) \)

\( \alpha*(< y_i >) = \alpha*(< x_i >). \)

This completes the proof.

**Corollary 3.1.** If \((x^* + 1 - x_1, x^* + 1 - x_2, \ldots, x^* + 1 - x_{q-1}, x^* + 1 - x_q, x_1, x_2, \ldots, x_q )\) represents super edge-magic sequence on \(2q\) edges satisfying the conditions in theorem 3.1 then the number of vertices is given by

\[
\alpha*(< x_i >) + 2q - \min_{1 \leq i \leq 2q} \{y_i + i\}
\]

Proof. Using the Theorem 3.1:

\[
\min_{1 \leq i \leq 2q} \{y_i + i\} = \min_{1 \leq i \leq q} \{y_i + i\} = \min_{1 \leq i \leq q} \{(x^* + 1 - x_i) + i\}
\]
(by 3.b) and we have

\[ p(<y_i>) = \alpha * (<x_i>) + 2q - \min_{1 \leq i \leq q} \{x^* - x_i + i + 1\} \]

\[ p(<y_i>) = \alpha * (<x_i>) + 2q - \min_{1 \leq i \leq 2q} \{y_i + i\} \]

This completes the proof.

Let us illustrate the theorem and corollary of 3.1, by the following:

Suppose \(<x_i> = (3, 2, 2, 1, 1)\), by the theorem 3.1, \(\alpha^* = 3\), \(p(G_1) = 4, x^* = 3\), \(<y_i> = (1, 2, 2, 3, 3, 3, 2, 2, 1, 1)\), and by corollary 3.1, \(p(G_2) = 11\). Then the graph as shown in Figure 3:

![Figure 3](image1)

![Figure 4](image2)

**Theorem 3.2.** If \((x_1, x_2, \ldots, x_q)\) represents a super edge-magic sequence of a graph \(G\) on \(q\) edges with

\[ \alpha * (<x_i>) = \max_{1 \leq i \leq q} \{2x_i + i\} + 1 - q \ldots \ldots \ (3.g) \]

\[ p(<x_i>) = \max_{1 \leq i \leq q} \{2x_i + i\} - \min_{1 \leq i \leq q} \{x_i + i\} + 1 \ldots \ldots \ (3.h) \]
Then \((x_1 + 1 - x_q, x_1 + 1 - x_{q-1}, \ldots, x^* + 1 - x_2, x^* + 1 - x_1, x_1, x_2, \ldots, x_q)\) represents super edge-magic sequence on 2\(q\) edges with same \(\alpha^*\), for \(x^* = \max\{x_i/1 \leq i \leq q\}\).

**Proof.** Let \(y_i = x^* + 1 - x_{q} - i + 1, 1 \leq i \leq q\) \(y_{q+i} = x_i, 1 \leq i \leq q\) The given sequence becomes: \((y_1, y_2, \ldots, y_q, y_{q+1}, \ldots, y_{2q})\)

\[
\max_{1 \leq i \leq q} \{2y_i + i\} = \max_{1 \leq i \leq q} \{2(x^* + 1 - x_{q-i+1} + i)\}
\]

\[
= 2x^* + 2 - \min_{1 \leq i \leq q} \{2x_{q-i+1} + i\}
\]

\[
\max_{1 \leq i \leq q} \{2y_i + i\} < 2x^* + q \ldots (3.i)
\]

Using theorem 3.1, \(2x^* + q < \alpha^* (\langle x_i \rangle) + 2q - 1\) and By (3.e)

\[
\max_{q+1 \leq i \leq 2q} \{2y_i + i\} \leq \alpha^* (\langle x_i \rangle) + 2q - 1\ldots (3.j)
\]

By (3.i) and (3.j),

\[
\max_{1 \leq i \leq 2q} \{2y_i + i\} = \alpha^* (\langle y_i \rangle) + 1 - 2q
\]

By applying (3.g) \(\alpha^* (\langle y_i \rangle) = \alpha^* (\langle x_i \rangle)\)

This completes the proof.

