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EXISTENCE OF SOLUTIONS FOR AN ELLIPTIC EQUATION WITH NONSTANDARD GROWTH

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Abstract: This paper deals with the existence of solutions for some elliptic equations with nonstandard growth under zero Dirichlet boundary condition. Using a direct variational method and the theory of the variable exponent Sobolev spaces, we set some conditions that ensures the existence of nontrivial weak solutions.

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1. Introduction

In the present paper we are concerned with the boundary value problem

$$\begin{cases} -\triangle_{p(x)}u = \lambda m(x)\,|u|^{q(x)-2}\,u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (\mathbf{P})$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\lambda > 0$; p, $q \in C\left(\overline{\Omega}\right)$ and m is a non-negative measurable real function.

The study of differential equations and variational problems with nonstandard growth equations have been a new and interesting topic. The main interest in studying such problems arises from the presence of the $p(x)$-Laplace operator represented as $\triangle_{p(x)}u = \operatorname{div}\left(|\nabla u|^{p(x)-2}\,\nabla u\right)$. This is a generalization of the p-Laplace operator $\triangle_p u = \operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right)$ obtained in the case when $p(x) \equiv p$ is a positive constant. Differential equations involving the $p(x)$-Laplace equations are not trivial generalizations of similar problems studied in the constant case since $p(x)$-Laplace operator is not homogeneous and, thus, some techniques which can be applied in the case of the p-Laplace operators will fail in that new situation, such as the theory of Sobolev spaces. On the other hand, problems involving nonstandard growth conditions are extremely attractive because they can model phenomenons which arise from the study of electrorheological fluids or elastic mechanics, stationary thermo-rheological viscous flows of non-Newtonian fluids and they also appear in the mathematical description of the processes filtration of an ideal barotropic gas through a porous medium [1, 14, 15]. We refer the reader to [2, 3, 4, 9, 10, 12, 13] and the references therein for the study of $p(x)$-Laplacian equations.

2. Preliminaries

We state some basic properties of the variable exponent Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain.In that context we refer to [5, 7, 8, 11] for the fundamental properties of these spaces.

Set

$$C_+\left(\overline{\Omega}\right) = \left\{p : \ p \in C\left(\overline{\Omega}\right), \ p(x) > 1 \text{ for any } x \in \overline{\Omega}\right\}.$$

For any $p \in C_+\left(\overline{\Omega}\right)$, denote $1 < p^- := \inf_{x\in\overline{\Omega}} p(x) \le p(x) \le p^+ := \sup_{x\in\overline{\Omega}} p(x) < \infty$, and define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{u \mid u : \Omega \to \mathbb{R} \text{ is measurable, } \int_\Omega |u(x)|^{\,p(x)}\,dx < \infty\right\}.$$

We define a norm, the so-called Luxemburg norm, on this space by the formula

$$|u|_{p(x)} = \inf\left\{\eta > 0 : \int_\Omega \left|\frac{u(x)}{\eta}\right|^{p(x)} dx \le 1\right\},$$

and $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ becomes a Banach space.

The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $\frac{1}{p'(x)} + \frac{1}{p(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have

$$\left|\int_\Omega uv dx\right| \le \left(\frac{1}{p^-} + \frac{1}{(p^-)'}\right) |u|_{p(x)} |v|_{p'(x)}, \tag{1.1}$$

which is known as Hölder inequality.The modular of $L^{p(x)}(\Omega)$ is $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx.$$

If $u, u_n \in L^{p(x)}(\Omega)$ $(n = 1, 2, ...)$ and $p^+ < \infty$, we have

$$(i)\ |u|_{p(x)} > 1 \Longrightarrow |u|^{p^-}_{p(x)} \le \rho_{p(x)}(u) \le |u|^{p^+}_{p(x)}; \tag{1.2}$$

$$(ii)\ |u|_{p(x)} < 1 \Longrightarrow |u|^{p^+}_{p(x)} \le \rho_{p(x)}(u) \le |u|^{p^-}_{p(x)}; \tag{1.3}$$

$$(iii)\ |u_n - u|_{p(x)} \to 0 \Longleftrightarrow \rho_{p(x)}(u_n - u) \to 0. \tag{1.4}$$

