

DIRICHLET PROBLEM FOR ELLIPTIC EQUATIONS IN WEIGHTED SOBOLEV SPACES

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Abstract: We study the Dirichlet problem for uniformly elliptic second order linear differential equations with discontinuous coefficients in an unbounded domain of \mathbb{R}^n , $n \geq 2$, in weighted Sobolev spaces. An a priori bound and an existence and uniqueness result are proved.

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1. Introduction

The suitability of weighted Sobolev spaces in the theory of PDEs with prescribed boundary conditions is well known. In particular, they find a natural field of application in the case of unbounded domains, thanks to the introduction of an opportune weight function that controls the behaviour of the solution at infinity.

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In this framework, given an unbounded open subset Ω of \mathbb{R}^n , $n \geq 2$, we study the uniformly elliptic second order linear differential operator with discontinuous coefficients

$$L = - \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} + a,$$

and the related Dirichlet problem

$$\begin{cases} u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}_s^{1,p}(\Omega), \\ Lu = f, \quad f \in L_s^p(\Omega), \end{cases} \tag{1.1}$$

where $p > 1$, $s \in \mathbb{R}$, and $W_s^{2,p}(\Omega)$, $\overset{\circ}{W}_s^{1,p}(\Omega)$ and $L_s^p(\Omega)$ are some classes of weighted Sobolev and Lebesgue spaces recently introduced in [18], where a complete account of their properties can be found. In these spaces, the weight ρ^s is a power of a function ρ of class $C^2(\bar{\Omega})$ such that $\rho : \bar{\Omega} \rightarrow \mathbb{R}_+$ and

$$\sup_{x \in \Omega} \frac{|\partial^\alpha \rho(x)|}{\rho(x)} < +\infty \quad \forall |\alpha| \leq 2,$$

$$\lim_{|x| \rightarrow +\infty} \left(\rho(x) + \frac{1}{\rho(x)} \right) = +\infty \quad \text{and} \quad \lim_{|x| \rightarrow +\infty} \frac{\rho_x(x) + \rho_{xx}(x)}{\rho(x)} = 0.$$

As an example, one can think of the function

$$\rho(x) = (1 + |x|^2)^t, \quad t \in \mathbb{R} \setminus \{0\}.$$

Concerning the hypotheses on the coefficients of the operator L , we consider the very general case that the a_{ij} are bounded, are locally VMO, and are “close” at infinity to certain functions e_{ij} having vanishing mean oscillation, that is a kind of continuity in the average sense and not in the pointwise sense. As for the lower-order terms coefficients we suppose that they verify suitable regularity hypotheses and have a prescribed behaviour at the infinity.

Our main result is the following weighted a priori estimate:

$$\|u\|_{W_s^{2,p}(\Omega)} \leq c \left(\|Lu\|_{L_s^p(\Omega)} + \|u\|_{L^p(\Omega_1)} \right) \quad \forall u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}_s^{1,p}(\Omega),$$

where $c \in \mathbb{R}_+$ is a constant whose dependence is fully described and Ω_1 is a bounded open subset of Ω . As an application of this a priori bound, an existence and uniqueness theorem for the solution of problem (1.1) has been established.

If $n \geq 3$, the no-weighted problem corresponding to (1.1) has been studied by several authors under various hypotheses on the a_{ij} . We quote here the first existence and uniqueness theorem regarding VMO coefficients, obtained in [13, 14], assuming that Ω is bounded, the $a_{ij} \in L^\infty(\Omega) \cap VMO(\Omega)$ and $a_i = a = 0$. This latter condition has been dropped later on in [30, 31]. Recently, these results have been extended to the case of unbounded open sets in no-weighted contests (see [8, 9] for $n \geq 3$, and [12] for $n = 2$) and also in weighted ones (see [2, 3, 4, 5, 6]). Variational problems can be found in [7, 16, 17, 19, 20, 21] and related weighted and no-weighted results are proved in [22, 23, 24].

