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# FUZZY SOFT PRIME IDEALS OVER RIGHT TERNARY NEAR-RINGS

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**Abstract:** Right ternary near-rings are generalization of their binary counterpart and fuzzy soft sets are generalization of soft sets which are parametrized family of subsets of a universal set. The authors in their earlier paper have introduced fuzzy soft right ternary near-rings, fuzzy soft ideals and studied their basic algebraic properties. In this paper fuzzy soft prime ideals, completely prime ideals and completely semi-prime ideals over a right ternary near-ring are defined and their basic algebraic properties are studied. The characterization theorems for fuzzy soft prime ideals, completely prime ideals and completely semi-prime ideals are given.

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## 1. Introduction

The concept of fuzzy set was introduced by Zadeh [18] in 1965. Since then many researchers are exploring the generalisation of the notion of fuzzy sets. In 1999, Molodtsov [11] introduced the soft set to deal with the uncertainties

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present in most of our real life situations. The parameterization tools of soft set theory enhance the flexibility of its application to different problems. In 2001, Maji et al. [8] expanded soft set theory to fuzzy soft set theory. In recent times, researches have contributed a lot towards fuzzification of soft set theory.

Fuzzy soft sets combine the strengths of both soft sets and fuzzy sets. They have applications in Medical diagnosis [3], decision making ([9]) knowledge representation and retrieval [10] etc. Aygungolu and Aygun [2] generalized the notion of soft groups and introduced fuzzy soft groups in 2009. Thereafter many researchers are applying fuzzy soft tools to other algebraic structures.

The first step towards near-ring was an axiomatic research done by Dickson in 1905. In 1936, it was Zassenhaus who used the name near-ring. Near-rings appear to have an application in characterizing endomorphisms of a group. In [12] Pilz mentions that near-rings are algebraic structures which arise in a natural way in the study of mappings from a group into itself where addition is defined point wise and multiplication is defined as composition of mappings. Many parts of the well established theory of rings are transferred to near-rings and new specific features of near-rings have been discovered.

The notion of fuzzy subnear-ring, fuzzy left (resp. right) ideals and prime ideals in near-ring was introduced by Abou-Zaid ([1]). Fuzzy prime ideals in near-rings are further discussed by Jun et al ([4]), Srinivas et al [15]. Ternary algebraic structures are generalization of their binary counterpart and they appear more or less naturally in various domains of theretical and mathematical physics. The notion of ternary algebraic system was introduced by Lehmer [7] in 1932. Ternary semigroups [13] and [14], ternary semirings [5] are some of the algebraic structures which involve ternary product. To deal with the concept of near-rings using ternary product Warud Nakkhasen and Bundit Pibaljommee [17] have applied the concept of ternary semiring to define left ternary near-rings, ternary subnear-rings and their ideals and investigated some properties of $L$-fuzzy ternary near subrings in 2012.

The authors [16] have introduced fuzzy soft right ternary near-rings, fuzzy soft ideals and studied their basic algebraic properties. In this paper fuzzy soft prime ideals, completely prime ideal and completely semi-prime ideal are defined and their basic structural properties are studied. The characterization theorems for these fuzzy soft ideals are also established.

## 2. Preliminaries

In this section we give the basic definitions that are necessary for the following sections of this paper.

**Definition 1.** [14] Let $N$ be a non-empty set and $[\ \ ]$ be an operation defined from $N \times N \times N$ to $N$ called a ternary operation. Then $(N, [\ \ ])$ is a *ternary semigroup* if for every $x, y, z, u, v \in N$, $[[xyz]uv] = [x[yzu]v] = [xy[zuv]]$.

**Definition 2.** [14] Let $A, B, C$ be non-empty subsets of a ternary semigroup $N$. Then $[ABC] = \{[abc] \in N | a \in A, b \in B, c \in C\}$.

**Definition 3.** [16] Let $N$ be a non-empty set together with a binary operation $+$ and a ternary operation $[\ \ ] : N \times N \times N \to N$. Then $(N, +, [\ \ ])$ is a *right ternary near-ring* (a right ternary near ring is written as RTNR) if
(RTNR-1) $(N, +)$ is a group (not necessarily abelian).
(RTNR-2) $(N, [\ \ ])$ is a ternary semigroup.
(RTNR-3) $[(a + b)cd] = [acd] + [bcd]$, for every $a, b, c, d$ in $N$.

Similarly left ternary near-rings and lateral ternary near rings are defined.

**Example 4.** [16] Let $\Gamma$ be a group under $+$. Let $M(\Gamma) = \{\theta | \Gamma \to \Gamma\}$. Define $+$ and $[\ \ ]$ on $M(\Gamma)$ by $(\theta + \eta)(x) = \theta(x) + \eta(x)$, $[\theta\xi\eta](x) = \theta(\xi(\eta(x)))$, for every $x \in \Gamma$. Then $(M(\Gamma), +, [\ \ ])$, is a right ternary near-ring.

**Definition 5.** [17] A non-empty subset $S$ of a ternary near-ring is called a *ternary subnear-ring* if (i) $x - y \in S$ if $x, y \in S$ (ii) $[SSS] \subseteq S$.

**Definition 6.** [17] Let $N$ and $N'$ be any two right ternary near rings. Then a mapping $h : N \to N'$ is called a *right ternary near ring homomorphism* if (i) $h(x+y) = h(x)+h(y)$, (ii) $h([xyz]) = [h(x)h(y)h(z)]$, for every $x, y, z \in N$.

**Definition 7.** [16] Let $N$ be a right ternary near-ring. Let $J$ be a normal subgroup $(N, +)$. Then $J$ is called (i) *a right ideal* of $N$ if $[JNN] \subseteq J$ (ii) *a left ideal* if $[tt'(t'' + i)] - [tt't''] \in J$ (iii) *a lateral ideal* if $[t(t' + i)t''] - [tt't''] \in J$ where $t, t', t'' \in N$, $i \in J$.
$J$ is an ideal of $N$ if it is a right, lateral and left ideal of $N$.

**Definition 8.** [18] If $X$ is a universal set then a *fuzzy subset* of $X$ is a map $\mu : X \to [0, 1]$ which is denoted by $\mu = \{(x, \mu(x)) | x \in X\}$.

**Definition 9.** [11] Let $U$ be a universal set. Let $A$ be a subset of a set of parameters $E$. Then $(F, A)$ is called a *soft set* over $U$ where $F : A \to \mathcal{P}(U)$ and $\mathcal{P}(U)$ is the set of subsets of $U$.

**Definition 10.** [8] Let $U$ be a universal set and let $A$ be a subset of a set of parameters $E$. Let $I^U$ (where $I = [0, 1]$) be the set of fuzzy subsets of $U$. Then $(f, A)$ is called a *fuzzy soft set* over $U$ where $f : A \to I^U$ and $f(a) = f_a : U \to I$ is a fuzzy subset of $U$. i.e., a fuzzy soft set is a parameterized family of fuzzy subsets of the set $U$.

Every fuzzy subset $f_e$, for every $e \in E$, can be considered as the fuzzy subset of $e$-approximate elements of the fuzzy soft set. According to this manner, a fuzzy soft set $(f, E)$ is given as a collection of approximations : $(f, E) = \{f_e : e \in E\}$.

**Definition 11.** [6] A fuzzy soft set $(f, A)$ over $U$ is called *null fuzzy soft* set if $f_a(x) = 0$ for every $x \in U$ and $a \in A$ and it is denoted by $(\phi, A)$ and if $f_a(x) = t$, for every $x \in U$, $t \in (0, 1]$ then $(f, A)$ is called a *constant fuzzy soft set*.

$(1, E)$ is called *absolute fuzzy soft set* where $1_e(x) = 1$ for every $x \in U$ and $e \in E$, parameter set of $U$.

**Definition 12.** [8] Let $U$ be a universal set and let $A$ and $B$ be any two non-empty subsets of a set of parameters $E$. Let $(f, A)$, $(g, B)$ be any two fuzzy soft sets over $U$. Then $(f, A)$ is a *fuzzy soft subset* of $(g, B)$ i.e., $(f, A)\tilde{\subseteq}(g, B)$ if $A \subseteq B$ and $f_a(x) \leq g_a(x)$ for every $a \in A$.