We also consider the weighted variable exponent Lebesgue spaces. Let $b : \Omega \to \mathbb{R}$ is a measurable real function such that $b(x) > 0$ a.e. $x \in \Omega$. We define

$$L^{p(x)}_{b(x)}(\Omega) = \left\{u \mid u : \Omega \to \mathbb{R} \text{ is measurable}, \int_\Omega b(x) |u(x)|^{p(x)} dx < \infty\right\}.$$

The space $L^{p(x)}_{b(x)}(\Omega)$ endowed with the above norm is a Banach space which has similar properties with the variable exponent Lebesgue spaces.The modular of $L^{p(x)}_{b(x)}(\Omega)$ is $\rho_{(p(x),b(x))} : L^{p(x)}_{b(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{(p(x),b(x))}(u) = \int_\Omega b(x) |u(x)|^{p(x)} dx.$$

If $u, u_n \in L^{p(x)}_{b(x)}(\Omega)$ $(n = 1, 2, ...)$ and $p^+ < \infty$, we have

$$(i)\ |u_n|_{(p(x),b(x))} > 1 \Longrightarrow |u|^{p^-}_{(p(x),b(x))} \le \rho_{p(x)}(u) \le |u|^{p^+}_{(p(x),b(x))}; \tag{1.5}$$

$$(ii)\ |u_n|_{(p(x),b(x))} < \quad 1 \Longrightarrow |u|^{p^+}_{(p(x),b(x))} \le \rho_{p(x)}(u) \le |u|^{p^-}_{(p(x),b(x))}. \tag{1.6}$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega);\quad |\nabla u| \in L^{p(x)}(\Omega)\},$$

then it can be equipped with the norm

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}\ , \forall u \in W^{1,p(x)}(\Omega).$$

The space $W_0^{1,p(x)}(\Omega)$ is denoted by the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. We will use $\|u\| = |\nabla u|_{p(x)}$ for $u \in W_0^{1,p(x)}(\Omega)$ in the following discussions.

Moreover, if $1 < p^- \le p^+ < \infty$ the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces [11].

Proposition 2.2. [7, 11] *Let $q \in C_+(\overline{\Omega})$. If $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous, where $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ and $p^*(x) = +\infty$ if $p(x) \ge N$.*

Proposition 2.3. [9] *Let X be a Banach space and $\Lambda(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx$. The functional $\Lambda : X \to \mathbb{R}$ is convex. The mapping $\Lambda' : X \to X^*$ is a strictly monotone, bounded homeomorphism, and of (S_+) type, namely*

$$u_n \rightharpoonup u \text{ (weakly) and } \overline{\lim_{n\to\infty}} \langle \Lambda'(u_n), u_n - u\rangle \le 0 \text{ implies } u_n \to u \text{ (strongly).}$$

3. The Main Results

We say that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of (**P**) if

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \varphi dx = \lambda \int_\Omega m(x) |u|^{q(x)-2} u\varphi dx,$$

where $\varphi \in W_0^{1,p(x)}(\Omega)$.

The energy functional corresponding to problem (**P**) is defined as $J_\lambda : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$,

$$J_\lambda(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx - \lambda \int_\Omega m(x) \frac{|u|^{q(x)}}{q(x)} dx,$$

It is not difficult to show that $J_\lambda \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$, and

$$\langle J'_\lambda(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx - \lambda \int_\Omega m(x)|u|^{q(x)-2} uvdx,$$

for any $u, v \in W_0^{1,p(x)}(\Omega)$. Hence, we can infer that critical points of functional J_λ are the weak solutions for problem (P).

We will prove:

Theorem 3.1. *Suppose the following conditions hold:*

(P_1) $m \in L^{\delta(x)}(\Omega)$, $m(x) > 0$ *and* $\delta \in C_+(\overline{\Omega})$ *such that* $\frac{1}{\delta(x)} + \frac{1}{\delta_0(x)} = 1$, $p(x) < \delta_0(x) q(x)$ *and* $1 < q(x) < \frac{1}{\delta_0(x)} p^*(x)$ $\forall x \in \overline{\Omega}$,

(P_2) $1 < q^- < p^- < q^+$, $p^+ < N$.

