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TOWARDS URYSOHN'S LEMMA IN MINIMAL STRUCTURES

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Abstract: With necessary modifications and corrections of some results in the existing literature on minimal structures we propose in this paper a development in the related theory that enables us, along with other important results, to establish a version of Urysohn's lemma in this setting. In doing so various separation axioms and different kinds of continuity in the replacement of topology, generalized topology or weak structure by minimal structure which are closely related to Urysohn's lemma have been studied.

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1. Introduction and Preliminaries

Minimal structure, as the name suggests, involves least possible requirements for a class of subsets of a nonempty set to have some "structure" from the view point of topological studies. This unique concept, emerged in 2000 (V. Popa and T. Noiri [7]), drew the attention of a number of researchers in the related field. Very recent, it was further studied by Carlos Carpintero, Ennis Rosas, Margot Salas [3], A. Pushpalatha, E. Subha [4], R. Parimelazhagan, K. Balachandran,

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N. Nagaveni [5], Sunisa Buadong, Chokchai Viriyapong, Chawalit Boonpok [6], Won Keun Min, Young Key Kim [7] etc.. We extend the field by introducing notions of various separation axioms, continuity and established the Urysohn's lemma in this context. At first we give some basic definitions and outline of the results relavent to this extension. As in [2, 3, 4, 5, 6, 7] a family m of subsets of a non-empty set X containing the empty subset and the whole set is called a minimal structure and we call the ordered pair (X, m) a minimal structured space (briefly MSS). Elements of a minimal structure m on X are called m-open sets and their complements are called m-closed sets. For any $A \subset X$, $i_m(A)$ ($m - Int$ in [7]) denotes the union of all m-open sets contained in A and $c_m(A)$ ($m - Cl$ in [7]) denotes the intersection of all m-closed sets containing A; presence of the empty set in m well-defines $i_m(A)$ and $c_m(A)$. i_m is contractive c_m is expansive, both are monotonic and idempotent and their relationship is $i_m(A) = X - c_m(X - A)$, for any $A \subset X$. $x \in i_m(A)$ iff there is an m-open set $B \subset A$ containing x and $x \in c_m(A)$ iff $B \cap A \neq \phi$ whenever $x \in B \in m$. All these are similar results to other topology-like structures (topology or generalized topology or weak structure) but unlike in topology or generalized topology $i_m(A)$ may not be m-open and $c_m(B)$ may not be m-closed, for $A, B \subset X$; though they will be so if A is m-open and B is m-closed. In the present work we have shown that how this dissimilarities propagate differences in separation axioms, continuity and Urysohn's lemma.

2. Correction of Some Results Appeared in [7]

Let's begin with the following example;

Example 2.1. Let us consider the minimal structure $m_X = \{\phi, \{a, b\}, \{b, c\}, X\}$ on $X = \{a, b, c\}$, $A = \{a, b\}$, $B = \{b, c\}$, $C = \{c\}$ and $D = \{a\}$ then $m - Int(A) = A$, $m - Int(B) = B$, $m - Int(A \cap B) = \phi$, $m - Cl(C) = C$, $m - Cl(D) = D$ and $m - Cl(C \cup D) = X$.

This shows that the statement in Theorem 2.5 (5) [7] is not true and it must be replaced by $m - Int(A \cap B) \subset m - Int(A) \cap m - Int(B)$ and $m - Cl(A) \cup m - Cl(B) \subset m - Cl(A \cup B)$.

In the proof of the Theorem 3.8 [7] and Theorem 4.20 [7] monotonicity of the operators I_m and Cl_m have been used but none of them is monotonic as shown in the following example;

Example 2.2. Consider the minimal structure $m_X = \{\phi, \{a, b\}, \{b, c\}, \{c, d\}, X\}$ on $X = \{a, b, c, d\}$, $A = \{a, b, c\}$, $B = \{a, b\}$ and $C = \{a\}$ then $I_m(A) = \phi$,

$I_m(B) = B$, $Cl_m(B) = B$ and $Cl_m(C) = X$.