**Definition 13.** [6] Let $X$ and $Y$ be any two non-empty sets and $E_1$ and $E_2$ be their parameter sets. Let $A \subseteq E_1$, $B \subseteq E_2$. Let $(f, A)$ and $(g, B)$ be any two non-empty fuzzy soft sets over $X$ and $Y$ respectively. Let $\phi : X \to Y$ and $\psi : A \to B$. Then $(\phi, \psi) : (f, A) \to (g, B)$ is called a *fuzzy soft function* and the *inverse image* of fuzzy soft set $(g, B)$ is defined by $(\phi, \psi)^{-1}((g, B)) = (\phi^{-1}(g), \psi^{-1}(B))$, where $(\phi^{-1}(g))_a(x) = g_{\psi(a)}(\phi(x))$, for every $a \in \psi^{-1}(B)$ and $x \in X$.

**Definition 14.** [16] A fuzzy soft set $(f, A)$ over $N$ is a fuzzy soft right ternary near-ring if

(i) $f_a(x + y) \geq \min\{f_a(x), f_a(y)\}$,

(ii) $f_a(-x) \geq f_a(x)$ $\forall\, a \in A$, $x, y \in X$ and

(iii) $f_a([xyz]) \geq \min\{f_a x), f_a(y), f_a(z)\}$ for every $a \in A$ and $x, y, z \in N$.

**Definition 15.** [16] A fuzzy soft set $(f, A)$ over a right ternary near-ring $N$ is fuzzy soft ideal over $N$ if

(i) $f_a(x - y) \geq \min\{f_a(x), f_a(y)\}$

(ii) $f_a(y+x-y) \geq f_a(x)$

(iii) $f_a([xyz]) \geq f_a(x)$

(iv) $f_a([xy(z+i)] - [xyz]) \geq f_a(i)$

(v) $f_a([x(y+i)z] - [xyz]) \geq f_a(i)$, for every $a \in A$ and $x, y, z, i \in N$.

A fuzzy soft set $(f, A)$ is called a *fuzzy soft right ideal* if it satisfies (i), (ii), (iii). $(f, A)$ is called a *fuzzy soft left ideal* if it satisfies (i), (ii), (iv). $(f, A)$ is called a *fuzzy soft lateral ideal* if it satisfies (i), (ii), (v). We also remark that $f_a(0) > 0$ if $(f, A)$ is a fuzzy soft ideal or RTNR.

**Example 16.** (i) Let $N = \{0, x, y, z\}$. Define $+$ as in Table 1 and $[\ \ ]$ on $N$ by $[xyz] = (x.y).z$ for every $x, y, z \in N$ where . is defined as in Table 2. Then $(N, +, [\ \ ])$ is a right ternary near-ring and $\{0, x\}$ and $\{0, y\}$ are ideals of $N$.

| $+$ | $0$ | $x$ | $y$ | $z$ |
|---|---|---|---|---|
| $0$ | $0$ | $x$ | $y$ | $z$ |
| $x$ | $x$ | $0$ | $z$ | $y$ |
| $y$ | $y$ | $z$ | $0$ | $x$ |
| $z$ | $z$ | $y$ | $x$ | $0$ |

Table 1

| $.$ | $0$ | $x$ | $y$ | $z$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $x$ | $0$ | $0$ | $0$ | $0$ |
| $y$ | $0$ | $0$ | $0$ | $0$ |
| $z$ | $0$ | $x$ | $y$ | $x$ |

Table 2

(ii) Let $N = \{0, x, y, z\}$. Define $+$ as in Table 3 and $[\ \ ]$ on $N$ by $[xyz] = (x.y).z$ for every $x, y, z$ in $N$ where . is defined as in Table 4. Then $(N, +, [\ \ ])$ is a right ternary near-ring and $\{0, x\}$ is an ideal of $N$ but $\{0, y\}$ is not an ideal of $N$ as $[yxx] = x \notin \{0, y\}$.

| $+$ | $0$ | $x$ | $y$ | $z$ |
|---|---|---|---|---|
| $0$ | $0$ | $x$ | $y$ | $z$ |
| $x$ | $x$ | $0$ | $z$ | $y$ |
| $y$ | $y$ | $z$ | $0$ | $x$ |
| $z$ | $z$ | $y$ | $x$ | $0$ |

Table 3

| $.$ | $0$ | $x$ | $y$ | $z$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $x$ | $x$ | $x$ | $x$ | $x$ |
| $y$ | $0$ | $x$ | $y$ | $z$ |
| $z$ | $x$ | $0$ | $z$ | $y$ |

Table 4

**Lemma 17.** *[16] If $(f, A)$ is a fuzzy soft right ternary near-ring over $N$ then $f_a(0) \geq f_a(x)$, for every $a \in A$ and $x \in N$.*

**Theorem 18.** *[16] If $(f, A)$ is a fuzzy soft set over $N$ and if for $a \in A$, $S = N_{f_a}^{f_a}(0) = \{x \in N | f_a(x) = f_a(0)\}$ then $S$ is an ideal of $N$ if $(f, A)$ is a fuzzy soft ideal over $N$.*

*We also note that the converse of the above theorem holds.*

**Corollary 19.** *[16] Let $(f, A)$ be a fuzzy soft set over $N$ and $T = N_{f_a}^1 = \{x \in N | f_a(x) = 1\}$. Then $T$ is an ideal of $N$ if $(f, A)$ is a fuzzy soft ideal over $N$.*

**Theorem 20.** *[16] Let $(f, A)$ be a fuzzy soft set over $N$ and $f_a(0) = 1$. Let $N_{f_a}^1 = \{x \in N | f_a(x) = 1\}$ be an ideal of $N$. Then $(f, A)$ is a fuzzy soft ideal over $N$.*

**Proposition 21.** *[16] The inverse homomorphic image of $(g, B)$ is a fuzzy soft ideal over $M$ if $(g, B)$ is a fuzzy soft ideal over $N$.*

**Theorem 22.** *[16] (i) A non-empty subset $L$ of $N$ is an ideal of $N$ iff $(f, A)$ is a fuzzy soft ideal over $N$, where $f : A \to I^N$ is defined by*

$$f_a(x) = \begin{cases} r & \text{if } x \in L \\ t & \text{if } x \in N - L \end{cases}, \text{ where } r > t, \text{ for every } a \in A.$$

*(ii) In particular, a non-empty subset $L$ of $N$ is a ideal of $N$ iff the characteristic function $(\psi_L, E)$ fuzzy soft ideal over $N$, where $\psi_L : E \to I^N$ is defined by*

$$(\psi_L)_e(x) = \begin{cases} 1, & \text{if } x \in L \\ 0 & \text{if } x \notin L \end{cases}, \text{ for every } e \in E.$$

**Proposition 23.** *[16] Let $N$ be a right ternary near-ring. Let $A$ be a subset of a parameter set $E$. Let $(f, A)$ be a fuzzy soft set over $N$. Then $(f, A)$ is a fuzzy soft ideal over $N$ iff for each $f_a$, each non-empty level subset $(f_a)_t$, $t \in Im\ f_a$ is an ideal of $N$.*

## 3. Fuzzy Soft Prime Ideal

In this section the definition of prime ideal of an RTNR, zero-symmetric RTNR, fuzzy soft prime ideal are given. Basic properties of these fuzzy soft ideals are studied. The characterization theorem for fuzzy soft prime ideals is established.

**Definition 24.** Let $N$ be an RTNR. Then an ideal $P$ of $N$ is a *prime ideal* if there are ideals $X, Y, Z$ in $N$ such that $[XYZ] \subseteq P \Rightarrow X \subseteq P$ or $Y \subseteq P$ or $Z \subseteq P$.

**Example 25.** Let $N$ be as in Example 16(i) and consider the ideals $X = \{0\}$, $Y = \{0, x\}$, $Z = \{0, y\}$ in $N$.
Then $[XYZ] \subseteq \{0\}$ and $X \subseteq \{0\}$ and hence $\{0\}$ is a prime ideal of $N$.

In the following right zero-symmetric part, lateral zero-symmetric part and left zero-symmetric part of $N$ denoted by $(N_0)_R$, $(N_0)_M$ and $(N_0)_L$ respectively are defined.

