International Journal of Pure and Applied Mathematics
Volume 85 No. 3 2013, 563-575
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v85i3.11
ijpam.eu

COUNTING NUMBER OF FUZZY SUBGROUPS OF SOME OF DIHEDRAL GROUPS

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Abstract: In this paper, we compute number of fuzzy subgroups of some dihedral groups such as D_{2p^n} where p is a prime number and $D_{2p_1\times p_2\times\cdots\times p_n}$ where $p_1, p_2, ..., p_n$ are distinct prime numbers. We use their chain diagram to determine the number of their fuzzy subgroups and present an explicit recursive formula to D_{2p^n} and at the result in specially case D_{2^n} and finally a formula to count number of fuzzy subgroups of dihedral group $D_{2p_1\times p_2\times\cdots\times p_n}$.

AMS Subject Classification: 03E72, 06D72, 20N25
Key Words: fuzzy groups, lattice, dihedral groups, chain

1. Introduction

One of the most important problems of fuzzy group theory is to classify the fuzzy subgroups of a finite group [1]. Several papers have treated the particular case of finite Abelian group. Laszlo [2] studied the construction of fuzzy subgroups of groups of the orders one to six. Zhang and Zou [3] have determined the number of fuzzy subgroups of cyclic groups of the order p^n where p is a prime number.

Received: February 11, 2013

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Murali and Makamba in [4] and [5], considering a similar problem, found the number of fuzzy subgroups of Abelian groups of the order $p^n q^m$ where p and q are different primes. In [6], Tarnauceanu and Bentea established a recurrence relation verified by the number of fuzzy subgroups of finite cyclic groups. Their result is the improving of Murali's work in [4] and [5]. In this paper, we work on particular case of dihedral groups such as D_{2p^n} where p is a prime number and $D_{2p_1 \times p_2 \times \cdots \times p_n}$, where $p_1, p_2, ..., p_n$ are distinct prime numbers.

2. Preliminaries

First of all, we present some basic notions and results of fuzzy theory that are required later in this paper (for more details, see [10] and [11]). In this section, a group G is assumed to be a finite group. Let $(G, ., e)$ be a group (e denotes the identity of G) and $\mu : G \to [0, 1]$ a fuzzy subset of G. We say that μ is a fuzzy subgroup of G if it satisfies the next two conditions:

(1) $\mu(xy) \geq min(\mu(x), \mu(y))$ for all $x, y \in G$

(2) $\mu(x^{-1}) \geq \mu(x)$ for any $x \in G$

In this situation we have $\mu(x^{-1}) = \mu(x)$ for any $x \in G$, and $\mu(e) = max\, \mu(G)$[9].

Theorem 1. (see [7]) *A fuzzy subset μ of G is a fuzzy subgroup of G if and only if there is a chain of subgroups of G, $P_1(\mu) \subset P_2(\mu) \subset \cdots \subset P_n(\mu) = G$ such that μ can be written as:*

$$\mu(x) = \begin{cases} \theta_1 & x \in P_1(\mu) \\ \theta_2 & x \in P_2(\mu) - P_1(\mu) \\ \vdots & \\ \theta_n & x \in P_n(\mu) - P_{n-1}(\mu) \end{cases} \tag{1}$$

Definition 2. (see [7]) Let μ, γ be fuzzy subgroups of G of the form

$$\mu(x) = \begin{cases} \theta_1 & x \in P_1 \\ \theta_2 & x \in P_2 - P_1 \\ \vdots & \\ \theta_n & x \in P_n - P_{n-1} \end{cases}$$

and

$$\gamma(x) = \begin{cases} \delta_1 & x \in M_1 \\ \delta_2 & x \in M_2 - M_1 \\ \vdots & \\ \delta_m & x \in M_m - M_{m-1} \end{cases}$$

Then, we say μ and γ are equivalent and write $\mu \sim \gamma$, if:

(1) $m = n$

(2) $P_i(\mu) = M_i(\gamma) \quad \forall i \in \{1, 2, ..., n\}$

It is easily checked that this relation is indeed an equivalence relation. Two fuzzy subgroups of G are said to be different if they are not equivalent.

