THE MINIMUM DOMINATING ENERGY OF A GRAPH

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Abstract: Recently Professor Chandrashekar Adiga et al [3] defined the minimum covering energy, \( E_C(G) \) of a graph which depends on its particular minimum cover \( C \). Motivated by this, we introduced minimum dominating energy of a graph \( E_D(G) \) and computed minimum dominating energies of a star graph, complete graph, crown graph and cocktail graphs. Upper and lower bounds for \( E_D(G) \) are established.

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1. Introduction

The concept of energy of a graph was introduced by I. Gutman [7] in the year 1978. Let \( G \) be a graph with \( n \) vertices and \( m \) edges and let \( A = (a_{ij}) \) be the adjacency matrix of the graph. The eigenvalues \( \lambda_1, \lambda_2, \cdots, \lambda_n \) of \( A \), assumed in non increasing order, are the eigenvalues of the graph \( G \). As \( A \) is real symmetric, the eigenvalues of \( G \) are real with sum equal to zero. The energy \( E(G) \) of \( G \) is...
defined to be the sum of the absolute values of the eigenvalues of G. i.e., \( E(G) = \sum_{i=1}^{n} |\lambda_i| \).

For details on the mathematical aspects of the theory of graph energy see the reviews [8], papers [4, 5, 9] and the references cited there in. The basic properties including various upper and lower bounds for energy of a graph have been established in [11, 12], and it has found remarkable chemical applications in the molecular orbital theory of conjugated molecules [6, 10].

2. The Minimum Dominating Energy

Let \( G \) be a simple graph of order \( n \) with vertex set \( V = \{v_1, v_2, ..., v_n\} \) and edge set \( E \). A subset \( D \) of \( V \) is called a dominating set of \( G \) if every vertex of \( V-D \) is adjacent to some vertex in \( D \). Any dominating set with minimum cardinality is called a minimum dominating set. Let \( D \) be a minimum dominating set of a graph \( G \). The minimum dominating matrix of \( G \) is the \( n \times n \) matrix defined by \( A_D(G) := (a_{ij}) \), where

\[
  a_{ij} = \begin{cases} 
  1 & \text{if } v_i v_j \in E, \\
  1 & \text{if } i = j \text{ and } v_i \in D, \\
  0 & \text{otherwise}.
\end{cases}
\]

The characteristic polynomial of \( A_D(G) \) is denoted by \( f_n(G, \lambda) = \det(\lambda I - A_D(G)) \). The minimum dominating eigenvalues of the graph \( G \) are the eigenvalues of \( A_D(G) \). Since \( A_D(G) \) is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \). The minimum dominating energy of \( G \) is defined as

\[
  E_D(G) := \sum_{i=1}^{n} |\lambda_i|.
\]

Note that the trace of \( A_D(G) = \text{Domination Number} = k \).

Example 1. The possible minimum dominating sets for the following graph \( G \) are:

i) \( D_1 = \{v_1, v_5\} \);

ii) \( D_2 = \{v_2, v_5\} \)

iii) \( D_3 = \{v_2, v_6\} \).
Minimum dominating eigen values are \( \lambda_1 \approx -1.6473, \lambda_2 \approx -1.1263, \lambda_3 \approx 0, \lambda_4 \approx 0.2546, \lambda_5 \approx 1.3261, \lambda_6 \approx 3.1929. \)

Minimum dominating energy, \( E_{D_1}(G) \approx 7.5471. \)

Minimum dominating eigen values are \( \lambda_1 \approx -1.4495, \lambda_2 \approx -1, \lambda_3 \approx 0, \lambda_4 \approx 0, \lambda_5 \approx 1, \lambda_6 \approx 3.4495. \)

Minimum domination energy, \( E_{D_2}(G) \approx 6.8990. \)

Minimum dominating energy depends on the dominating set.


**Definition 3.1.** The Cocktail party graph is denoted by \( K_{n \times 2}, \) is a graph having the vertex set \( V = \bigcup_{i=1}^{n} \{u_i, v_i\} \) and the edge set \( E = \{u_iu_j, v_iv_j : i \neq j, 1 \leq i, j \leq n\}. \)
Theorem 3.1. The minimum dominating energy of Cocktail party graph $K_{n \times 2}$ is $(2n - 3) + \sqrt{4n^2 - 4n - 9}$.