2. Weight Functions and Weighted Spaces

This section is devoted to a class of weighted Sobolev spaces recently introduced in [18], where all the details concerning the properties stated below can be found. Once the definition is given, we recall a fundamental lemma allowing to use no-weighted results in order to pass to the weighted contest. Finally, we remind some auxiliary functions, somehow related to the weight and satisfying opportune properties, needed throughout this work.

From now on let Ω be an open subset of \mathbb{R}^n , $n \geq 2$, not necessarily bounded, and consider a weight function $\rho : \Omega \rightarrow \mathbb{R}_+$ such that $\rho \in C^2(\bar{\Omega})$ and

$$\sup_{x \in \Omega} \frac{|\partial^\alpha \rho(x)|}{\rho(x)} < +\infty \quad \forall |\alpha| \leq 2. \tag{2.1}$$

To fix the ideas, one can think of ρ as the function

$$\rho(x) = (1 + |x|^2)^t, \quad t \in \mathbb{R}.$$

For $k \in \mathbb{N}_0$, $p \in [1, +\infty[$ and $s \in \mathbb{R}$, and given a weight ρ satisfying (2.1), we define $W_s^{k,p}(\Omega)$ as the space of distributions u on Ω such that

$$\|u\|_{W_s^{k,p}(\Omega)} = \sum_{|\alpha| \leq k} \|\rho^s \partial^\alpha u\|_{L^p(\Omega)} < +\infty, \tag{2.2}$$

endowed with the norm in (2.2). Furthermore, we denote the closure of $C^\infty(\Omega)$ in $W_s^{k,p}(\Omega)$ by $\overset{\circ}{W}_s^{k,p}(\Omega)$ and put $W_s^{0,p}(\Omega) = L_s^p(\Omega)$.

One of the most important characteristics of the above mentioned weighted spaces is contained in the following lemma:

Lemma 2.1. *Let $k \in \mathbb{N}_0$, $p \in [1, +\infty[$ and $s \in \mathbb{R}$. If Ω has the segment property, then the map*

$$u \longrightarrow \rho^s u$$

defines a topological isomorphism from $W_s^{k,p}(\Omega)$ to $W^{k,p}(\Omega)$ and from $\overset{\circ}{W}_s^{k,p}(\Omega)$ to $\overset{\circ}{W}^{k,p}(\Omega)$.

In the sequel the additional assumptions below are also needed

$$\lim_{|x| \rightarrow +\infty} \left(\rho(x) + \frac{1}{\rho(x)} \right) = +\infty \quad \text{and} \quad \lim_{|x| \rightarrow +\infty} \frac{\rho_x(x) + \rho_{xx}(x)}{\rho(x)} = 0. \quad (2.3)$$

As an example, we can therefore consider

$$\rho(x) = (1 + |x|^2)^t, \quad t \in \mathbb{R} \setminus \{0\}.$$

To our purpose it is essential to associate to the weight ρ the function σ defined by

$$\begin{cases} \sigma = \rho & \text{if } \rho \rightarrow +\infty & \text{for } |x| \rightarrow +\infty, \\ \sigma = \frac{1}{\rho} & \text{if } \rho \rightarrow 0 & \text{for } |x| \rightarrow +\infty. \end{cases}$$

It is easily seen that σ verifies (2.1) too, and moreover

$$\lim_{|x| \rightarrow +\infty} \sigma(x) = +\infty, \quad \lim_{|x| \rightarrow +\infty} \frac{\sigma_x(x) + \sigma_{xx}(x)}{\sigma(x)} = 0.$$