**Definition 26.** Let $N$ be an RTNR. Then

$$\begin{aligned} (N_0)_R &= \{n \in N | [0nn'] = 0, \ \forall\, n' \in N\}, \\ (N_0)_M &= \{n \in N | [n0n'] = 0, \ \forall\, n' \in N\}, \\ (N_0)_L &= \{n \in N | [nn'0] = 0, \ \forall\, n' \in N\} \end{aligned}$$

**Definition 27.** If $N = (N_0)_M$ then $N$ is called a *lateral zero-symmetric RTNR*, if $N = (N_0)_L$ then $N$ is called a *left zero-symmetric RTNR* and if $N = (N_0)_R$ then $N$ is called a *right zero-symmetric RTNR*.

It can be noted that an RTNR is always a right zero-symmetric RTNR.

$N_0 = \{n \in N | [n00] = 0\}$ is the *zero-symmetric part* of $N$ and if $N = N_0$ then $N$ is called a *zero-symmetric RTNR*.

**Example 28.** (i) Let $N = \mathbb{Q}$. Then under the usual addition of rational numbers and ternary multiplication defined as below, $N$ is a zero-symmetric RTNR. For $x, y, z \in N$ define

$$[xyz] = \begin{cases} \frac{x}{y} z & \text{if } x, y, z \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

(ii) Let $N$ be as in Example 16(ii). Then $N$ is not zero-symmetric, as $[x00] = x \neq 0$.

**Definition 29.** Let $N$ be an RTNR and $L, B, C, D$ be subsets of a parameter set $E$ of $N$. If $(g, B)$, $(h, C)$, $(r, D)$ are fuzzy soft sets over $N$, then $(g, B) \circ (h, C) \circ (r, D) = (k, L)$ where $L = B \cap C \cap D$ and for every $\ell \in L$ and

$$k_\ell(u) = (g_b \circ h_c \circ r_d)(u) = \begin{cases} \bigvee_{u=[xyz]} (g_b(x) \wedge h_c(y) \wedge r_d(z)) & \text{if } u = [xyz] \\ 0 & \text{otherwise.} \end{cases}$$

Now fuzzy soft prime ideals over $N$ are defined. Throughout this section $N$ is assumed to be zero-symmetric and $E$ denotes a parameter set of $N$.

**Definition 30.** Let $N$ be an RTNR and $B, C, D$ be subsets of $A \subseteq E$ where $E$ is a parameter set of $N$. Then a fuzzy soft ideal $(f, A)$ is a *fuzzy soft prime ideal* over $N$ if it is non-constant and if there exists non-absolute fuzzy soft ideals $(g, B)$, $(h, C)$, $(r, D)$ over $N$, such that $(g, B) \circ (h, C) \circ (r, D) \subseteq (f, A) \Rightarrow$ either $(g, B) \subseteq (f, A)$ or $(h, C) \subseteq (f, A)$ or $(r, D) \subseteq (f, A)$.

**Example 31.** Let $N$ be as in Example 16(i) and $E = N$. Let $A = E$, $B = \{0\}$, $C = \{0, x\}$ and $D = \{0, y\}$. Let $(f, A)$, $(g, B)$, $(h, C)$ and $(r, D)$ be defined by

$$f_a(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0.9 & \text{if } x \neq 0 \end{cases}, \quad \text{for every } a \in A$$

$$g_b(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0.1 & \text{if } x \neq 0 \end{cases} \quad h_c(t) = \begin{cases} 1 & \text{if } t = 0, x \\ 0.7 & \text{if } t = y, z \end{cases} \quad r_d(t) = \begin{cases} 1 & \text{if } t = 0, y \\ 0.8 & \text{if } t = x, z \end{cases},$$

for every $b \in B$, $c \in C$, $d \in D$, then $(g, B)$, $(h, C)$, $(r, D)$ are all fuzzy soft ideals over $N$. Obviously if $(g, B) \circ (h, C) \circ (r, D) = (k, L)$ where $L = B \cap C \cap D = \{0\}$, then for $l \in L$

$$k_l(u) = \begin{cases} 1 & \text{if } u = [xyz] \\ 0 & \text{otherwise} \end{cases}$$

Hence $(g, B) \circ (h, C) \circ (r, D) \subseteq (f, A)$ and $(g, B) \subseteq (f, A)$. Thus $(f, A)$ is a fuzzy soft prime ideal over $N$.

**Theorem 32.** *If $(f, A)$ is a fuzzy soft set over $N$ and if for $a \in A$, $S = N_{f_a}^{f_a(0)} = \{x \in N | f_a(x) = f_a(0)\}$ then $S$ is a prime ideal of $N$ if $(f, A)$ is a fuzzy soft prime ideal over $N$.*

*Proof.* Let $A$ be a subset of a parameter set $E$ of $N$. Since $(f, A)$ is a fuzzy soft ideal over $N, S$ is an ideal of $N$. We now prove that $S$ is a prime ideal if $(f, A)$ is fuzzy soft prime ideal over $N$. Let $X, Y, Z$ be ideals of $N$ such that $[XYZ] \subseteq S \ldots\ldots\ldots$ (**P**). Let $(g, B)$, $(h, C)$, $(r, D)$ be fuzzy soft sets over $N$ where $B, C, D$ are subsets of $A$ defined as below. For $x, y, z \in N$ define

$$g_b(x) = \begin{cases} f_a(0) & \text{if } x \in X \\ 0 & \text{otherwise} \end{cases} \qquad h_c(y) = \begin{cases} f_a(0) & \text{if } y \in Y \\ 0 & \text{otherwise} \end{cases}$$

$$r_d(z) = \begin{cases} f_a(0) & \text{if } z \in Z \\ 0 & \text{otherwise} \end{cases}, \quad \text{for every } b \in B, c \in C, d \in D.$$

Then by Theorem 22(i), $(g, B)$, $(h, C)$, $(r, D)$ are fuzzy soft ideals over $N$. Now to prove that $(g, B) \circ (h, C) \circ (r, D) \subseteq (f, A) \ldots\ldots$ (**Q**) (i.e) to prove that

$(g_b \circ h_c \circ r_d)(u) \leq f_a(u)$, for every $u \in N$. Obviously $B \cap C \cap D \subseteq A$.
If $u \in N$ then there are two cases (i) $u \in S$ (ii) $u \notin S$.
**Case (i):** Let $u \in S$. Then $u = 0$ or $u \neq 0$. If $u = 0$, then as $N$ is zero-symmetric $(g_b \circ h_c \circ r_d)(u) = f_a(0)$. Hence $(g_b \circ h_c \circ r_d)(u) = f_a(u)$, for every $u \in N$. If $u \neq 0$ then $(g_b \circ h_c \circ r_d)(u) = f_a(0)$ or 0, depending on $u$ is expressed as $[xyz]$ or not. In either case as $u \in S$, $f_a(u) = f_a(0) > 0$. Hence $(g_b \circ h_c \circ r_d)(u) \leq f_a(u)$.
**Case (ii):** Suppose $u \notin S$. Then $f_a(u) \neq f_a(0)$. Since $f_a(0) > 0$ and $f_a(u) \neq f_a(0)$ we have $f_a(u) = 0$ or $f_a(u) > 0$. Also since $u \notin S$ if $u = [xyz]$ then by (**P**) either $x, y, z$ are not in $X, Y, Z$ respectively or atleast one of $x, y, z$ is not respectively in $X, Y, Z$. Thus $(g_b \circ h_c \circ r_d)(u) = \sup\{g_b(x) \wedge h_c(y) \wedge r_d(z)\} = 0 \leq f_a(u)$.
Hence in both the cases we have $(g_b \circ h_c \circ r_d)(u) \leq f_a(u)$, for every $u \in N$.
Hence $(g, B) \circ (h, C) \circ (r, D) \subseteq (f, A)$, proving (**Q**).
Since $(f, A)$ is a fuzzy soft prime ideal over $N$ either $(g, B) \subseteq (f, A)$ or $(h, C) \subseteq (f, A)$ or $(r, D) \subseteq (f, A)$.