The statement of the Theorem 3.8 [7] is wrong because (3) $\Leftrightarrow$ (4) there. The following examples are supporting wittiness.

Example 2.3. Let $X = \{a, b, c\}$, $Y = \{1, 2\}$, $m_X = \{\phi, \{b\}, X\}$, $m_Y = \{\phi, \{2\}, Y\}$ and $f : X \to Y$ be a function defined by $f(a) = f(c) = 1, f(b) = 2$; f is $M^{\square}$-continuous and Theorem 3.8 (3) [7] holds but $f(Cl_{m_X}(A)) = f(X) = \{1, 2\} \nsubseteq \{1\} = Cl_{m_Y}(\{1\}) = Cl_{m_Y}(f(A))$, where $A = \{c\}$, shows that Theorem 3.8 (4) [7] does not hold.

Example 2.4. Let $X = \{a, b, c\}$, $Y = \{1, 2, 3\}$, $m_X = \{\phi, \{b\}, X\}$ and $m_Y = \{\phi, \{2\}, Y\}$. Then the function $f : X \to Y$ defined by $f(a) = f(b) = 1, f(c) = 2$ is not $M^{\square}$-continuous and Theorem 3.8 (3) [7] is not true for this function but it can be verified that Theorem 3.8 (4) [7] holds.

The Theorem 4.20 [7] seems to be incorrect, we put it in the revised form in the following

Theorem 2.1. *A bijection $f : (X, m_X) \to (Y, m_Y)$, where m_X and m_Y are minimal structures on X and Y respectively, is $M^{\square}$-open if and only if $I_m(f^{-1}(B)) \subset f^{-1}(I_m(B))$ for each $B \subset Y$.*

A corrected version of the Theorem 3.8 [7] is included in the Theorem 3.11 of the following Section 3.

3. Separation Axioms, Continuity and Urysohn's Lemma

Definition 3.1. A minimal structured space (X, m) is called $m - T_1$ if for any two distinct points $x, y \in X$ there are m-open sets U and V so that $x \in U$, $y \in V$, $x \notin V$ and $y \notin U$.

Example 3.1. Let $X = \{a, b, c\}$. On this X, $m_1 = \{\Phi, X, \{a\}, \{b\}, \{c\}\}$, $m_2 = \{\Phi, X, \{a, b\}, \{b, c\}, \{c, a\}\}$ are MSs and the corresponding MSSs are $m - T_1$.

The above MSSs in Example 3.1 show that a finite MSS may be $m - T_1$ without being $P(X)$.

Definition 3.2. Let (X, m) be an MSS. An element A of $P(X)$ is called a fixed point of c_m if $c_m(A) = A$.

Theorem 3.1. *A MSS (X, m) is $m - T_1$ iff every singleton is a fixed point of c_m.*

Proof. If $x \neq y \in c_m(\{x\})$ then every m-open set containing y will contain x and (X, m) would not be $m - T_1$. Thus in a $m - T_1$ MSS (X, m) every singleton is a fixed point of c_m. Conversely for any two distinct points x and y in a MSS (X, m), where every singleton is a fixed point of c_m, $y \notin c_m(\{x\})$ and $x \notin c_m(\{y\})$; this implies there are m-open sets U and V containing x and y respectively such that $U \bigcap \{y\} = \Phi$ and $V \bigcap \{x\} = \Phi$, so (X, m) is $m - T_1$. □

Note that those singletons may not be m-closed, (X, m_1) in Example 3.1 is one such wetness.

Definition 3.3. An MSS (X, m) is called $m - T_2$ if for any two distinct points $x, y \in X$ there exists disjoint m-open sets U and V so that $x \in U$, $y \in V$.

In Example 3.1 (X, m_1) is a $m - T_2$ but (X, m_2) is not $m - T_2$ though it is $m - T_1$. Obviously an $m - T_2$ MSS is $m - T_1$.