Lemma 3. *The number of fuzzy subgroups of G is equal to the number of chain on the lattice subgroups of G.*

Sulaiman and Abd Ghafur [8] present some definition for symbolizing the number of fuzzy subgroups. We mention some of their results and theorems without any proof (for more details see [8]). Considering (1), we can denote the number of fuzzy subgroups of G as $O(F_G)$, while the number of fuzzy subgroups μ of G with $P_1(\mu) = H$ is denoted by $O(F_{P_1=H})$. From Theorem 1 we have:

$$O(F_G) = \sum_{H \leq G} O(F_{P_1=H}) \tag{2}$$

and

$$O(F_{P_1=G}) = 1 \tag{3}$$

Theorem 4. *Let H be a subgroup of G, and the set of all subgroups of G which contain H (but are not equal to H) be $\{H_1, H_2, \cdots, H_n = G\}$. Then $O(F_{P_1=H}) = \sum_{1 \leq i \leq n} O(F_{P_1=H_i})$.*

Theorem 5. *Let e be identity element of a group G. Then $O(F_G) = 2.O(F_{P_1=\{e\}})$.*

2.1. Dihedral Groups

The dihedral groups have a presentation by generators and relations as:

$$D_{2n} =< a, b; a^n = 1, b^2 = 1, ba = a^{-1}b > \tag{4}$$

Also, we can say the dihedral group is generated by the elements a and b as:

$$D_{2n} = \{1, a, a^2, ..., a^{n-1}, b, ab, a^2b, ..., a^{n-1}b\} \tag{5}$$

Theorem 6. *every subgroup of D_{2n} is cyclic or dihedral. A complete listing of the subgroups is as follows:*

(1) Cyclic subgroups $< a^d >$, where $d|n$, with index $2d$.

(2) Dihedral subgroups $< a^d, a^i b >$, where $d|n$ and $0 \leq i \leq d-1$, with index d.

every subgroup of occurs exactly once in this listing [12].

3. Number of Fuzzy Subgroups of D_{2p^n}

In this section we present a recursive formula to count number of fuzzy subgroups of group D_{2p^n} that is called G_n in this section.

Theorem 7. *Let G_n be a dihedral group of form $G_n = D_{2p^n}$, where p is a prime number, the number of fuzzy subgroups of G_n is:*

$$O(F_{G_n}) = 2.O(F_{\{e\}_n})$$

$$O(F_{\{e\}_n}) = 2 + \sum_{i=1}^{n-1} O(F_{P_1=\{e\}_i}) + \sum_{i=1}^{n} 2^{i-1}p^i, \qquad O(F_{P_1=\{e\}_1}) = p+2 \tag{6}$$

Proof. we focused to determine a relation between this number and lattice subgroup. The lattice subgroup of $G_1 = D_{2p}$ is shown in Figure 1. According to relation (3) we have $O(F_{P_1=G}) = 1$ and using Theorem 4 we have:

$$O(F_{P_1=<b>}) = O(F_{P_1=<ab>}) = \cdots = O(F_{P_1=<a^{p-1}b>}) = 1, \qquad O(F_{P_1=<a>}) = 1$$

Therefore:

$$O(F_{P_1=<b>}) = 1 + 1 + p(1) = p + 2.$$

Similarly, the lattice subgroup of $G_2 = D_{2p^2}$ is shown in Figure 2 and According to relation (3) we have $O(F_{P_1=G}) = 1$ and using Theorem 4 we have:

$$O(F_{P_1=<a^p,b>}) = O(F_{P_1=<a^p,ab>}) = O(F_{P_1=<a^p,a^2b>}) = \cdots = O(F_{P_1=<a^p,a^{p-1}b>}) = 1,$$

$$O(F_{P_1=<b>}) = O(F_{P_1=<ab>}) = O(F_{P_1=<a^2b>}) = \cdots = O(F_{P_1=<a^{p-1}b>}) = 2,$$