Proof. Let $K_{n \times 2}$ be the Cocktail party graph with vertex set $V = \bigcup_{i=1}^{n}\{u_i, v_i\}$. The minimum dominating set is $D = \{u_1, v_1\}$. Then

$$A_D(K_{n \times 2}) = \begin{pmatrix}
1 & 0 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & \ldots & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & \ldots & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & \ldots & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & \ldots & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & \ldots & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & \ldots & 1 & 1 & 0 & 0
\end{pmatrix}.$$

Characteristic polynomial is

$$\lambda^n - 1 \lambda^{-1} - 1 - 1 - 1 \ldots - 1 - 1 - 1 - 1 - 1.$$

Minimum dominating eigen values are:

$$\lambda = 0 \ [\text{one time}],$$

$$\lambda = 1 \ [\text{one time}],$$

$$\lambda = -2 \ [\text{n-2 times}], \ \lambda = \frac{(2n - 3) \pm \sqrt{4n^2 - 4n + 9}}{2} \ [\text{one time each}].$$

Minimum dominating energy $E_D(K_{n \times 2})$.
\[= 0 + 1 + \lfloor -2(n - 2) + \frac{(2n - 3) + \sqrt{4n^2 - 4n + 9}}{2} \rfloor + \frac{(2n - 3) - \sqrt{4n^2 - 4n + 9}}{2} \]

\[= 1 + 2(n - 2) + \sqrt{4n^2 + 4n - 7} = 2n - 3 + \sqrt{4n^2 - 4n + 9}. \square \]

**Theorem 3.2.** For \( n \geq 2 \), the minimum dominating energy of Star graph \( K_{1,n-1} \) is equal to \( \sqrt{4n - 3} \).

*Proof.* Consider the Star graph \( K_{1,n-1} \) with vertex set \( V = \{v_0, v_1, v_2, ..., v_{n-1}\} \). Minimum dominating set is \( D = \{v_0\} \). Then

\[
A_D(K_{1,n-1}) = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & 0 & 0 & \ldots & 0 \\
1 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \ldots & 0
\end{pmatrix}_{n \times n},
\]

Characteristic polynomial is

\[
\begin{vmatrix}
\lambda - 1 & -1 & -1 & \ldots & -1 \\
-1 & \lambda & 0 & \ldots & 0 \\
-1 & 0 & \lambda & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & 0 & 0 & \ldots & \lambda
\end{vmatrix}
\]

Characteristic equation is \( \lambda^{n-2}(\lambda^2 - \lambda - (n - 1)) = 0 \).

The minimum dominating eigen values are:

\[ \lambda = 0 \quad [(n-2) \text{ times}], \quad \lambda = \frac{1 \pm \sqrt{4n - 3}}{2} \quad [\text{one time each}]. \]

Minimum dominating energy is

\[ E_D(K_{1,n-1}) = |0|(n - 2) + \frac{1 + \sqrt{4n - 3}}{2} + \frac{1 - \sqrt{4n - 3}}{2} = \sqrt{4n - 3}. \square \]

**Theorem 3.3.** For \( n \geq 2 \), the minimum dominating energy of complete graph \( K_n \) is \( (n - 2) + \sqrt{n^2 - 2n + 5} \).

*Proof.* \( K_n \) is Complete graph with vertex set \( V = \{v_1, v_2, ..., v_n\} \). The minimum dominating set is \( D = \{v_1\} \). Then:

\[
A_D(K_n) = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 & 1 \\
1 & 0 & 1 & \ldots & 1 & 1 \\
1 & 1 & 0 & \ldots & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 1 & \ldots & 1 & 0
\end{pmatrix}_{n \times n},
\]
Characteristic polynomial is
\[ \lambda - 1 - 1 \ldots - 1 - 1 \]
Characteristic equation is \((\lambda + 1)^{n-2}(\lambda^2 - (n - 1)\lambda - 1) = 0\).
The minimum dominating eigen values are
\[ \lambda = -1 \text{ [(n-2) times]}, \lambda = \frac{(n - 1) \pm \sqrt{n^2 - 2n + 5}}{2} \text{ [one time each]}. \]
Minimum dominating energy is
\[ E_D(K_n) = \left| -1 \right|(n-2) + \left| \frac{(n - 1) + \sqrt{n^2 - 2n + 5}}{2} \right| + \left| \frac{(n - 1) - \sqrt{n^2 - 2n + 5}}{2} \right| \]
\[ = (n - 2) + \sqrt{n^2 - 2n + 5}. \]

**Definition 3.2.** The crown graph \(S^0_n\) for an integer \(n \geq 2\) is the graph with vertex set \(\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}\) and edge set \(\{u_i v_j : 1 \leq i, j \leq n, i \neq j\}\).
\(S^0_n\) coincides with the complete bipartite graph \(K_{n,n}\) with horizontal edges removed.

**Theorem 3.4.** For \(n \geq 2\), the minimum dominating energy of the crown graph \(S^0_n\) is equal to \(2(n - 2) + \sqrt{n^2 - 2n + 5} + \sqrt{n^2 + 2n - 3}\).

**Proof.** For the crown graph \(S^0_n\) with vertex set \(V = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}\), minimum dominating set is \(S = \{u_1, v_1\}\). Then

\[ A_D(S^0_n) = \left( \begin{array}{ccccccccccc} 1 & 0 & 0 & \ldots & 0 & 0 & 1 & 1 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 & 1 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 1 & 0 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 & 1 & 1 & \ldots & 0 \\ 0 & 1 & 1 & \ldots & 1 & 1 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 1 & \ldots & 1 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 0 & \ldots & 1 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \end{array} \right) \]
\((2n \times 2n)\)
Characteristic polynomial is

\[
\begin{vmatrix}
\lambda - 1 & 0 & 0 & \ldots & 0 & 0 & -1 & -1 & \ldots & -1 \\
0 & \lambda & 0 & \ldots & 0 & -1 & 0 & -1 & \ldots & -1 \\
0 & 0 & \lambda & \ldots & 0 & -1 & -1 & 0 & \ldots & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \lambda & -1 & -1 & -1 & \ldots & 0 \\
0 & -1 & -1 & \ldots & -1 & \lambda - 1 & 0 & 0 & \ldots & 0 \\
-1 & 0 & -1 & \ldots & -1 & 0 & \lambda & 0 & \ldots & 0 \\
-1 & -1 & 0 & \ldots & -1 & 0 & 0 & \lambda & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & -1 & \ldots & 0 & 0 & 0 & 0 & \ldots & \lambda \\
\end{vmatrix}
\]

Characteristic equation is

\[
(\lambda - 1)^{n-2}(\lambda + 1)^{n-2}(\lambda^2 - (n - 1)\lambda - 1)(\lambda^2 + (n - 3)\lambda - (2n - 3)) = 0
\]

Minimum dominating eigen values are \(\lambda = 1\) \([(n - 2)\text{times}]\),
\[
\lambda = -1 \quad \text{[}(n - 2)\text{times}]\]
\[
\lambda = \frac{(n - 1) \pm \sqrt{n^2 - 2n + 5}}{2}, \quad \text{[one time each]},
\]
\[
\lambda = \frac{(3 - n) \pm \sqrt{n^2 + 2n - 3}}{2}, \quad \text{[one time each]}
\]

Minimum dominating energy

\[
E_D(S^0_n) = 1(n - 2) + | - 1| (n - 2)
\]
\[
+ \frac{(n - 1) + \sqrt{n^2 - 2n + 5}}{2} + \frac{(n - 1) - \sqrt{n^2 - 2n + 5}}{2}
\]
\[
+ \frac{(3 - n) + \sqrt{n^2 + 2n - 3}}{2} + \frac{(3 - n) - \sqrt{n^2 + 2n - 3}}{2}
\]
\[
= 2(n - 2) + \sqrt{n^2 - 2n + 5} + \sqrt{n^2 + 2n - 3}.
\]