Definition 3.4. We say that a sequence $\{x_n\}$ in an MSS (X, m) converges to $x \in X$ if for any m-open set U there exists a positive integer N so that $x \in U$ for all $n \geq N$; in this case $\{x_n\}$ is called a convergent sequence and x is called limit of it.

Theorem 3.2. *In an $m - T_2$ MSS every convergent sequence has unique limit.*

Proof. If x, y are two distinct limits of a sequence $\{x_n\}$ in an $m - T_2$ MSS (X, m), then for any two m-open sets U and V containing x and y respectively there exists a positive integer N so that $x_n \in U \cap V$ for all $n \geq N$ and this contradicts the fact that the MSS is $m - T_2$. □

Definition 3.5. An MSS (X, m) is called m-regular if for any point $x \in X$ and any m-closed set C not containing x there exists disjoint m-open sets U and V so that $x \in U$ and $C \subset V$.

Theorem 3.3. *If (X, m) is m-regular MSS then for any $x \in U \in m$ there exists $V \in m$ so that $x \in V \subset c_m(V) \subset U$.*

Proof. Let (X, m) be an m-regular MSS and U be any m-open set, then for any point $x \in U$ the m-closed set $X - U$ does not contain x, so there exists m-open sets V and E such that $x \in V$, $X - U \subset E$ and $V \cap E = \phi$. This implies $V \subset X - E$ and hence $c_m(V) \subset X - E \subset U$, since $X - E$ is m-closed. Thus for any $x \in U \in m$ there exists $V \in m$ so that $x \in V \subset c_m(V) \subset U$. □

But the converse of the Theorem 3.3 is not true as seen in the MSS (X, m_1) of the Example 3.1 which is true in regular topological spaces.

We call the family $M(x) = \{U; x \in U \in m\}$, where m is a minimal structure on X, a minimal-star at x and by the help of the family $\{M(x); x \in X\}$ of minimal-stars we define $i^{\star}{}_m(A) = \{x \in A; A \in M(x)\}(= I_m(A)$ in [7]) and $c^{\star}{}_m(A) = \{x \in X; X - A \notin M(x)\}(= Cl_m(A)$ in [7]). Now $i^{\star}{}_m$ agrees with i_m in m and $c^{\star}{}_m$ agrees with c_m in $m^c = \{X - U; U \in m\}$; also if $A \notin m$ then $i^{\star}{}_m(A) = \Phi$ and $c^{\star}{}_m(X - A) = X$. So, $i^{\star}{}_m(A) \subset i_m(A)$, $c_m(A) \subset c^{\star}{}_m(A)$. Let $X = \{a, b, c\}$ and $m = \{\Phi, X, \{a\}, \{b, c\}\}$ then $i^{\star}{}_m(\{a\}) = \{a\} \supset \Phi = i^{\star}{}_m(\{a, b\})$ and $c^{\star}{}_m(\{b, c\}) = \{b, c\} \subset X = c^{\star}{}_m(\{b\})$, hence $i^{\star}$ and $c^{\star}$ are not monotonic on the power set $P(X)$ of the underlying set X. Now let $x \notin X - i^{\star}{}_m(X - A) \Leftrightarrow x \in i^{\star}{}_m(X - A) \Leftrightarrow x \in X - A \in m \Leftrightarrow X - A \in M(x) \Leftrightarrow x \notin c^{\star}{}_m(A)$. Thus $c^{\star}{}_m(A) = X - i^{\star}{}_m(X - A)$ and similarly $i^{\star}{}_m(A) = X - c^{\star}{}_m(X - A)$ for all $A \subset X$. The purpose of the definitions of $i^{\star}{}_m$ and $c^{\star}{}_m$ is the fact that $i^{\star}{}_m(A) = A \Rightarrow A \in m$ and $c^{\star}{}_m(A) = A \Rightarrow X - A \in m$. Thus we have