Then, there exists $\lambda^* > 0$ *such that* (P) *has a nontrivial weak solution for any* $\lambda \in (0, \lambda^*)$.

The proof of Theorem 3.1 is broken into several parts.

Lemma 3.2. *Assume that* (P_1) *and* (P_2) *hold.Then there exist positive real numbers* γ, r *and* $\lambda^* > 0$ *such that for any* $\lambda \in (0, \lambda^*)$, *we have*

$J_\lambda(u) \geq r > 0$, $u \in W_0^{1,p(x)}(\Omega)$ *with* $\|u\| = \gamma$.

Proof. By using assumption (P_1), and the arguments developed in [12, Theorem 2.8], we can write

$$\int_\Omega m(x)|u|^{q(x)} dx \leq C\left(\|u\|^{q^-} + \|u\|^{q^+}\right). \tag{3.1}$$

Consider $\gamma \in (0, 1)$. Then the above relation implies

$$\int_\Omega m(x)|u|^{q(x)} dx \leq C\|u\|^{q^-}, \forall u \in W_0^{1,p(x)}(\Omega). \tag{3.2}$$

Using (P_1), (1.2) and (3.2), we obtain that for any $u \in W_0^{1,p(x)}(\Omega)$ with $\|u\| = \gamma$ the following inequalities hold true

$$\begin{aligned} J_\lambda(u) &= \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx - \lambda \int_\Omega \frac{m(x)}{q(x)} |u|^{q(x)} dx \\ &\geq \frac{1}{p^+}\|u\|^{p^+} - \frac{\lambda C}{q^-}\|u\|^{q^-} \\ &= \gamma^{q^-}\left(\frac{1}{p^+}\gamma^{p^+-q^-} - \frac{\lambda C}{q^-}\right). \end{aligned} \tag{3.3}$$

In the last inequality above, if we choose

$$\lambda^* = \frac{q^-}{Cp^+}\gamma^{p^+-q^-},$$

then it is clear that there exists $r > 0$ such that for any $\lambda \in (0, \lambda^*)$ we have

$$J_\lambda(u) \geq r, \ \forall u \in W_0^{1,p(x)}(\Omega) \text{ with } \|u\| = \gamma.$$

The proof is complete. □

Lemma 3.3. *Assume that* (P_1) *and* (P_2) *hold. Then there exists* $\varphi \in W_0^{1,p(x)}(\Omega)$ *such that* $\varphi \geq 0, \varphi \neq 0$ *and* $J_\lambda(t\varphi) < 0$ *for* $t > 0$ *small enough.*

Proof. From assumption (P_2) we know that $q^- < p^-$. Let $\epsilon_0 > 0$ be such that $q^- + \epsilon_0 < p^-$. On the other hand, since $q \in C\left(\overline{\Omega}\right)$ it follows that there exists an open set $\Omega_0 \subset \Omega$ such that $|q(x) - q^-| < \epsilon_0$ for all $x \in \overline{\Omega}_0$. Thus, we conclude that $q(x) \leq q^- + \epsilon_0 < p^-$ for all $x \in \overline{\Omega}_0$.

Let $\varphi \in C_0^\infty(\Omega)$ be such that $supp(\varphi) \supset \overline{\Omega}_0$, $\varphi(x) = 1$ for all $x \in \overline{\Omega}_0$ and $0 \leq \varphi(x) \leq 1$ in Ω. Then from the above facts for any $t \in (0,1)$ it follows

$$\begin{aligned} J_\lambda(t\varphi) &= \int_\Omega \frac{|\nabla t\varphi|^{p(x)}}{p(x)}dx - \lambda \int_\Omega \frac{m(x)}{q(x)} |t\varphi|^{q(x)} dx \\ &< \frac{t^{p^-}}{p^-}\int_\Omega |\nabla\varphi|^{p(x)} dx - \frac{\lambda t^{q^-+\epsilon_0}}{q^+}\int_{\Omega_0} m(x) |\varphi|^{q(x)} dx. \end{aligned}$$