Suppose $(g, B) \subseteq (f, A)$ and $X$ is not a subset of $S$. Then there exists $x \in X$ and $x \notin S \Rightarrow f_a(x) \neq f_a(0)$. But $f_a(0) \geq f_a(x)$, by Lemma 17(ii). Thus $f_a(x) < f_a(0)$. Now $g_b(x) = f_a(0) > f_a(x)$ which contradicts the assumption $(g, B) \subseteq (f, A)$. Hence $X \subseteq S$. Similarly $(h, C) \subseteq (f, A)$ or $(r, D) \subseteq (f, A) \Rightarrow Y \subseteq S$ or $Z \subseteq S$. Hence $S$ is a prime ideal. □

**Corollary 33.** *Let $(f, A)$ be a fuzzy soft set over $N$ and $T = N^1_{f_a} = \{x \in N | f_a(x) = 1\}$. Then $T$ is a prime ideal of $N$ if $(f, A)$ is a fuzzy soft prime ideal over $N$.*

*Proof.* By taking $f_a(0) = 1$ in $N^{f_a(0)}_{f_a}$ in the above theorem the proof follows. □

The following lemma and theorem give the characteristics of a fuzzy soft prime ideal.

**Lemma 34.** *If $(f, A)$ is a fuzzy soft prime ideal over $N$ then $f_a(0) = 1$, for every $a \in A$.*

*Proof.* Suppose $f_a(0) < 1$. Since $(f, A)$ is non-constant,there exists $x \in N$ such that $f_a(x) < f_a(0)$. We define $(g, B)$, $(h, C)$, $(r, D)$ where $B, C, D$ are subsets of $A$ as below. For $x, y, z \in N$

$$g_b(x) = \begin{cases} 1 & \text{if } f_a(x) = f_a(0) \\ 0 & \text{otherwise} \end{cases} \quad h_c(y) = f_a(0) \quad r_d(z) = \begin{cases} 1 & \text{if } f_a(z) = f_a(0) \\ 0 & \text{otherwise} \end{cases}$$

for every $b \in B$, $c \in C$, $d \in D$. Then $(g, B)$, $(h, C)$, $(r, D)$ are all fuzzy soft ideals over $N$. Since $(g_b)(0) = 1 > f_a(0) > f_a(x)$, $(h_c)(0) = f_a(0) > f_a(x)$, $(r_d)(0) = 1 > f_a(0)$, $(g, B)$, $(h, C)$, $(r, D)$ are not subsets of $(f, A) \ldots\ldots$ (**P**). Now to prove that $(g, B) \circ (h, C) \circ (r, D) \subseteq (f, A)$. (i.e) to prove that $(g_b \circ h_c \circ r_d)(u) \leq f_a(u)$, for every $u \in N$. Obviously $B \cap C \cap D \subseteq A$. If $u \in N$ then there are two cases (i) $u = 0$ (ii) $u \neq 0$.
**Case (i):** If $u = 0$ then $f_a(u) = f_a(0)$. Now, $(g_b \circ h_c \circ r_d)(u) = \sup\{f_a(0), 0\}$. Then $(g_b \circ h_c \circ r_d)(u) = f_a(0) = f_a(u)$, for every $u \in N$.
**Case (ii):** Let $u \neq 0$ and $u \neq [xyz]$. Then $(g_b \circ h_c \circ r_d)(u) = 0 \leq f_a(u)$. If $u = [xyz]$, $(g_b \circ h_c \circ r_d)(u) \neq 0$ and if $g_b(x)$ or $h_c(y)$ is zero, then $g_b(x) \wedge h_c(y) \wedge r_d(z) = 0$ and hence $(g_b \circ h_c \circ r_d)(u) = 0 \leq f_a(u)$. Also if $(g_b)(x)$ and $r_d(z)$ are non-zero $g_b(x) \wedge h_c(y) \wedge r_d(z) = f_a(0) = f_a(x) \leq f_a(u))$ by the definitions of $(g, B)$, $(h, C)$ and $(f, A)$ is a fuzzy soft ideal. Hence in both the cases $(g_b \circ h_c \circ r_d)(u) \leq f_a(u)$, for every $u \in N$.

Thus $(g, B) \circ (h, C) \circ (r, D) \subseteq (f, A)$. Since $(f, A)$ is a fuzzy soft prime ideal either $(g, B) \subseteq (f, A)$ or $(h, C) \subseteq (f, A)$ or $(r, D) \subseteq (f, A)$ which contradicts (**P**). Hence $f_a(0) = 1$. □

If the range of $f_a$ is singleton then $(f, A)$ becomes a constant ideal. Thus if $(f, A)$ is a non-constant fuzzy soft ideal then $f_a$ should take more than one value on $N$.

In the following theorem it is established that if $(f, A)$ is a fuzzy soft prime ideal then the cardinality of $Im\ f_a$ is two.

**Theorem 35.** *If $(f, A)$ is a fuzzy soft prime ideal over $N$ then for $a \in A$, $f_a(x)$ can take only two values.*

*Proof.* Let $(f, A)$ be a fuzzy soft prime ideal over $N$. Then $f_a(0) = 1$. To prove that $f_a(x)$ takes only two values. Suppose $f_a(x)$ takes more than two values. Then there exists $x, y \in N$ such that $f_a(x) = r$, $f_a(y) = t$. For the sake of definiteness, let $0 < r < t < 1$. Now, let

$$X = \{x \in N | f_a(x) \geq r\}; \quad Z = \{z \in N | f_a(z) \geq t\}$$

Then $X$ and $Z$ are ideals of $N$ by Proposition 23. Let $B, C, D$ be subsets of $A$ and define $(g, B)$, $(h, C)$ and $(r, D)$ over $N$ as below

$$g_b(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{otherwise} \end{cases} \quad h_c(y) = \frac{r + t}{2} = s(say) \quad r_d(z) = \begin{cases} 1 & \text{if } z \in Z \\ 0 & \text{otherwise} \end{cases}$$

for every $b \in B$, $c \in C$, $d \in D$. Then by Theorem 22(ii), $(g, B)$, $(h, C)$ and $(r, D)$ are fuzzy soft ideals over $N$. Obviously $(h, C)$ is a fuzzy soft ideal over $N$. Since $r > 0$, $(g, B) \neq (1, E)$.

To establish that $(g,B)\circ(h,C)\circ(r,D)\subseteq(f,A)$. i.e., $(g_b\circ h_c\circ r_d)(u)\leq f_a(u)$, $\forall\ u\in N$.
Since $u\in N$ there are two cases (i) $u=0$, (ii) $u\neq 0$.
**Case (i):** Let $u=0$. Then $f_a(0)=f_a(u)=1$. Since $r<1=f_a(0)$, $f_a(0)>r$ and hence $0\in X$ and therefore $g_b(0)=1$. Obviously $h_c(0)=s$. Since $t<1=f_a(0)$, $f_a(0)>t$ and hence $0\in Z$ and therefore $r_d(0)=1$.
Thus if $u=0$, $(g_b\circ h_c\circ r_d)(u)=1$ or $s$. Since $s<1=f_a(0)=f_a(u)$, $(g_b\circ h_c\circ r_d)(u)\leq f_a(u)$.
**Case (ii):** Let $u$ be any non-zero element of $N$ such that $f_a(u)\geq t$. Then $u\in Z\Rightarrow r_d(u)=1$. Obviously $h_c(u)=s$. Since $f_a(u)\geq t>r$, $g_b(u)=1$. Thus, $g_b(x)\wedge h_c(y)\wedge r_d(z)=1\wedge s\wedge 1=s\Rightarrow(g_b\circ h_c\circ r_d)(u)=s<t\leq f_a(u)$.

Now, let $u$ be any non-zero element of $N$ such that $f_a(u)<t$. Then $r_d(u)=0\Rightarrow(g_b\circ h_c\circ r_d)(u)=0\leq f_a(u)$.

Thus, in both the cases $(g_b\circ h_c\circ r_d)(u)\leq f_a(u)$, $\forall\ u\in N$. Hence $(g,B)\circ(h,C)\circ(r,D)\subseteq(f,A)$. Now, since $(f,A)$ is a fuzzy soft prime ideal, either

$$(g,B)\subseteq(f,A)\ \text{ or }\ (h,C)\subseteq(f,A)\ \text{ or }\ (r,d)\subseteq(f,A) \tag{1}$$

Since $f_a(x)=r$, $g_b(x)=1>r\Rightarrow g_b(x)>f_a(x)\Rightarrow(g,B)\nsubseteq(f,A)$. Similarly $(r,D)\nsubseteq(f,A)$, $(h,C)\nsubseteq(f,A)$ which contradicts (1). Hence $f_a(x)$ can not take more than two values in $[0,1]$. □

In the following theorem the converse of Corollary 33 is established if $|Imf_a|=2$.