Theorem 3.4. *Let m be a minimal structure on X. Then*
1. $i^{\star}{}_m(A) = i_m(A)\ \forall\ A \in m$
2. $c^{\star}{}_m(X - A) = c_m(X - A)\ \forall\ A \in m$
3. $i^{\star}{}_m(A) = \Phi$ *and* $c^{\star}{}_m(X - A) = X$ *whenever* $A \in P(X) - m$.
4. $i^{\star}{}_m(A) \subset i_m(A)$, *and* $c_m(A) \subset c^{\star}{}_m(A)\ \forall\ A \subset X$.
5. $i^{\star}{}_m$ *and* $c^{\star}{}_m$ *are monotonic on* m *but not on the power set* $P(X)$ *(if* $m \neq P(X)$*) of the underlying set* X.
6. $c^{\star}{}_m(A) = X - i^{\star}{}_m(X - A)\ \forall\ A \subset X$.
7. $i^{\star}{}_m(A) = A \Rightarrow A \in m$ *and* $c^{\star}{}_m(A) = A \Rightarrow X - A \in m$.

Theorem 3.5. *An MSS (X, m) is m-regular if for any $x \in U \in m$ there exists $V \in m$ so that $x \in V \subset c^{\star}{}_m(V) \subset U$.*

Proof. Now in an MSS (X, m), let for any $x \in U \in m$ there exists $V \in m$ so that $x \in V \subset c^{\star}{}_m(V) \subset U$. Let x be any point of X and C be any m-closed set not containing x, so $x \in X - C$ and $X - C$ is an m-open set and therefore by the hypothesis there exists m-open set V such that $x \in V \subset c^{\star}{}_m(V) \subset X - C \Rightarrow x \in V$ and $C \subset X - c^{\star}{}_m(V) = T$ (say). Then since $V \cap T = \phi$ hence the MSS (X, m) is m-regular. Thus an MSS (X, m) is m-regular if for any $x \in U \in m$ there exists $V \in m$ so that $x \in V \subset c^{\star}{}_m(V) \subset U$ □

Example 3.2. The MSS $(\mathbb{R}, m)$ where m is the collection of all right and left open rays including ϕ and $\mathbb{R}$ shows that the condition in Theorem 3.5 is not necessary.

Theorem 3.6. *If an MSS (X, m) is m-regular then for any $x \in U \in m$ there exists $V \in m$ so that $x \in V \subset c_m(V) \subset U$.*

The MSS (X, m_1) in the Example 3.1 ensures that this condition is not sufficient. We search a necessary as well as sufficient condition for an MSS (X, m) to be m-regular.

Theorem 3.7. *An MSS (X, m) is m-regular if and only if for any point $x \in X$ and any m-open set U containing x there exist m-open set V and m-closed set $V^{\square}$ so that $x \in V \subset V^{\square} \subset U$.*

Proof. Let an MSS (X, m) be m-regular, x be any point in X and U be any m-open set containing x. Then $X - U$ is m-closed set not containing x. So, there are m-open sets E and F such that $x \in E$, $(X - U) \subset F$ and $E \cap F = \phi$. Hence $E \subset (X - F) \subset U$. Taking $V = E$ and $V^{\square} = (X - F)$ we have V is m-open, $V^{\square}$ is m-closed and $x \in V \subset V^{\square} \subset U$. Conversely, let for any point $x \in X$ and any m-open set U containing x there exists m-open set V and m-closed set $V^{\square}$ so that $x \in V \subset V^{\square} \subset U$. Let $x \in X$ and E be any m-closed set not containing x. Then $x \in (X - E)$ and $(X - E)$ is m-open. Hence by the hypothesis there is an m-open set V and an m-closed set $V^{\square}$ so that $x \in V \subset V^{\square} \subset (X - E)$. Therefore $E \subset (X - V^{\square})$, also $V \cap (X - V^{\square}) \subset V \cap (X - V) = \phi$ hence $V \cap (X - V^{\square}) = \phi$ which implies (X, m) is m-regular. □

Definition 3.6. An MSS (X, m) is called m-normal if for any two disjoint m-closed sets C and D there exists two disjoint m-open sets U and V so that $C \subset U$ and $D \subset V$.