**Corollary 3.2.** If \((x^* + 1 - x_q, x^* + 1 - x_{q-1}, \ldots, x^* + 1 - x_2, x^* + 1 - x_1, x_1, x_2, \ldots, x_q)\) represents super edge-magic sequence on 2\(q\) edges satisfying the conditions in theorem 3.2 then the number of vertices is given by

\[
\alpha^* (\langle x_i \rangle) + 2q - \min_{1 \leq i \leq 2q} \{y_i + i\}
\]

**Proof.** The sequence \(\langle y_i \rangle\) and other details are followed by theorem 3.2.

\[
\min_{1 \leq i \leq 2q} \{y_i + i\} = \min_{1 \leq i \leq q} \{(x^* + 1 - x_{q-i+1})\}
\]

\[
= (x^* + 1) - \max_{1 \leq i \leq q} \{(x_{q-i+1})\}
\]

\[
p(\langle y_i \rangle) = \max_{1 \leq i \leq 2q} \{2y_i + i\} - \min_{1 \leq i \leq 2q} \{y_i + i\} + 1
\]
By using (3.h)

\[ p(<yi>) = \alpha*(<xi>) + 2q - \min_{1 \leq i \leq 2q} \{y_i + i\} \]

This completes the proof.

\[ \text{Concrete Example for the theorem and corollary of 3.2:} \]
Suppose \(<x_i> = (3,2,2,1,1)\), by using theorem 3.2, \(\alpha* = 3\), \(p(G_1) = 4\), \(x^* = 3\), \(<y_i> = (3,3,2,2,1,3,2,2,1,1)\), and by corollary 3.2, \(p(G_2) = 9\). Then the graph as shown in Figure 4.

\textbf{Theorem 3.3.} If \((x_1, x_2, \ldots, x_q)\) represents a super edge-magic sequence with

\[ \alpha*(<x_i>) = \max_{1 \leq i \leq q} \{2x_i + i\} + 1 - q \]

and

\[ p(<x_i>) = \max_{1 \leq i \leq q} \{2x_i + i\} - \min_{1 \leq i \leq q} \{x_i + i\} + 1 \]

Then \((\alpha* + q - 1 - x_1, \alpha* + q - 2 - x_2, \ldots, \alpha* - x_q, \lambda, x_1, x_2, \ldots, x_q)\) represents a super edge-magic sequence on \(2q+1\) edges with same \(\alpha*\), choose \(\lambda\) such that \(1 \leq \lambda < q\).

\textbf{Proof.} This is an immediate consequence of definition and by applying theorem 3.1.

\textbf{Remark 3.1.} Based on the parameter \(<x_i>, q, \alpha*\), At least one \(\lambda\) such that \(1 \leq \lambda < q\) would give SEMG.

\[ \text{Let us illustrate the theorem and corollary of 3.3 as follows:} \quad <x_i> = (2,1,2,2,1), \quad \alpha* = 4, \quad p(G_1) = 6, \quad <y_i> = (6,6,4,3,3,2,1,2,2,1), \lambda = 3, \quad p(G_2) = 8. \]

Figure 5 shows the graph.
Remark 3.2. Theorems from 3.1 to 3.3 have the same property that the sequence \( < x_i > \) and \( < y_i > \) have same \( \alpha^* \). Each sequence gives one super edge-magic graph say \( G_1, G_2 \) respectively then the graph \( G_2 \) contains \( G_1 \) always. \( G_1 \) indicated by block color and \( G_2 \) indicated by pink color.

Theorem 3.4. If \( (x_1, x_2, \ldots, x_q) \) represents a super edge-magic sequence of a graph \( G \) on \( "q" \) edges with

\[
\alpha^*(<x_i>) = \max_{1 \leq i \leq q} \{2x_i + i\} + 1 - q
\]

\[
p(<x_i>) = \max_{1 \leq i \leq q} \{2x_i + i\} - \min_{1 \leq i \leq q} \{x_i + i\} + 1
\]

Then \( (2x_1 + 1, 2x_1, 2x_2 + 1, 2x_2, ..., 2x_q + 1, 2x_q) \) represents super edge-magic sequence of a graph \( H \) such that \( \alpha^*(H) = 2\alpha^*(G) \) and \( p(H) = 2p(G) \).