**Theorem 36.** *If $(f,A)$ is a fuzzy soft set over $N$ such that for $a\in A$, $|Im\ f_a|=2$ and $f_a(0)=1$ and $T=\{x\in N|f_a(x)=1\}$ is a prime ideal of $N$ then $(f,A)$ is a fuzzy soft prime ideal over $N$.*

*Proof.* By Theorem 20, if $T$ is an ideal of $N$ then $(f,A)$ is a fuzzy soft ideal over $N$. To prove that $(f,A)$ is a fuzzy soft prime ideal over $N$. Let $(g,B)$, $(h,C)$, $(r,D)$ be fuzzy soft ideals over $N$ where $B,C,D$ are subsets of $A\subseteq E$ with

$$(g,B)\circ(h,C)\circ(r,D)\subseteq(f,A). \tag{2}$$

Suppose $(g,B)\nsubseteq(f,A)$, $(h,C)\nsubseteq(f,A)$, $(r,D)\nsubseteq(f,A)$. Then

$$g_b(x)>f_a(x),h_c(y)>f_a(y),r_d(z)>f_a(z) \tag{3}$$

for some $x,y,z$ in $N$. But $f_a(u)=1=f_a(0)\ \forall\ u\in N$.

From (3) $x, y, z \notin T$ ($\because$ if $x, y, z \in T$ then $f_a(x) = 1 = f_a(y) = f_a(z)$ which from (3) will imply that $g_b(x) > 1$, $h_c(y) > 1$, $r_d(z) > 1$ which is impossible). Since $x, y, z \notin T$, $[xyz] \notin T \Rightarrow f_a([xyz]) \neq 1$. Now, consider,

$$(g_b \circ h_c \circ r_d)(u) = \bigvee_{u=[xyz]} g_b(x) \wedge h_c(y) \wedge r_d(z)$$

From (2),

$$\begin{aligned} f_a(u) = f_a([xyz]) &\geq \bigvee_{u=[xyz]} g_b(x) \wedge h_c(y) \wedge r_d(z) \\ &\geq g_b(x) \wedge h_c(y) \wedge r_d(z) \\ &> f_a(x) \wedge f_a(y) \wedge f_a(z) \end{aligned} \tag{4}$$

Since $|Im\ f_a| = 2$ and $f_a(x) \neq 1$, $f_a(x) = r$. Similarly $f_a(y) = r = f_a(s)$. Also $f_a(u) \neq 1$ and hence $f_a(u) = r$. Thus from (4) we get $r > r$, a contradiction. Hence $(g, B) \subseteq (f, A)$ or $(h, C) \subseteq (f, A)$ or $(r, D) \subseteq (f, A)$ and hence $(f, A)$ is a fuzzy soft prime ideal over $N$. □

Corollary 33 and Theorem 36 can be combined to obtain the following characterization theorem for a fuzzy soft prime ideal over $N$.

**Theorem 37.** *$(f, A)$ is a fuzzy soft prime ideal iff for $a \in A$, $|Im\ f_a| = 2$ and $f_a(0) = 1$ and $T = \{x \in R | f_a(x) = 1\}$.*

**Theorem 38.** *(i) A non empty subset $J$ of $N$ is a prime ideal iff $(f, A)$ is a fuzzy soft prime ideal over $N$ where*

$$f_a(x) = \begin{cases} 1 & \text{if } x \in J \\ t & \text{if } x \notin J \end{cases}, \quad 0 \leq t < 1$$

*for every $a \in A$.*

*(ii) In particular an ideal $L$ of $N$ is a prime ideal iff $(\psi_L, E)$ is a fuzzy soft prime ideal over $N$ where $\psi_L : E \to I^N$ is defined by*

$$(\psi_L)_e(x) = \begin{cases} 1 & \text{if } x \in L \\ 0 & \text{otherwise} \end{cases}, \quad 0 \leq t < 1$$

*for every $e \in E$.*

*Proof.* Let $J$ be a prime ideal of $N$. To prove that $(f, A)$ is a fuzzy soft prime ideal over $N$. Let $(g, B)$, $(h, C)$, $(r, D)$ be fuzzy soft ideals over $N$ such that

$$(g, B) \circ (h, C) \circ (r, D) \subseteq (f, A). \tag{5}$$

Suppose $(g,B) \nsubseteq (f,A)$, $(h,C) \nsubseteq (f,A)$, $(r,D) \nsubseteq (f,A)$. Then $g_b(x) > f_a(x)$, $h_c(y) > f_a(y)$, $r_d(z) > f_a(z)$ for $x,y,z \in N$. Obviously $f_a(x) \neq 1$, $f_a(y) \neq 1$ and $f_a(z) \neq 1$. Hence by the definition of $(f,A)$, $f_a(x) = t$, $f_a(y) = t$, $f_a(z) = t$

$$\Rightarrow x,y,z \notin J \Rightarrow [xyz] \notin J \tag{6}$$

$$\begin{aligned} f_a(u) = f_a([xyz]) &\geq \bigvee_{u=[xyz]} g_b(x) \wedge h_c(y) \wedge r_d(z) \\ &> g_b(x) \wedge h_c(y) \wedge r_d(z) \\ &> t \end{aligned}$$

$\Rightarrow f_a([xyz]) = 1 \Rightarrow [xyz] \in J$ which is a contradiction to (6) and hence $(g,B) \subseteq (f,A)$ or $(h,C) \subseteq (f,A)$ or $(r,D) \subseteq (f,A)$. Thus $(f,A)$ is a fuzzy soft prime ideal over $N$.

Conversely let $(f,A)$ be a fuzzy soft prime ideal over $N$. We prove that $J$ is a prime ideal of $N$. Suppose there exists ideals $X,Y,Z$ of $N$ such that $[XYZ] \subseteq J$. Let $(g,B)$, $(h,C)$, $(r,D)$ be fuzzy soft sets over $N$ defined by

$$g_b(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{otherwise} \end{cases} \quad h_c(y) = \begin{cases} 1 & \text{if } y \in Y \\ 0 & \text{otherwise} \end{cases} \quad r_d(z) = \begin{cases} 1 & \text{if } z \in Z \\ 0 & \text{otherwise} \end{cases}$$

for every $b \in B$, $c \in C$ and $d \in D$. Then $(g,B)$, $(h,C)$, $(r,D)$ are fuzzy soft ideals over $N$ and it can be established that

$$(g,B) \circ (h,C) \circ (r,D) \subseteq (f,A) \tag{7}$$

$$\Rightarrow (g,B) \subseteq (f,A) \text{ or } (h,C) \subseteq (f,A) \text{ or } (r,D) \subseteq (f,A) \tag{8}$$

Now, suppose $X \nsubseteq J$, $Y \nsubseteq J$, $Z \nsubseteq J$. Then there exists $x \in X$, $y \in Y$, $z \in Z$ such that $x \notin J$, $y \notin J$, $z \notin J$. $\Rightarrow g_b(x) = 1 = h_c(y) = r_d(z)$ $\Rightarrow g_b(x) = 1 > t = f_a(x) \Rightarrow (g,B) \nsubseteq (f,A)$. Similarly $(h,C) \nsubseteq (f,A)$, $(r,D) \nsubseteq (f,A)$ contradicting (8). Hence $J$ is a prime ideal of $N$.
The particular case (ii) follows from (i) by taking $J = L$ and $t = 0$. □

**Theorem 39.** *A fuzzy soft set $(f,A)$ over $N$ is a fuzzy soft prime ideal over $N$ iff for each $f_a$, where $a \in A$, $(f_a)_t$, $t \in Im\ f_a$ is a prime ideal of $N$.*

*Proof.* Let $(f,A)$ be a fuzzy soft prime ideal over $N$. To prove that $(f_a)_t$ is a prime ideal of $N$.