$$O(F_{P_1=<a^p>}) = 2 + p, \qquad O(F_{P_1=<a>}) = 1.$$

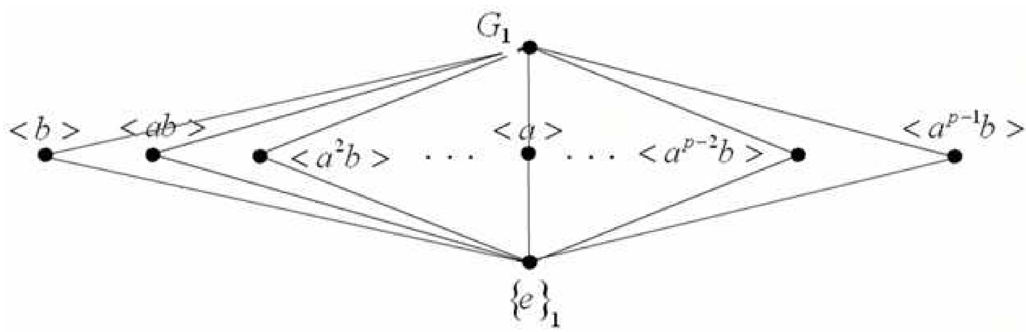

Figure 1: Lattice subgroup of $G_1 = D_{2p}$

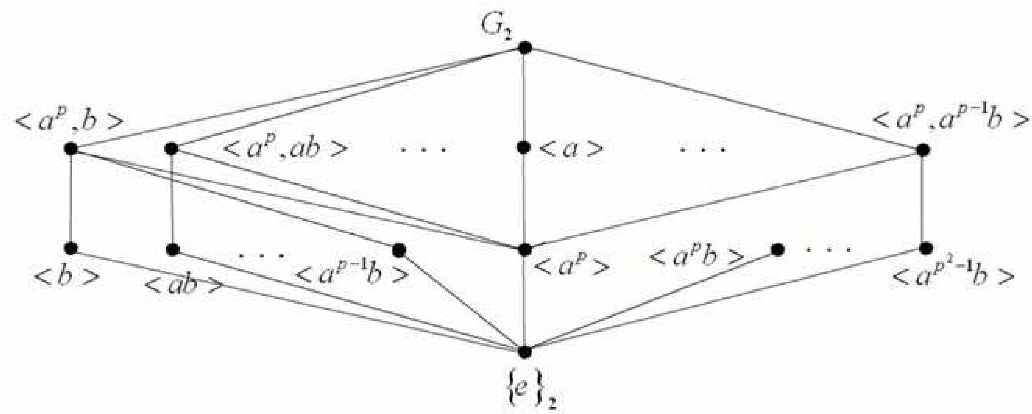

Figure 2: lattice Subgroup of $G_2 = D_{2p}$

Therefore:

$$O(F_{P_1=<a>}) = 1 + 1 + p(1) + p^2(2) + (2 + p) = 2p^2 + 2p + 4$$

We can similarly continue this approach and draw diagrams for others G_n and find $O(F_{\{e\}_n})$. That is the sum of all $O(F_{P_1=H_i})$ and $O(F_G)$ which come in upper levels of it in lattice. This is the key to find the formula.

As can be seen in each level we have two types of subgroups. One of them is cyclic subgroups $< a^{p^k} >$ that their number is equal to: $O(F_{P_1=\{e\}_i})$ and the others are dihedral subgroups in form $< a^p, a^k b >$, that their number in level i is equal to: $2^{i-1}p^i$.

So, we can write:

$$O(F_{\{e\}_n})$$
$$= O(F_G) + O(F_{<a>}) + \sum \Big\{\text{all of the number of cyclic subgroups of } G_n\Big\} + \sum_{i=1}^{n} \Big\{\text{numbers of dihedral subgroups in level i}\Big\}.$$

Therefore:

$$O(F_{\{e\}_n}) = 2 + \sum_{i=1}^{n-1} O(F_{P_1=\{e\}_i}) + \sum_{i=1}^{n} 2^{i-1}p^i.$$

Now, from theorem 5 we have: $O(F_{G_n}) = 2.O(F_{P_1=\{e\}_n})$. □

This formula can simplify as follows:

Corollary 8. *Let G be a dihedral group of form $G = D_{2p^n}$, where p is a prime number. The number of fuzzy subgroups of G is:*

$$O(F_{D_{2p^n}}) = 2^{n+1} + \sum_{j=0}^{n-1} 2^n p^{n-j}$$

Proof. First, from (6) we have:

$$O(F_{\{e\}_n}) = 2 + \sum_{i=1}^{n-1} O(F_{P_1=\{e\}_i}) + \sum_{i=1}^{n} 2^{i-1}p^i$$
$$\Longrightarrow \sum_{i=1}^{n-1} O(F_{P_1=\{e\}_i}) = O(F_{\{e\}_n}) - 2 - \sum_{i=1}^{n} 2^{i-1}p^i.$$

From writing relation (6) for $n+1$, we have:

$$O(F_{\{e\}_{n+1}}) = 2 + \sum_{i=1}^{n} O(F_{P_1=\{e\}_i}) + \sum_{i=1}^{n+1} 2^{i-1}p^i$$
$$= 2 + \sum_{i=1}^{n-1} O(F_{P_1=\{e\}_i}) + O(F_{P_1=\{e\}_n}) + \sum_{i=1}^{n+1} 2^{i-1}p^i$$

and by simplfy:

$$O(F_{\{e\}_{n+1}}) = 2 + (O(F_{\{e\}_n}) - 2 - \sum_{i=1}^{n} 2^{i-1}p^i) + O(F_{\{e\}_n}) + \sum_{i=1}^{n+1} 2^{i-1}p^i$$
$$= 2O(F_{\{e\}_n}) + \sum_{i=1}^{n+1} 2^{i-1}p^i - \sum_{i=1}^{n} 2^{i-1}p^i = 2O(F_{\{e\}_n}) + 2^n p^{n+1}.$$

Therefore:

$$\begin{aligned} O(F_{\{e\}_{n+1}}) =& 2O(F_{\{e\}_n}) + 2^n p^{n+1} = 2(2O(F_{\{e\}_{n-1}}) + 2^{n-1}p^n) + 2^n p^{n+1} \\ =& 2^2 O(F_{\{e\}_{n-1}}) + 2^n p^n) + 2^n p^{n+1} \\ =& 2^i O(F_{\{e\}_{n-(i-1)}}) + 2^n p^{n-(i-2)} + ... + 2^n p^{n-1} + 2^n p^n + 2^n p^{n+1} \\ =& 2^{n+1} O(F_{\{e\}_0}) + \sum_{j=0}^{n-1} 2^n p^{n-j+1} = 2^{n+1} + \sum_{j=0}^{n-1} 2^n p^{n-j+1}, \end{aligned}$$

and now, from theorem 4 we have:

$$O(F_{D_{2p^{n+1}}}) = 2O(F_{\{e\}_{n+1}}) = 2^{n+2} + \sum_{j=0}^{n} 2^{n+1} p^{n-j+1}$$

and similarly for n:

$$O(F_{D_{2p^n}}) = 2^{n+1} + \sum_{j=0}^{n-1} 2^n p^{n-j}.$$

□

It is clear, that the dihedral groups D_{2^n} is a specially case of G_n. See bellow corollary.

Corollary 9. *Let G be a dihedral group of form $G = D_{2^n}$, then the number of fuzzy subgroups of G is:*

$$O(F_{D_{2^n}}) = 2^{2n-1}.$$

Proof. by substitution $p = 2$ in above relation we have:

$$O(F_{D_{2.2^n}}) = 2^{n+1} + \sum_{j=0}^{n-1} 2^n 2^{n-j}$$

$$= 2^{n+1} + \sum_{j=0}^{n-1} 2^{2n-j} = 2^{n+1} + 2^{2n} + 2^{2n-1} + ... + 2^{2n-(n-1)}.$$

Therefore:

$$O(F_{D_{2^{n+1}}}) = 2^{2n+1}$$

easily can find, for $G = D_{2^n}$:

$$O(F_{D_{2^n}}) = 2^{2n-1}.$$

□

4. Number of Fuzzy Subgroups of $D_{2p_1 \times p_2 \times \cdots \times p_n}$

Theorem 10. *If $n \geq 1$, then the number of fuzzy subgroups of dihedral group $D_{2p_1 \times p_2 \times \cdots \times p_n}$ where $p_1, p_2, \cdots, p_n$ are distinct prime numbers is:*