Theorem 3.8. *Let C be any m-closed set and U be any m-open set containing C. Then there exists an m-open set V so that $C \subset V \subset c^{\square}{}_m(V) \subset U \Rightarrow$ MSS (X, m) is m-normal $\Rightarrow$ there exists an m-open set V so that $C \subset V \subset c_m(V) \subset U$.*

Theorem 3.9. *A necessary and sufficient condition for an MSS (X, m) is m-normal if for any m-closed set C and any m-open set U containing C there is an m-open set V and an m-closed set $V^{\square}$ such that $C \subset V \subset V^{\square} \subset U$.*

Definition 3.7. A function $f : X \to Y$ from an MSS (X, m_X) into an MSS (Y, m_Y) is said to be M-continuous at a point $x \in X$ if for any $V \in M(f(x))$, there is $U \in M(x)$ such that $f(U) \subset V$; f is called M-continuous if it is so at each point of its domain.

Theorem 3.10. *For any function $f : X \to Y$ from an MSS (X, m_X) into an MSS (Y, m_Y), as in general case, it is observed that the following are equivalent:*

(1) f is M-continuous.

(2) $f(c_{m_X}(A)) \subset c_{m_Y}(f(A))$ for $A \subset X$.

(3) $c_{m_X}(f^{-1}(B)) \subset f^{-1}(c_{m_Y}(B))$ *for* $B \subset Y$.

(4) $f^{-1}(i_{m_Y}(B)) \subset i_{m_X}(f^{-1}(B))$ *for* $B \subset Y$.

Definition 3.8. A function $f : X \to Y$ from an MSS (X, m_X) into an MSS (Y, m_Y) is said to be $M^{\star}$-continuous if for every $B \in m_Y$, $f^{-1}(B) \in m_X$.

Obviously $M^{\star}$-continuity implies M-continuity but the converse is not true as seen in the following Example 3.3.

Example 3.3. Let $X = Y = \{a, b, c\}$, $m_X = \{\phi, X, \{a\}, \{b\}\}$,and $m_Y = \{\phi, X, \{a, b\}\}$ then the identity function is M-continuous but not $M^{\star}$-continuous.

If $x \in i^{\star}_m(A)$ then there is a $U \in M(x)$ contained in A but $x \in c^{\star}{}_m(A)$ does not mean that every $V \in M(x)$ intersects A as shown in the following Example 3.4.

Example 3.4. Let $X = \{a, b, c\}$, $m = \{\phi, X, \{a\}, \{a, b\}, \{b, c\}\}$ and $A = \{b\}$ then, $a \in c^{\star}{}_m(A) = X$ and $V = \{a\} \in M(a)$ but $V \cap A = \phi$.

This fact together with the non-monotonicity of the operators $i^{\star}$ and $c^{\star}$ cause the difference between M-continuity and $M^{\star}$-continuity.

Theorem 3.11. *Let* $f : X \to Y$ *be a function from an MSS* (X, m_X) *into an MSS* (Y, m_Y)*, then the following are equivalent:*

(i) f *is* $M^{\star}$*-continuous.*

(ii) $f^{-1}(i^{\star}_{m_Y}(B)) \subset i^{\star}_{m_X}(f^{-1}(B))$ *for* $B \subset Y$.

(iii) $c^{\star}_{m_X}(f^{-1}(B)) \subset f^{-1}(c^{\star}_{m_Y}(B))$ *for* $B \subset Y$.

If further f *is a bijection, the necessity of it is given below, then all these three statements are equivalent to*

(iv) $f(c^{\star}_{m_X}(A)) \subset c^{\star}_{m_Y}(f(A))$ *for* $A \subset X$.