Let $X, Y, Z$ be ideals of $N$ such that $[XYZ] \subseteq (f_a)_t$.
Let $(g, B)$, $(h, C)$, $(r, D)$ be fuzzy soft sets over $N$ defined by

$$g_b(x) = \begin{cases} t & \text{if } x \in X \\ 0 & \text{otherwise} \end{cases} \quad h_c(y) = \begin{cases} t & \text{if } y \in Y \\ 0 & \text{otherwise} \end{cases} \quad r_d(z) = \begin{cases} t & \text{if } z \in Z \\ 0 & \text{otherwise} \end{cases}$$

where $t \in Im\ f_a$, $B, C, D$ are subsets of $A \subseteq E$ and $b \in B$, $c \in C$, $d \in D$. Then $(g, B)$, $(h, C)$, $(r, D)$ are fuzzy soft ideals over $N$ and also $(g, B) \circ (h, C) \circ (r, D) \subseteq (f, A)$. Since $(f, A)$ is a fuzzy soft prime ideal over $N$,

$$(g, B) \subseteq (f, A) \text{ or } (h, C) \subseteq (f, A) \text{ or } (r, D) \subseteq (f, A) \tag{9}$$

Suppose $X \not\subseteq (f_a)_t$. Then there exists $x \in X$ but $x \notin (f_a)_t$.
$\Rightarrow f_a(x) < t$. Now $g_b(x) = t$ ($\because x \in X$) $> f_a(x)$. $\Rightarrow (g, B) \not\subseteq (f, A)$. Similarly $(h, C) \not\subseteq (f, A)$, $(r, D) \not\subseteq (f, A)$ a contradiction to (9). Hence $(f_a)_t$ is a prime ideal of $N$.

Conversely let $(f_a)_t$ be a prime ideal of $N$. To prove that $(f, A)$ is a fuzzy soft prime ideal over $N$. Let $B, C, D \subseteq A \subseteq E$ be such that $(g, B), (h, C), (r, D)$ are fuzzy soft ideals and

$$(g, B) \circ (h, C) \circ (r, D) \subseteq (f, A). \tag{10}$$

Suppose $(g, B) \not\subseteq (f, A)$, $(h, C) \not\subseteq (f, A)$, $(r, D) \not\subseteq (f, A)$.
Then

$$g_b(x) > f_a(x), h_c(y) > f_a(y), r_d(z) > f_a(z) \tag{11}$$

for $x, y, z \in N$. Obviously, $f_a(x) \neq 1$, $f_a(y) \neq 1$, $f_a(z) \neq 1$.
Let $f_a(x) = r = f_a(y) = f_a(z) \Rightarrow g_b(x) > r$, $h_c(y) > r$, $r_d(z) > r$. But

$$\begin{aligned} f_a(u) &\geq \bigvee_{u=[xyz]} g_b(x) \wedge h_c(y) \wedge r_d(z) \\ &> g_b(x) \wedge h_c(y) \wedge r_d(z) \\ &> r \end{aligned}$$

$\Rightarrow [xyz] \in (f_a)_r$
$\Rightarrow [(g_b)_r (h_c)_r (r_d)_r] \subseteq (f_a)_r$ which is a prime ideal
$\therefore (g_b)_r \subseteq (f_a)_r$ or $(h_c)_r \subseteq (f_a)_r$ or $(r_d)_r \subseteq (f_a)_r$
$\Rightarrow x \in (g_b)_r \Rightarrow x \in (f_a)_r \Rightarrow g_b(x) \geq r \Rightarrow f_a(x) \geq r$ which contradicts (11).
Hence $(g, B) \subseteq (f, A)$ or $(h, C) \subseteq (f, A)$ or $(r, D) \subseteq (f, A)$, proving that $(f, A)$ is a fuzzy soft prime ideal over $N$. ∎

## 4. Fuzzy Soft Completely Prime Ideal

In this section first completely prime ideals in an RTNR $N$ and define fuzzy soft completely prime ideal over $N$ are defined and some of their algebraic properties are derived.

Srinivas et al [15] have mentioned that the image of a fuzzy completely prime ideal can be greater than 2. However for a fuzzy completely prime ideal to be a fuzzy prime ideal cardinality of $Im\ f_a$ should only be two and $f_a(0) = 1$.

**Definition 40.** An ideal $J$ of an RTNR $N$ is called a *completely prime ideal* of $N$ if for $x, y, z \in N$ $[xyz] \in J \Rightarrow$ either $x \in J$ or $y \in J$ or $z \in J$.

**Example 41.** (i) Let $N = \{0, x, y, z\}$ and $+$ be defined as in Table 5 and $[\ \ ]$ be defined by $[xyz] = (x.y).z$ where . is defined as in Table 6. Then $\{0, x\}$ is a completely prime ideal of $N$.

| + | 0 | x | y | z |
|---|---|---|---|---|
| 0 | 0 | x | y | z |
| x | x | 0 | z | y |
| y | y | z | 0 | x |
| z | z | y | x | 0 |

Table 5

| . | 0 | x | y | z |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| x | 0 | 0 | 0 | 0 |
| y | 0 | 0 | y | y |
| z | 0 | 0 | y | y |

Table 6

(ii) Let $N = \{0, 1, 2, 3, 4, 5\}$. Let $+$ be defined as in Table 7 and for $x, y, z \in N$, let $[xyz] = (x.y).z$ where . is defined as in Table 8.

Then $J = \{0, 3\}$ is a completely prime ideal of $N$.

| + | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 1 | 2 | 0 | 4 | 5 | 3 |
| 2 | 2 | 0 | 1 | 5 | 3 | 4 |
| 3 | 3 | 5 | 4 | 0 | 2 | 1 |
| 4 | 4 | 3 | 5 | 1 | 0 | 2 |
| 5 | 5 | 4 | 3 | 2 | 1 | 0 |

Table 7

| . | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 3 | 1 | 5 | 3 | 1 | 5 |
| 2 | 0 | 2 | 4 | 0 | 2 | 4 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 0 | 4 | 2 | 0 | 4 | 2 |
| 5 | 3 | 5 | 1 | 3 | 5 | 1 |

Table 8

(iii) If $N = \{0, x, y, z\}$ and $+$ and $[\ \ ]$ be defined as in Example 16(i). Then

$\{0, x\}$ is not a completely prime ideal as $[yyz] = (y.y).z = 0.z = 0 \in \{0, x\}$ but $y \notin \{0, x\}$ and $z \notin \{0, x\}$.

**Proposition 42.** *If an ideal $P$ is a completely prime ideal of an RTNR $N$ then $P$ is a prime ideal of $N$.*

*Proof.* Let $P$ be a completely prime ideal of an RTNR $N$. Let $X, Y, Z$ be ideals of $N$ such that $[XYZ] \subseteq P$. Then to prove that either $X \subseteq P$ or $Y \subseteq P$ or $Z \subseteq P$. Supose not i.e., $X \nsubseteq P$, $Y \nsubseteq P$, $Z \nsubseteq P$. Then there exists $x \in X$ but $x \notin P$, $y \in Y$ but $y \notin P$ and $z \in Z$ but $z \notin P$. Since $x \in X$, $y \in Y$, $z \in Z$, $[xyz] \in [XYZ] \subseteq P \Rightarrow x \in P$ or $y \in P$ or $z \in P$ as $P$ is a completely prime ideal of $N$ which contradicts $x \notin P$, $y \notin P$, $z \notin P$. Hence $P$ is a prime ideal of $N$. □

**Definition 43.** A fuzzy soft ideal $(f, A)$ over $N$ is called a *fuzzy soft completely prime ideal* if for all $x, y, z \in N$, $f_a([xyz]) \leq \max\{f_a(x), f_a(y), f_a(z)\}$.

**Example 44.** Let $N$ be as in Example 41(ii). Let $E = N$. Let $A = \{0, 3\}$. Define $(f, A)$ as follows. For every $x \in N$

$$(f_a)(x) = \begin{cases} 1 & \text{if } x = 0, 3 \\ 0.8 & \text{if } x \neq 0, 3 \end{cases}, \quad \text{for every } a \in A$$

Then $(f, A)$ is a fuzzy soft completely prime ideal over $N$.