Therefore,

$$J_\lambda(t\varphi) < 0,$$

for $0 < t < \sigma^{1/p^- - q^- - \epsilon_0}$ with

$$0 < \sigma < \min\left\{1, \frac{\lambda p^-}{q^+}.\frac{\int_{\Omega_0} m(x) |\varphi|^{q(x)} dx}{\int_\Omega |\nabla\varphi|^{p(x)} dx}\right\}.$$

Finally, we remark that $\int_\Omega |\nabla\varphi|^{p(x)} dx > 0$. Indeed, it is obvious that

$$\int_{\Omega_0} m(x) |\varphi|^{q(x)} dx \leq \int_\Omega m(x) |\varphi|^{q(x)} dx \leq \int_{\Omega_0} m(x) |\varphi|^{q^-} dx.$$

On the other hand, from (3.1) we know that there exists a positive constant c such that

$$\int_{\Omega_0} m(x) |\varphi|^{q^-} dx \leq c\|\varphi\|.$$

From the above inequalities we get $\|\varphi\| > 0$. Using the relations (1.2) – (1.3), we deduce that $\int_\Omega |\nabla\varphi|^{p(x)}\,dx > 0$. The proof is complete. $\square$

Proof of Theorem 3.1. From Lemma 3.2, we infer that there exists a ball centered at the origin $\overline{B_\rho(0)} \subset W_0^{1,p(x)}(\Omega)$, such that

$$\inf_{\partial B_\rho(0)} J_\lambda > 0.$$

Furthermore, by Lemma 3.3, we know that there exists $\varphi \in W_0^{1,p(x)}(\Omega)$ such that $J_\lambda(t\varphi) < 0$ for $t > 0$ small enough. Therefore, considering also inequality (3.3), we obtain that

$$-\infty < c := \inf_{\overline{B_\rho(0)}} J_\lambda < 0.$$

Let choose $\varepsilon > 0$. Then, it follows

$$0 < \varepsilon \leq \inf_{\partial B_\rho(0)} J_\lambda - \inf_{\overline{B_\rho(0)}} J_\lambda.$$

Now, if we apply the Ekeland‘s variational principle [6] to the functional $J_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$, it follows that there exists $u_\varepsilon \in \overline{B_\rho(0)}$ such that

$$\begin{aligned} J_\lambda(u_\varepsilon) &< \inf_{\overline{B_\rho(0)}} J_\lambda + \varepsilon, \\ J_\lambda(u_\varepsilon) &< J_\lambda(u) + \varepsilon\|u - u_\varepsilon\|, \quad u_\varepsilon \neq u. \end{aligned}$$

By the fact that

$$J_\lambda(u_\varepsilon) < \inf_{\overline{B_\rho(0)}} J_\lambda + \varepsilon < \inf_{B_\rho(0)} J_\lambda + \varepsilon < \inf_{\partial B_\rho(0)} J_\lambda,$$

we can infer that $u_\varepsilon \in B_\rho(0)$.

Now, let define $\Phi_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by $\Phi_\lambda(u) = J_\lambda(u) + \varepsilon\|u - u_\varepsilon\|$. It is not difficult to see that u_ε is a minimum point of Φ_λ, and thus

$$\frac{\Phi_\lambda(u_\varepsilon + t\cdot v) - \Phi_\lambda(u_\varepsilon)}{t} \geq 0,$$

for $t > 0$ small enough and any $v \in B_1(0)$. By the above expression, we have

$$\frac{J_\lambda(u_\varepsilon + t\cdot v) - J_\lambda(u_\varepsilon)}{t} + \varepsilon\|v\| \geq 0.$$

Letting $t \to 0$, we have

$$\langle J_\lambda'(u_\varepsilon), v\rangle + \varepsilon\|v\| > 0,$$

and this implies that $\|J'_\lambda(u_\varepsilon)\| \leq \varepsilon$. So, we deduce that there exists a sequence $\{u_n\} \subset B_\rho(0)$ such that

$$J_\lambda(u_n) \to c = \inf_{\overline{B_\rho(0)}} J_\lambda < 0 \quad \text{and} \quad J'_\lambda(u_n) \to 0.$$