4. Properties of Minimum Dominating Eigen Values

**Theorem 4.1.** Let \(G\) be a simple graph with vertex set \(V = \{v_1, v_2, \ldots, v_n\}\), edge set \(E\) and \(D = \{u_1, u_2, \ldots, u_k\}\) be a minimum dominating set. If \(\lambda_1, \lambda_2, \ldots, \lambda_n\) are the eigen values of minimum dominating matrix \(A_D(G)\) then:
\( (i) \sum_{i=1}^{n} \lambda_i = |D|; \)

\( (ii) \sum_{i=1}^{n} \lambda_i^2 = 2|E| + |D|. \)

**Proof.** (i) We know that the sum of the eigen values of \( A_D(G) \) is the trace of \( A_D(G) \)

\[
\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = |D| = k.
\]

(ii) Similarly the sum of squares of the eigen values of \( A_D(G) \) is trace of \([A_D(G)]^2\)

\[
\sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}a_{ji} \\
= \sum_{i=1}^{n} (a_{ii})^2 + \sum_{i \neq j} a_{ij}a_{ji} \\
= \sum_{i=1}^{n} (a_{ii})^2 + 2\sum_{i<j} (a_{ij})^2 \\
= |D| + 2|E|. \]

\[\square\]

5. Bounds for Minimum Dominating Energy

Similar to McClelland’s [12] bounds for energy of a graph, bounds for \( E_D(G) \) are given in the following theorem.

**Theorem 5.1.** Let \( G \) be a simple graph with \( n \) vertices and \( m \) edges. If \( D \) is the minimum dominating set and \( P = |detA_D(G)| \) then

\[
\sqrt{(2m + k) + n(n - 1)P^2} \leq E_D(G) \leq \sqrt{n(2m + k)},
\]

where \( k \) is domination number.

**Proof.**

Cauchy Schwarz inequality is

\[
\left( \sum_{i=1}^{n} a_i b_i \right)^2 \leq \left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{i=1}^{n} b_i^2 \right)
\]
If \( a_i = 1, b_i = |\lambda_i| \) then
\[
\left( \sum_{i=1}^{n} |\lambda_i| \right)^2 \leq \left( \sum_{i=1}^{n} 1 \right) \left( \sum_{i=1}^{n} \lambda_i^2 \right)
\]
\[
[E_D(G)]^2 \leq n(2m + k) \quad \text{[Theorem 4.1]}
\]
\[
\implies E_D(G) \leq \sqrt{n(2m + k)}
\]

Since arithmetic mean is not smaller than geometric mean we have

\[
\frac{1}{n(n-1)} \sum_{i \neq j} |\lambda_i| |\lambda_j| \geq \left[ \prod_{i \neq j} |\lambda_i| \right] \frac{1}{n(n-1)}
\]
\[
= \left[ \prod_{i=1}^{n} |\lambda_i| \right]^{\frac{2(n-1)}{n}} \frac{1}{n(n-1)}
\]
\[
= \left[ \prod_{i=1}^{n} |\lambda_i| \right]^{\frac{2}{n}}
\]
\[
= |detA_D(G)|^{\frac{2}{n}} = P_\pi^2
\]
\[
\sum_{i \neq j} |\lambda_i| |\lambda_j| \geq n(n-1)P_\pi^2 \quad (1)
\]

Now consider,
\[
[E_D(G)]^2 = \left( \sum_{i=1}^{n} |\lambda_i| \right)^2
\]
\[
= \sum_{i=1}^{n} |\lambda_i|^2 + \sum_{i \neq j} |\lambda_i| |\lambda_j|
\]
\[
[E_D(G)]^2 \geq (k + 2m) + n(n-1)P_\pi^2 \quad \text{[From (5.1)]}
\]
\[
i.e., \quad E_D(G) \geq \sqrt{(k + 2m) + n(n-1)P_\pi^2}.
\]

**Theorem 5.2.** If \( \lambda_1(G) \) is the largest minimum dominating eigen value of \( A_D(G) \), then \( \lambda_1(G) \geq \frac{2m + k}{n} \) where \( k \) is the domination number.
Proof. Let $X$ be any nonzero vector. Then by [1],

We have $\lambda_1(A) = \max_{X \neq 0} \left\{ \frac{X'AX}{X'X} \right\}$.