Proof. This is an immediate consequence on modification of \( < x_i > \) by definition.

Remark 3.3. \( \alpha^* \) of \( H \) is twice number of \( \alpha^* \) of \( G \), so that the structure of the graph \( G \) is embedded in \( H \) and also the labeling of \( G \) is exactly doubled.

Let us illustrate the theorem 3.4 by the following: In the succeeding figures the color green indicates the graph \( G \) and blue indicates \( H \).

- If \( G \) has one connected component then the corresponding \( H \) as follows: \( < x_i > = (4,4,2,1,2,3,1,3,1) \), \( \alpha^*(G) = 6 \), \( p(G) = 10 \), \( q(G) = 9 \). The sequence for \( H = (9,8,9,8,5,4,3,2,5,4,7,6,3,2,7,6,3,2) \), \( p(H) = 20 \), \( q(H) = 18 \), \( \alpha^*(H) = 12 \). Then the graph as shown in Figure 6

![Figure 6](image-url)
• If G has more than one connected components the corresponding H as follows: \(< x_i > = (4,5,6,6,6,6) \alpha^*(G)=13, p(G)=14, q(G)=6. The sequence for H=(9,8,11,10,13,12,13,12,13,12,13,12,13,12), \alpha^*(H)=26, p(H)=28, q(H)=12. Then the graph as shown in Figure 7

**Proposition 3.1.** If any super edge-magic sequence does not contain the element '1', then \((m-1)\) number of deficiency can be reduced by subtracting \((m-1)\) in each element of that sequence, where \(m\) is minimum element of the given super edge-magic sequence.

*Proof.* This is an immediate consequence of theorem 3.4.

4. Monotonic Sequences and their Behavior in Super Edge-Magic Graph

**Proposition 4.1.** If the sequence \(< x_i >, 1 \leq i \leq q\) defines a super edge-magic graph with \(x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_q (= 1)\) then the feasible range of \(\alpha^*\) is \(x_q + 2 \leq \alpha^* \leq x_1 + 2\).

The range of \(\alpha^*\) is \((x_q + 2, x_1 + 2)\) but all \(\alpha^*\) need not give super edge-magic graph. To find \(\alpha^*\) which will give super edge-magic graph from the following.

**Proposition 4.2.** Let \(< x_i >, 1 \leq i \leq q\) be super edge-magic sequence such that \(x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_q\) with \(x_q = 1\),

\[
\alpha = \max_{1 \leq i \leq q} \{2x_i + i\} and \beta = \min_{1 \leq i \leq q} \{x_i + i\}
\]

Then the number of super edge-magic graph is \((q + 1) - (\alpha - \beta)\) which is denoted by 'r'. Moreover, the value of \(\alpha^*\) is \(\alpha - q + j, 1 \leq j \leq r\) and the corresponding value of \(p\) is at least \(\alpha^* - \beta + q\).

**Definition 4.1.** Let \(< x_i >, 1 \leq i \leq q\) be super edge-magic sequence such that \(x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_q\) then number of digits appeared in the sequence \(< x_i >, 1 \leq i \leq q\) is said to be "order of these sequence" and denoted by 'n'.

**Proposition 4.3.** Let \(< x_i >, 1 \leq i \leq q\) be super edge-magic sequence such that \(x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_q\) with \(x_1 = 1\). Then \(\alpha^* = 2x_q + 1, p=2x_q + q - 1\) must be super edge-magic graph with deficiency[13] is \(n-1\).

5. Conclusion

The relevance of this paper is two fold. First, we introduced definition of SEMS and construction method for SEMG from SEMS. In a consequent step,
the limitations and upshots of SEMS were also discussed. Second, Fabrication of New SEMS and behavior of monotonic sequences of super edge-magic graph were discussed.

References