Example 3.5. Let $X = \{a, b, c\}$, $Y = \{1, 2\}$, $m_X = \{\phi, X, \{b\}\}$, $m_Y = \{\phi, Y, \{2\}\}$ and $f : X \to Y$ be a function defined by $f(a) = f(c) = 1, f(b) = 2$; f is $M^{\star}$-continuous hence (iii) holds but $f(c^{\star}_{m_X}(A)) = f(X) = \{1, 2\} \nsubseteq \{1\} = c^{\star}_{m_Y}(\{1\}) = c^{\star}_{m_Y}(f(A))$ where $A = \{c\}$ shows that (iv) does not hold. Also let $X = \{a, b, c\}$, $Y = \{1, 2, 3\}$, $m_X = \{\phi, X, \{b\}\}$ and $m_Y = \{\phi, Y, \{2\}\}$. Then the function $f : X \to Y$ defined by $f(a) = f(b) = 1, f(c) = 2$ is not $M^{\star}$-continuous, so (iii) can not be true but it can be verified that (iv) holds.

Definition 3.9. Intersection of any right open ray (a, ∞) in the linearly ordered set of real numbers with $[0, 1]$ is called a right open ray in $[0, 1]$; similarly define left open ray in $[0, 1]$. A function $f : X \to [0, 1]$ from an MSS (X, m_X) into the subspace $[0, 1]$ of $\mathbb{R}$ with usual topology is said to be m_X-upper-semi-continuous (resp. m_X-lower-semi-continuous) [1] at a point $x \in X$ if for any left (resp. right) open ray R in $[0,1]$ containing $f(x)$, there is $U \in M(x)$ such that $f(U) \subset R$; f is called m_X-upper-semi-continuous (resp. m_X-lower-semi-continuous) if it is so at each point of its domain.

Example 3.6. Let m_l and m_r be the collections of all left-open rays and right-open rays respectively in $[0,1]$ together with ϕ and $[0,1]$. Then the identity mapping e on $[0,1]$ is m_l-upper-semi-continuous, m_r-lower-semi-continuous and $m_l \cup m_r$-upper-lower-semi-continuous but neither it is m_l-lower-semi-continuous nor m_r-upper-semi-continuous even $e : ([0,1], m_l \cup m_r) \to [0,1]$ is not M-continuous.

We end with the following generalization of Urysohn's lemma.

Theorem 3.12. *For any pair of disjoint m-closed sets C and D in an m-normal MSS (X, m) there is a m-upper-lower-semi-continuous function $f : X \to [0,1]$ so that $f(x) = 0 \ \forall \ x \in C$ and $f(x) = 1 \ \forall \ x \in D$.*