**Theorem 45.** *Let $(f, A)$ be a fuzzy soft set over $N$. Then for $a \in A$, $S = \{x \in N | f_a(x) = f_a(0)\}$ is a completely prime ideal of $N$ if $(f, A)$ is a fuzzy soft completely prime ideal over $N$.*

*Proof.* Let $(f, A)$ be a fuzzy soft completely prime ideal over $N$. Then $S$ is an ideal of $N$. To prove $S$ is a completely prime ideal let $[xyz] \in S$. Then $f_a([xyz]) = f_a(0)$. But $f_a([xyz]) \leq \max\{f_a(x), f_a(y), f_a(z)\}$
$\Rightarrow f_a(0) \leq \max\{f_a(x), f_a(y), f_a(z)\}$
$\Rightarrow f_a(0) \leq f_a(x)$ or $f_a(0) \leq f_a(y)$ or $f_a(0) \leq f_a(z)$
But $f_a(0) \geq f_a(x)$ for every $x \in N$.
Hence $f_a(x) = f_a(0)$ or $f_a(y) = f_a(0)$ or $f_a(z) = f_a(0)$
$\Rightarrow x \in S$ or $y \in S$ or $z \in S$. Hence $S$ is a prime ideal. □

**Remark 46.** The converse of the above theorem holds if range of $f_a$ consists of only two elements. i.e., $|Im\ f_a| = 2$, $a \in A$.

**Theorem 47.** *Let $(f, A)$ be a fuzzy soft set over $N$. Let $|Im\ f_a| = 2$ and $f_a(0) = 1$ and if $T = \{x \in N | f_a(x) = 1\}$ is a completely prime ideal then $(f, A)$ is a fuzzy soft completely prime ideal over $N$.*

*Proof.* As $T$ is an ideal by Theorem 20, $(f, A)$ is a fuzzy soft ideal over $N$. Suppose, $(f, A)$ is not a fuzzy soft completely prime ideal. Then there exists $x, y, z \in N$ such that $f_a([xyz]) > \max\{f_a(x), f_a(y), f_a(z)\}$, $\forall\ a \in A$. Since $|Im\ f_a| = 2$ and $f_a(0) = 1$, $f_a(x) = f_a(y) = f_a(z) = r \neq 1 \ldots\ldots(\mathbf{P})$ where $r \in [0, 1)$. Thus, $f_a([xyz]) > r \Rightarrow f_a([xyz]) = 1 = f_a(0) \Rightarrow [xyz] \in T$. Since $T$ is a completely prime ideal either $x \in T$ or $y \in T$ or $z \in T$. Hence $f_a(x) = 1$, $f_a(y) = 1$, $f_a(z) = 1$, a contradiction to $(\mathbf{P})$. Therefore, $f_a([xyz]) \leq \max\{f_a(x), f_a(y), f_a(z)\}$, for every $a \in A$. Hence $(f, A)$ is a fuzzy soft completely prime ideal over $N$. □

The following theorem is a characterization theorem for a fuzzy soft completely prime ideal.

**Theorem 48.** *Let $(f, A)$ be a fuzzy soft set over $N$. Then $(f, A)$ is a fuzzy soft completely prime ideal over $N$ iff for $a \in A$, $|Im\ f_a| = 2$, $f_a(0) = 1$ and $T = \{x \in N | f_a(x) = 1\}$ is a completely prime ideal of $N$.*

*Proof.* The proof follows from Remark 46, by taking $f_a(0) = 1$ and Theorem 47. □

**Theorem 49.** *A non empty subset $L$ of $N$ is a completely prime ideal of $N$. If $(f, A)$ is a fuzzy soft completely prime ideal over $N$ where $f : A \to I^N$ is defined by*

$$f_a(x) = \begin{cases} r & \text{if } x \in L \\ t & \text{if } x \notin L \end{cases}$$

*where $r > t$ and $t \in [0, 1]$ and for every $a \in A$.*

*Proof.* Let $(f, A)$ be a fuzzy soft completely prime ideal over $N$ then by Theorem 22(i), $L$ is an ideal of $N$. To establish that $L$ is a completely prime ideal of $N$ let us consider $[xyz] \in L$. Then $f_a([xyz]) = r$
$\Rightarrow \max\{f_a(x), f_a(y), f_a(z)\} \geq r$
$\Rightarrow$ either $f_a(x) \geq r$ or $f_a(y) \geq r$ or $f_a(z) \geq r \Rightarrow x \in L$ or $y \in L$ or $z \in L$.
Thus $I$ is a completely prime ideal of $N$.

Conversely let $L$ be a completely prime ideal. By Theorem 22(i), $(f, A)$ is a fuzzy soft ideal over $N$. Suppose $f_a([xyz]) > \max\{f_a(x), f_a(y), f_a(z)\}$. Then by the definition of $(f, A)$, $f_a([xyz]) = r$ and $f_a(x) = t = f_a(y) = f_a(z)$.
Thus $[xyz] \in L$ but $x \notin L$, $y \notin L$, $z \notin L$ which contradicts that $L$ is a completely prime ideal of $N$. Hence $(f, A)$ is a fuzzy soft completely prime ideal over $N$. □

The proof of the following corollary and the theorem is straight forward.

**Corollary 50.** *A non empty subset $L$ of $N$ is a completely prime ideal iff the characteristic function $(\psi_L, E)$ is a fuzzy soft completely prime ideal over $N$, where $\psi_L : E \to I^N$ is defined by*

$$(\psi_L)_e(x) = \begin{cases} 1 & \text{if } x \in L \\ 0 & \text{if } x \notin L \end{cases} \quad \text{for every } e \in E.$$

**Theorem 51.** *Let $A$ be a subset of a parameter $E$ of an RTNR $N$. Then a fuzzy soft set $(f, A)$ over $N$ is a fuzzy soft completely prime ideal implies for each $f_a$, where $a \in A$, $(f_a)_t$, $t \in Im\ f_a$ is a completely prime ideal of $N$.*

**Lemma 52.** *Let $J$ be a completely prime ideal of $N$. For $t \in (0, 1)$ there exists a fuzzy soft completely prime ideal $(f, A)$ over $N$ such that $(f_a)_t = J$ for each $a \in A$.*

*Proof.* Let $J$ be a completely prime ideal of $N$. Let $t \in (0, 1)$. Define $(f, A)$ over $N$ by

$$f_a(x) = \begin{cases} t & \text{if } x \in J \\ 0 & \text{if } x \notin J \end{cases}$$

Then by Theorem 22, $(f, A)$ is a fuzzy soft ideal over $N$. We now prove that $(f, A)$ is fuzzy soft completely prime ideal over $N$.

Suppose $(f, A)$ is not fuzzy soft completely prime ideal then there exists $x, y, z \in N$ such that $f_a([xyz]) > \max\{f_a(x), f_a(y), f_a(z)\}$. Now, by the definition of $(f, A)$ it follows that $f_a(x) = 0 = f_a(y) = f_a(z)$ and $f_a([xyz]) = t$. Thus, $[xyz] \in J$ but $x \notin J$, $y \notin J$, $z \notin J$ which contradicts that $J$ is a completely prime ideal of $N$. Hence $(f, A)$ is a fuzzy soft completely prime ideal over $N$. Obviously, $(f_a)_t = J$ for each $a \in A$. ☐

**Proposition 53.** *The inverse homomorphic image of $(g, B)$ is a fuzzy soft completely prime ideal if $(g, B)$ is a fuzzy soft completely prime ideal over $N$.*

*Proof.* Let $\phi : N \to M$ be an onto right ternary near-ring homomorphism.
Let $E_1, E_2$ be parameter sets of $N$ and $M$ respectively.
Let $\psi : A \to B$, $A \subseteq E_1$, $B \subseteq E_2$.
Let $(g, B)$ be a fuzzy completely prime ideal over $N$.
To prove that $(\phi, \psi)^{-1}(g, B) = (\phi^{-1}(g), \psi^{-1}(B))$ is a fuzzy soft completely prime ideal over $N$.
Let $\phi^{-1}(g) = h$, $\psi^{-1}(B) = C$. Then $h : C \to I^N$ and $h_c : N \to I$. Now, $h_c(x) = g_{\psi(c)}\phi(x)$. By Proposition 21, $(\phi, \psi)^{-1}(g, B)$ is a fuzzy soft ideal. Consider,

$$h_c([xyz]) = g_{\phi(c)}([xyz]) \leq \max\{g_{\phi(c)}(x), g_{\phi(c)}(y), g_{\phi(c)}(z)$$

$$= \max\{h_c(x), h_c(y), h_c(z)\}$$

Thus $(h, C)$ is in $(\phi, \psi)^{-1}(g, B)$ is a fuzzy soft completely prime ideal over $N$. □

**Lemma 54.** *If $(f, A)$ is a fuzzy soft completly prime ideal over $N$ with $|Im\ f_a| = 2$, $f_a(0) = 1$ for each $a \in A$, then $(f, A)$ is a fuzzy soft prime ideal over $N$.*

*Proof.* Let $(f, A)$ be a fuzzy soft compeltely prime ideal over $N$ such that $|Im\ f_a| = 2$, $f(0) = 1$, Then by Theorem 47, $T = \{x \in N | f_a(x) = 1\}$ is a completely prime ideal. By Proposition 42, $T$ is a prime ideal of $N$. Hence by Theorem 48, $(f, A)$ is a fuzzy soft prime ideal over $N$. □

## 5. Fuzzy Soft Completely Semi-Prime Ideal

In this section semi-prime ideal and completely semi-prime ideal of an RTNR $N$ and fuzzy soft completely semi-prime ideal are defined. A characterization theorem for a fuzzy soft completely semi-prime ideal is also given.