$$O(F_{D_{2p_1 \times p_2 \times \cdots \times p_n}}) = 2.A_{p_1 \times p_2 \times \cdots \times p_n},$$

$$A_{p_1 \times p_2 \times \cdots \times p_n} = 2 + \sum_{i=0}^{n-1} \Big\{ \sum_{1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_i \leq n} A_{p_{\alpha_1} \times p_{\alpha_2} \times \cdots \times p_{\alpha_i}} \Big\} + \sum_{i=0}^{n} \Big\{ \sum_{1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_i \leq n} p_{\alpha_1} \times p_{\alpha_2} \times \cdots \times p_{\alpha_i} \Big\} t_i, \tag{10}$$

when, t_i is defined as:

$$t_i = 1 + \sum_{k=1}^{i-1} \binom{i}{k} t_k \tag{11}$$

Proof. We show this relation satisfy for this type of dihedral groups by draw their chain and find their number step by step.

First, consider $D_{2p_1} = D_{2p}$. As shown in Figure 1, using relation (3) and theorem 4 we have:

$$O(F_{\{e\}_1}) = 2 + p$$

and we define:

$$A_p = O(F_{\{e\}_1}) = 2 + p \tag{12}$$

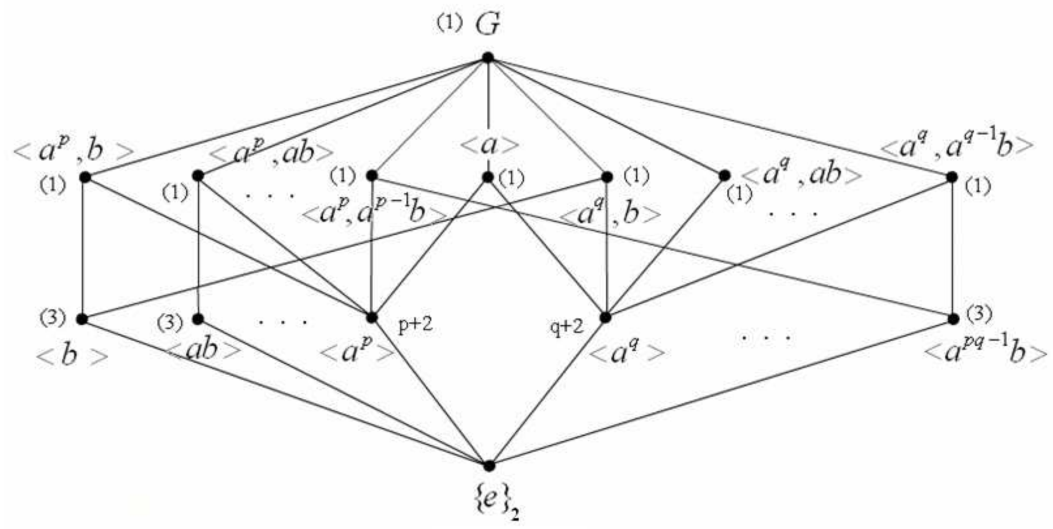

Figure 3: Lattice subgroup of $G = D_{2p\times q}$

Similarly, for $D_{2p_1\times p_2} = D_{2p\times q}$ as shown in Figure 3, according to relation (3) we have $O(F_{p_1=G}) = 1$ and using Theorem 4 for first level of lattice we have:

$$O(F_{P_1=<a^p,b>}) = O(F_{P_1=<a^p,ab>}) = \cdots = O(F_{P_1=<a^p,a^{p-1}b>}) = 1$$

$$O(F_{P_1=<a^q,b>}) = O(F_{P_1=<a^q,ab>}) = \cdots = O(F_{P_1=<a^q,a^{q-1}b>}) = 1,$$

$$O(F_{P_1=<a>}) = 1,$$

and, for second level:

$$O(F_{P_1=<b>}) = O(F_{P_1=<ab>}) = \cdots = O(F_{P_1=<a^{pq-1}b>}) = 3$$

$$O(F_{P_1=<a^p>}) = p + 2,\ O(F_{P_1=<a^q>}) = q + 2$$

Notice, in second level, each $< a^i, b >$ is a subgroup of three subgroups $< a^p, a^i b >$, $< a^q, a^i b >$ and G for $0 \le i \le pq - 1$ on upper levels.