Proof. Let C and D be any disjoint m-closed sets in an m-normal MSS (X, m), $V_1 = X - D$ and $V_0{}^{\square} = C$. Since the m-closed set $V_0{}^{\square}$ is contained in the m-open set V_1, therefore, by using m-normality there is an m-open set $V_{\frac{1}{2}}$ and an m-closed set $V_{\frac{1}{2}}{}^{\square}$ so that $V_0{}^{\square} \subset V_{\frac{1}{2}} \subset V_{\frac{1}{2}}{}^{\square} \subset V_1$. Applying the hypothesis on (X, m) to each pair $V_0{}^{\square}$, $V_{\frac{1}{2}}$ and $V_{\frac{1}{2}}{}^{\square}$, V_1, we have m-open sets $V_{\frac{1}{4}}$, $V_{\frac{3}{4}}$ and m-closed sets $V_{\frac{1}{4}}{}^{\square}$, $V_{\frac{3}{4}}{}^{\square}$ so that $V_0{}^{\square} \subset V_{\frac{1}{4}} \subset V_{\frac{1}{4}}{}^{\square} \subset V_{\frac{1}{2}} \subset V_{\frac{1}{2}}{}^{\square} \subset V_{\frac{3}{4}} \subset V_{\frac{3}{4}}{}^{\square} \subset V_1$. Continuing this process one can define m-open sets V_s, V_t and m-closed sets $V_s{}^{\square}$, $V_t{}^{\square}$ for any dyadic rational s and t in $[0,1]$ of the form $\frac{k}{2^n}$, $k = 1, 2, 3 \ldots\ldots 2^n - 1$ and $n \in \mathbb{N}$ so that $s < t \Rightarrow V_0{}^{\square} \subset V_s \subset V_s{}^{\square} \subset V_t \subset V_t{}^{\square} \subset V_1$. If s is any other dyadic rational, let $V_s = \phi$ for $s \leq 0$, $V_s = X$ for $s > 1$, and $V_s{}^{\square} = \phi$ for $s < 0$, $V_s{}^{\square} = X$ for $s \geq 1$. Now consider a function $f : X \to [0,1]$ defined by $f(x) = \inf\{s; x \in V_s\} = \inf\{s; x \in V_s{}^{\square}\} \ \forall \ x \in X$. Then $f(x) = 0, \ \forall \ x \in C$ and $f(x) = 1, \ \forall \ x \in D$. Definition of the function f and construction of the sets V_s and $V_s{}^{\square}$ for dyadic rational number s show that $x \in V_s{}^{\square} \Rightarrow f(x) \leq s$ and $x \notin V_s \Rightarrow f(x) \geq s$. Now for the ray $[0,1]$ (which is left as well as right open ray in $[0,1]$) U is an m-open set containing x_0 so that $f(U) \subset [0,1]$ where $U = V_1$ if $f(x_0) = 0$ and if $f(x_0) = 1$ then $U = X - V_0{}^{\square}$. For any left open ray $[0, d)$ in $[0,1]$ containing $f(x_0)$ choose a dyadic rational q such that $f(x_0) < q < d$ and consider the V_q. Since $0 < q < 1$ therefore V_q is m-open set containing x_0 because otherwise $f(x_0) \geq q$. Also $f(V_q) \subset [0, d)$ since $x \in V_q \Rightarrow f(x) \leq q$. And for any right open ray $(c, 1]$ in $[0,1]$ containing $f(x_0)$ select a dyadic rational p such that $c < p < f(x_0)$ then, $X - V_p{}^{\square}$ is an m-open set containing x_0 so that $f(X - V_p{}^{\square}) \subset (c, 1]$. □

Remark. Here we have a comment on Theorem 3.12. In the Theorem 3.12, if the minimal structure m is a topology then C and D become disjoint closed sets in the corresponding normal topological space, m-upper-lower-semi-continuity corresponds to continuity and Urysohn's lemma of point set topology will follow. Thus it can be stated that the Theorem 3.12 is not a translation of Urysohn's lemma, it is a generalization of Urysohn's lemma.

References

[1] N. Levine, Semi-open sets and semicontinuity in topological spaces, *Ams. Math. Monthly*, **70** (1963), 36-41.

[2] V. Popa, T. Noiri, On M-continuous functions, *Anal. Univ. "Dunarea de Jos" Galati, Ser. Mat. Fiz. Mec. Teor., Fasc. II*, **18**, No. 23 (2000), 31-41.

[3] Carlos Carpintero, Ennis Rosas, Margot Salas, Minimal structures and separations properties, *Int. J. Pure Appl. Math.*, **34**, No. 4 (2007), 473-488.

[4] A. Pushpalatha and E. Subha, Strongly generalized closed sets in minimal structures, *Int. Journal of Math. Analysis*, **3**, No. 26 (2009), 1259-1263.

[5] R. Parimelazhagan, K. Balachandran, N. Nagaveni, Weakly generalized closed sets in minimal structure, *Int. J. Contemp. Math. Sciences*, **4**, No. 27 (2009), 1335-1343.

[6] Sunisa Buadong, Chokchai Viriyapong, Chawalit Boonpok, On generalized topology and minimal structure spaces, *Int. Journal of Math. Analysis*, **5**, No. 31 (2011), 1507-1516.

[7] W. K. Min, Y. K. Kim, M*-Continuity and product minimal structure on minimal structures, *Int. J. Pure Appl. Math.*, **69**, No. 3 (2011), 329-339.