$\lambda_1(A) \geq \frac{J'AJ}{J'J} = \frac{2m + k}{n}$ where $J$ is a unit matrix.

Similar to Koolen and Moulton’s [13] upper bound for energy of a graph, upper bound for $E_D(G)$ is given in the following theorem.

**Theorem 5.3.** If $G$ is a graph with $n$ vertices and $m$ edges and $(2m + k) \geq n$ then

$$E_D(G) \leq \frac{2m + k}{n} + \sqrt{(n - 1) \left[ (2m + k) - \left( \frac{2m + k}{n} \right)^2 \right]}$$

where $k$ is a domination number.

**Proof.** Cauchy-Schwartz inequality is

$$\left[ \sum_{i=2}^{n} a_i b_i \right]^2 \leq \left( \sum_{i=2}^{n} a_i^2 \right) \left( \sum_{i=2}^{n} b_i^2 \right)$$

Put $a_i = 1, b_i = |\lambda_i|$, then

$$\left( \sum_{i=2}^{n} |\lambda_i| \right)^2 = \sum_{i=2}^{n} 1 \sum_{i=2}^{n} \lambda_i^2$$

$$\Rightarrow [E_D(G) - \lambda_1]^2 \leq (n - 1)(2m + k - \lambda_1^2)$$

$$\Rightarrow E_D(G) \leq \lambda_1 + \sqrt{(n - 1)(2m + k - \lambda_1^2)}$$

Let $f(x) = x + \sqrt{(n - 1)(2m + k - x^2)}$

For decreasing function $f'(x) \leq 0 \Rightarrow 1 - \frac{x(n - 1)}{\sqrt{(n - 1)(2m + k - x^2)}} \leq 0$

$$\Rightarrow x \geq \sqrt{\frac{2m + k}{n}}$$

Since $(2m + k) \geq n$, we have $\sqrt{\frac{2m + k}{n}} \leq \frac{2m + k}{n} \leq \lambda_1$

$$f(\lambda_1) \leq f\left( \frac{2m + k}{n} \right)$$

i.e. $E_D(G) \leq f(\lambda_1) \leq f\left( \frac{2m + k}{n} \right)$
i.e. \( E_D(G) \leq f\left(\frac{2m+k}{n}\right) \)

i.e. \( E_D(G) \leq \frac{2m+k}{n} + \sqrt{(n-1)\left[2m+k - \left(\frac{2m+k}{n}\right)^2\right]} \). \( \square \)

Bapat and S.Pati [2] proved that if the graph energy is a rational number then it is an even integer. Similar result for minimum dominating energy is given in the following theorem.

**Theorem 5.4.** Let \( G \) be a graph with a minimum dominating set \( D \). If the minimum dominating energy \( E_D(G) \) is a rational number, then \( E_D(G) \equiv |D| \pmod{2} \).

**Proof.** Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be minimum dominating eigen values of a graph \( G \) of which \( \lambda_1, \lambda_2, \ldots, \lambda_r \) are positive and the rest are non-positive, then

\[ \sum_{i=1}^{n} |\lambda_i| = (\lambda_1 + \lambda_2 + \ldots + \lambda_r) - (\lambda_{r+1} + \ldots + \lambda_n) \]

\[ \sum_{i=1}^{n} |\lambda_i| = 2(\lambda_1 + \lambda_2 + \ldots + \lambda_r) - (\lambda_1 + \lambda_2 + \ldots + \lambda_n) \]

i.e. \( E_D(G) = 2(\lambda_1 + \lambda_2 + \ldots + \lambda_r) - \sum_{i=1}^{n} \lambda_i \)

i.e. \( E_D(G) = 2(\lambda_1 + \lambda_2 + \ldots + \lambda_r) - |D| \)

\( E_D(G) \equiv |D| \pmod{2} \).

Hence the theorem holds true. \( \square \)

**References**