So, from theorem 4:

$$O(F_{<a^i,b>}) = 3; \qquad 0 \le i \le pq - 1$$

and in this level, $< a^p >$ is a subgroup of subgroups $< a^p, a^i b >$ for $0 \le i \le p-1$, $< a >$ and G on upper levels. So, from theorem 4:

$$O(F_{<a^p>}) = p + 1 + 1 = p + 2$$

and similarly:

$$O(F_{<a^q>}) = q + 2$$

Therefore, we have:

$$O(F_{\{e\}_2}) = 1 + 1 + p(1) + q(1) + (p + 2) + (q + 2) + pq(3) = 2p + 2q + 3pq + 6$$

And we define:

$$A_{p\times q} = 2p + 2q + 3pq + 6 \tag{13}$$

In each level we denote $O(F_{H_i})$ of dihedral subgroups in form

$$< a^{p_1p_2\cdots p_k}, a^j b > \qquad ; \qquad k = 1, 2, \cdots, n \quad and \quad 0 \le j \le p_1p_2\cdots p_k - 1$$

using by t_i. This is same number for all dihedral subgroups in level i.

From considering the lattice can easily find:

$$t_i = 1 + \sum_{j=1}^{i-1} \binom{i}{j} t_j \qquad ; \qquad t_1 = 1$$

Then, we can write $A_{p\times q}$ as follows:

$$A_{p\times q} = 2 + p + q + (p + 2) + (q + 2) + pqt_2$$

or:

$$A_{p\times q} = 2 + p + q + A_p + A_q + pqt_2 \tag{14}$$

Similarly, the subgroup lattice of $G_3 = D_{2p_1\times p_2\times p_3} = D_{2p\times q\times r}$ is shown in Figure 4 and according to relation (3) we have $O(F_{P_1=G}) = 1$ and using Theorem 4 we have:

$$O(F_{P_1=<a^p,b>}) = O(F_{P_1=<a^p,ab>}) = \cdots = O(F_{P_1=<a^p,a^{p-1}b>}) = 3,$$
$$O(F_{P_1=<a^q,b>}) = O(F_{P_1=<a^q,ab>}) = \cdots = O(F_{P_1=<a^q,a^{q-1}b>}) = 3,$$
$$O(F_{P_1=<a^r,b>}) = O(F_{P_1=<a^r,ab>}) = \cdots = O(F_{P_1=<a^r,a^{r-1}b>}) = 3,$$
$$O(F_{P_1=<a>}) = 1,$$

and for second level:

$$O(F_{P_1=<a^{pq},b>}) = O(F_{P_1=<a^{pq},ab>}) = \cdots = O(F_{P_1=<a^{pq},a^{pq-1}b>}) = 1,$$
$$O(F_{P_1=<a^{pr},b>}) = O(F_{P_1=<a^{pr},ab>}) = \cdots = O(F_{P_1=<a^{pr},a^{pr-1}b>}) = 1,$$

$$O(F_{P_1=<a^{qr},b>}) = O(F_{P_1=<a^{qr},ab>}) = \cdots = O(F_{P_1=<a^{qr},a^{qr-1}b>}) = 1,$$
$$O(F_{P_1=<a^p>}) = p+2, O(F_{P_1=<a^q>}) = q+2,\ O(F_{P_1=<a^r>}) = r+2.$$

Also, for third level:

$$O(F_{P_1=<b>}) = O(F_{P_1=<ab>}) = O(F_{P_1=<a^2b>}) = \cdots = O(F_{P_1=<a^{pqr-1}b>}) = 13,$$
$$O(F_{P_1=<a^{pq}>}) = 2p+2q+3pq+6 = A_{p\times q}, O(F_{P_1=<a^{pr}>}) = 2p+2r+3pr+6 = A_{p\times r}$$
$$O(F_{P_1=<a^{qr}>}) = 2q+2r+3qr+6 = A_{q\times r}$$