Hence, we have that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Thus, there exists $u \in W_0^{1,p(x)}(\Omega)$ such that, up to a subsequence, $\{u_n\}$ converges weakly to u in $W_0^{1,p(x)}(\Omega)$ so $\langle J'_\lambda(u_n), u_n - u\rangle \to 0$. Therefore, we can write

$$\begin{aligned}\langle J'_\lambda(u_n), u_n - u\rangle = &\int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u)\, dx \\ &- \lambda \int_\Omega m(x) |u_n|^{q(x)-2} u_n (u_n - u)\, dx \to 0.\end{aligned}$$

Using (P_1), (1.1) and Proposition 2.2, we get the compact embedding

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}_{m(x)}(\Omega)$$

(see [12], Theorem 2.8). Then, we obtain that

$$\int_\Omega m(x) |u_n|^{q(x)-2} u_n (u_n - u)\, dx \to 0,$$

and hence,

$$\int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u)\; dx \to 0.$$

Therefore, by Proposition 2.3, we get $u_n \to u$ (strongly) in $W_0^{1,p(x)}(\Omega)$, so we conclude that u is a nontrivial weak solution for problem (P).

The proof of Theorem 3.1 is complete. □

References

[1] S.N. Antontsev, S.I. Shmarev, A model porous medium equation with variable exponent of nonlinearity: existence, uniqueness and localization properties of solutions, *Nonlin. Anal.*, **60** (2005), 515-545.

[2] M. Avci, Ni-Serrin type equations arising from capillarity phenomena with non-standard growth, *Boundary Value Problems* (2013), 55.

[3] M. Avci, Existence and multiplicity of solutions for Dirichlet problems involving the $p(x)$-Laplace operator, *Electronic Journal of Differential Equations*, **2013**, No. 14 (2013), 1-9.

[4] B. Cekic, R.A. Mashiyev, Existence and localization results for p(x)-Laplacian via topological methods, *Fixed Point Theory Appl.* (2010), Article ID 120646.

[5] L. Diening, P. Harjulehto, P. Hasto, M. Ruzicka, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Mathematics, **2017** (2011), **doi:** 10.1007/978-3-642-18363-8.

[6] I. Ekeland, On the variational principle, *J. Math. Anal. Appl.*, **47** (1974), 324–353.

[7] X.L. Fan, J.S. Shen, D. Zhao, Sobolev embedding theorems for spaces $W^{k,p(\cdot)}(\Omega)$, *J. Math. Anal. Appl.*, **262** (2001), 749-760.

[8] X.L. Fan, D. Zhao, On the spaces $L^{p(x)}$ and $W^{m,p(x)}$, *J. Math. Anal. Appl.*, **263** (2001), 424-446.

[9] X. Fan, Q. H. Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problems, *Nonlinear Anal.*, **52** (2003), 1843-1852.

[10] P. Hästö, The $p(x)$-Laplacian and applications, *J. Anal.*, **15** (2007), 53-62.

[11] O. Kováčik, J. Răkosnik, On spaces $L^{p(x)}$ and $W^{1,p(x)}$, *Czechoslovak Math. J.*, **41**, No. 116 (1991), 592-618.

[12] R.A. Mashiyev, S. Ogras, Z. Yucedag, M. Avci, The Nehari manifold approach for Dirichlet problem involving the $p(x)$-Laplacian equation, *J. Korean Math. Soc.*, **47**, No. 4 (2010), 845-860.

[13] M. Mihăilescu, V. Rădulescu, On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent, *Proceedings Amer. Math. Soc.*, **135**, No. 9 (2007), 2929-2937.

[14] M. Růžička, *Electrorheological fluids: Modeling and Mathematical Theory*, Lecture Notes in Mathematics, **1748**, Springer-Verlag, Berlin (2000).

[15] V.V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, *Math. USSR. Izv.*, **9** (1987), 33-66.