**Definition 55.** An ideal $J$ of $N$ is called a *semi-prime ideal* if there is an ideal $X$ of $N$ such that $[XXX] \subseteq J \Rightarrow X \subseteq J$

**Definition 56.** An ideal $J$ of $N$ is called a *completely semi-prime ideal* if $[xxx] \in J \Rightarrow x \in J$.

$[xxx]$ is denoted by $x^3$.

**Example 57.** (i) Let $N = \{0, x, y, z\}$ and $+$ be defined as in Table 9 and $[\ ]$ be defined by $[xyz] = (x.y).z$ where . is defined as in Table 10. Then $\{0, x\}$ is a completely semi prime ideal of $N$.

| + | 0 | x | y | z |
|---|---|---|---|---|
| 0 | 0 | x | y | z |
| x | x | 0 | z | y |
| y | y | z | 0 | x |
| z | z | y | x | 0 |

Table 9

| . | 0 | x | y | z |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| x | 0 | x | x | 0 |
| y | 0 | x | y | z |
| z | 0 | 0 | z | z |

Table 10

(ii) If $N = \{0, x, y, z\}$ and $+$ and $[\ \ ]$ be defined as in Example 16(i). Then $J = \{0, x\}$ is not a completely semi prime ideal since $[yyy] = 0 \in J$ but $y \notin J$.

**Proposition 58.** *Every completely prime ideal of $N$ is a completely semi-prime ideal of $N$.*

*Proof.* By taking $y$ and $z$ in Definition 40 as $x$ the proof follows. □

**Definition 59.** A fuzzy soft ideal $(f, A)$ over $N$ is a *fuzzy soft completely semi-prime* ideal if $f_a(x^3) \leq f_a(x)$, $\forall\ a \in A$ and $x \in N$.

**Example 60.** Let $N$ be as in Example 57(i). Let $A = \{0, x\}$. Define $(f, A)$ by

$$f_a(x) = \begin{cases} 1 & \text{if } t = 0, x \\ 0.6 & \text{otherwise} \end{cases}, \quad \text{for each } a \in A.$$

then $(f, A)$ is fuzzy soft completely semi prime ideal.

**Lemma 61.** *A fuzzy soft ideal $(f, A)$ over $N$ is a fuzzy soft completely semi-prime ideal iff $f_a(x^3) = f_a(x)$, for every $a \in A$, $x \in N$.*

*Proof.* Let $(f, A)$ be a fuzzy soft ideal over $N$ with $f_a(x^3) = f_a(x)$, $\forall\ a \in A$ and $x \in N$. Then by Definition 59, $(f, A)$ is a fuzzy soft completely semi-prime ideal over $N$.

Conversely, if $(f, A)$ is a fuzzy soft completely semi-prime ideal by Definition 59, $f_a(x^3) \leq f_a(x)$ and as $(f, A)$ is a fuzzy soft ideal over $N$, $f_a(x^3) \geq f_a(x)$. Hence $f_a(x^3) = f_a(x)$ for every $a \in A$, $x \in N$. □

The following is a characterization theorem for a fuzzy soft completely semi-prime ideal which does not have any condition to be imposed on the cardinality of $Im\ f_a$.

**Theorem 62.** *A fuzzy soft set $(f, A)$ over $N$ is a fuzzy soft completely semi-prime ideal over $N$ iff for $a \in A$, $S = \{x \in N | f_a(x) = f_a(0)\}$ is a completely semi-prime ideal of $N$.*

*Proof.* Let $(f, A)$ be a fuzzy soft completely semi-prime ideal over $N$. Then $f_a(x^3) = f_a(x)$. Now let $x^3 \in S$. Then $f_a(x^3) = f_a(0) \Rightarrow f_a(x) = f_a(0) \Rightarrow x \in S$. Hence $S$ is a completely semi-prime ideal of $N$.

Conversely, if $S$ is a completely semi-prime ideal of $N$. Then $x^3 \in S \Rightarrow x \in S$. Since $x^3 \in S$, $f_a(x^3) = f_a(0)$ and $f_a(x) = f_a(0)$, $f_a(x^3) = f_a(x)$. Hence by Lemma 61, $(f, A)$ is a fuzzy soft completely semi-prime ideal over $N$. □

**Theorem 63.** *A fuzzy soft set $(f, A)$ is a fuzzy soft completely semi-prime ideal over $N$ iff $(f_a)_t$, $a \in A$, is a completely semi-prime ideal of $N$ where $t \in Im\ f_a$.*

*Proof.* Let $(f, A)$ be a fuzzy soft completely semi-prime ideal over $N$. Then $f_a(x^3) = f_a(x)$. Now, let $x^3 \in (f_a)_t \Rightarrow f_a(x^3) \geq t \Rightarrow f_a(x) \geq t \Rightarrow x \in (f_a)_t$ and hence $(f_a)_t$ is a completely semi-prime ideal of $N$.

Conversely let $(f_a)_t$ be a completely semi-prime ideal of $N$. Then $x^3 \in (f_a)_t \Rightarrow x \in (f_a)_t$. i.e., $f_a(x^3) \geq t \Rightarrow f_a(x) \geq t$. We claim that $f_a(x^3) = f_a(x)$. Suppose not. i.e., $f_a(x^3) \neq f_a(x)$. Let $f_a(x) = \alpha$. Then $x^3 \notin (f_a)_\alpha$ but $x \in (f_a)_\alpha$ which is a contradiction as $(f_a)_t$, $t \in Im\ f_a$ is a completely semi-prime ideal of $N$. □

**Theorem 64.** *The inverse homomorphic image of $(g, B)$ is a fuzzy soft completely semi-prime ideal over $N$ if $(g, B)$ is a fuzzy soft completely semi-prime ideal over $N$.*

*Proof.* Let $\phi : N \to M$ an onto right ternary near-ring homomorphism. Let $E_1, E_2$ be parameter sets of $N$ and $M$ respectively. Let $\psi : A \to B$, $A \subseteq E_1$, $B \subseteq E_2$. Let $(g, B)$ be a fuzzy soft completely semi-prime ideal over $M$. To prove that $(\phi, \psi)^{-1}(g, B) = (\phi^{-1}(g), \psi^{-1}(B))$ is a fuzzy soft completely semi-prime ideal over $N$. Let $\phi^{-1}(g) = h$, $\psi^{-1}(B) = C$. Then $h : C \to I^N$ and $h_C : N \to I$. Now, $h_c(x) = g_{\psi(c)}(\phi(x))$. By Proposition 21, $(\phi, \psi)^{-1}(g, B)$ is a fuzzy soft ideal over $N$. Consider, $h_c(x^3) = g_{\psi(c)}(x^3) \leq g_{\psi(c)}(x) = h_c(x)$. Since $(h, C)$ is a fuzzy soft ideal over $N$, $h_c(x^3) \geq h_c(x)$. Hence $h_c(x^3) = h_c(x)$, proving that $(h, C)$ is a fuzzy soft completely semi-prime ideal over $N$. □

## 6. Conclusion

In this paper the basic algebraic properties of fuzzy soft prime ideals, fuzzy soft completely prime ideals and fuzzy soft completely semi-prime ideals were studied and their characterization theorems were given. A similar approach can be extended to other subsystems of RTNR and the fuzziness of these structures can suitably be studied.

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