Therefore:

$$A_{p\times q\times r} = O(F_{\{e\}_3}) = 1+1+p+q+r+(p+2)+(q+2)+(r+2)+(pq+pr+qr)\times 3$$
$$+ (2p+2q+3pq+6) + (2p+2r+3pr+6) + (2q+2r+3qr+6) + pqr \times 13,$$

or

$$A_{p\times q\times r} = 2 + (p+q+r)t_1 + (A_p + A_q + A_r) + (pq+pr+qr)t_2 + (A_{p\times q} + A_{p\times r} + A_{q\times r}) + (pqr)t_3. \quad (15)$$

We can similarly continue this approach and draw lattices for others G. As can be seen in each level i we have two types of subgroups. One of them is cyclic subgroups that their number is equal to:$A_{p_1\times p_2\times\cdots\times p_i}$ and the others are dihedral subgroups, that their number for all of them in each level i is:

$$\sum_{each\ \alpha_j\ \text{is number between 1 and n}} (p_{\alpha_1} \times p_{\alpha_2} \times \cdots \times p_{\alpha_i})t_i; \qquad 1 \le i \le n.$$

So, we can write:

$$A_{p_1\times p_2\times\cdots\times p_n} = O(F_G) + O(F_{<a>}) + \sum \Big\{\text{all of the number of cyclic subgroups of G}\Big\} + \sum_{i=1}^{n} \Big\{\text{numbers of dihedral subgroups in level i}\Big\}.$$

Therefore:

$$A_{p_1\times p_2\times\cdots\times p_n} = 2 + \sum_{i=0}^{n-1} \Big\{ \sum_{1\le\alpha_1<\alpha_2<\cdots<\alpha_i\le n} A_{p_{\alpha_1}\times p_{\alpha_2}\times\cdots\times p_{\alpha_i}} \Big\}$$

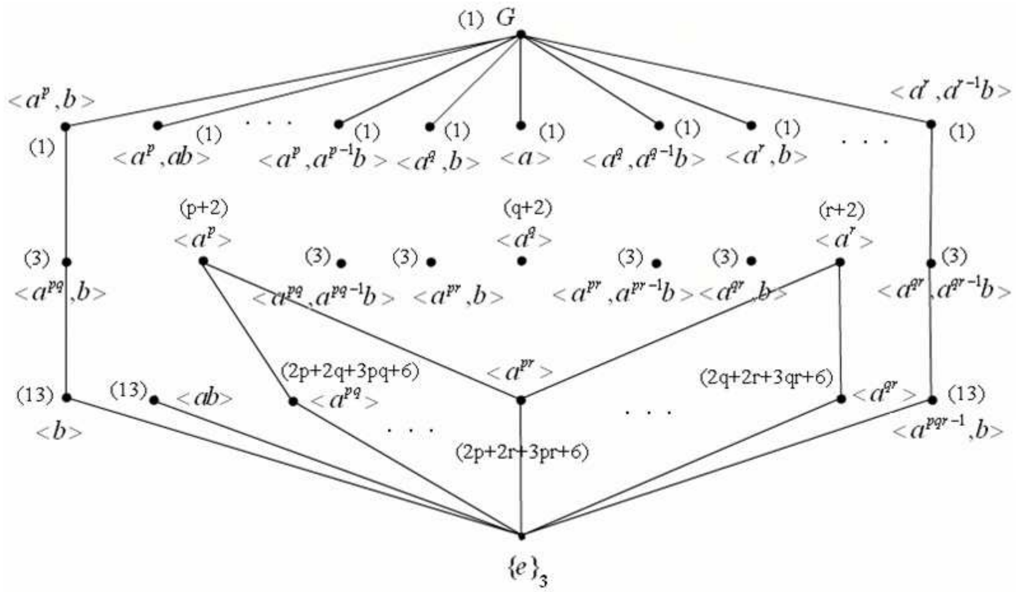

Figure 4: Lattice subgroup of $G = D_{2p\times q\times r}$

$$+\sum_{i=0}^{n}\Big\{\sum_{1\leq\alpha_1<\alpha_2<\cdots<\alpha_i\leq n} p_{\alpha_1}\times p_{\alpha_2}\times\cdots\times p_{\alpha_i}\Big\}t_i$$

Now, from theorem 5 we have:

$$O(F_{G_n}) = 2.O(F_{P_1=\{e\}_n}) = 2.A_{p_1\times p_2\times\cdots\times p_n}$$

□

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