Now, fix a cutoff function $f \in C^\infty(\bar{\mathbb{R}}_+)$ such that

$$0 \leq f \leq 1, \quad f(t) = 1 \text{ if } t \in [0, 1], \quad f(t) = 0 \text{ if } t \in [2, +\infty[$$

and set

$$\zeta_k : x \in \bar{\Omega} \longrightarrow f\left(\frac{\sigma(x)}{k}\right), \quad k \in \mathbb{N},$$

and

$$\Omega_k = \{x \in \Omega : \sigma(x) < k\}, \quad k \in \mathbb{N}.$$

One has that, introduced the sequence

$$\eta_k : x \in \bar{\Omega} \longrightarrow 2k \zeta_k(x) + (1 - \zeta_k(x))\sigma(x), \quad k \in \mathbb{N},$$

for any $k \in \mathbb{N}$

$$\sigma \sim \eta_k. \quad (2.4)$$

Furthermore, for any $k \in \mathbb{N}$,

$$\frac{(\eta_k)_x}{\eta_k} \leq c'_1 \sup_{\Omega \setminus \Omega_k} \frac{\sigma_x}{\sigma} \quad \text{in } \bar{\Omega}, \tag{2.5}$$

$$\frac{(\eta_k)_{xx}}{\eta_k} \leq c'_2 \sup_{\Omega \setminus \Omega_k} \frac{\sigma_x^2 + \sigma \sigma_{xx}}{\sigma^2} \quad \text{in } \bar{\Omega}, \tag{2.6}$$

and moreover

$$\lim_{k \rightarrow +\infty} \sup_{\Omega \setminus \Omega_k} \frac{\sigma_x(x) + \sigma_{xx}(x)}{\sigma(x)} = 0. \tag{2.7}$$

3. The Spaces of the Coefficients

Here we recall the definitions and the main properties of the functional spaces where the coefficients of our operator belong.

Let us give some notation. Let G be any Lebesgue measurable subset of \mathbb{R}^n and $\Sigma(G)$ the collection of all Lebesgue measurable subsets of G . For $F \in \Sigma(G)$, $|F|$ denotes its Lebesgue measure, χ_F its characteristic function and $D(F)$ the class of restrictions to F of functions $\zeta \in C^\infty(\mathbb{R}^n)$ with $\bar{F} \cap \text{supp } \zeta \subseteq F$. If $X(F)$ is a space of functions defined on F , $X_{\text{loc}}(F)$ denotes the class of all functions $g : F \rightarrow \mathbb{R}$ such that $\zeta g \in X(F)$ for any $\zeta \in D(F)$. Finally, for any $x \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$, we put $B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\}$, $B_r = B(0, r)$ and $F(x, r) = F \cap B(x, r)$.

In the sequel let us assume that Ω is an unbounded open subset of \mathbb{R}^n , $n \geq 2$.

If Ω is such that

$$|\Omega(x, r)| \geq A r^n \quad \forall x \in \Omega, \quad \forall r \in]0, 1], \tag{3.1}$$

where A is a positive constant independent of x and r , then we can consider the space $BMO(\Omega, \tau)$ ($\tau \in \mathbb{R}_+$) of functions $g \in L^1_{\text{loc}}(\bar{\Omega})$ such that

$$[g]_{BMO(\Omega, \tau)} = \sup_{\substack{x \in \Omega \\ r \in]0, \tau]}} \int_{\Omega(x, r)} |g - \int_{\Omega(x, r)} g| < +\infty,$$

with

$$\int_{\Omega(x, r)} g = |\Omega(x, r)|^{-1} \int_{\Omega(x, r)} g.$$

If $g \in BMO(\Omega) = BMO(\Omega, \tau_A)$, and

$$\tau_A = \sup \left\{ \tau \in \mathbb{R}_+ : \sup_{\substack{x \in \Omega \\ r \in]0, \tau]} \frac{r^n}{|\Omega(x, r)|} \leq \frac{1}{A} \right\},$$

we say that $g \in VMO(\Omega)$ if $[g]_{BMO(\Omega, \tau)} \rightarrow 0$ for $\tau \rightarrow 0^+$.

For more details on VMO functions we refer the reader to [25], where they were defined for the first time, and to [29] for details in the case of unbounded domains.

A function

$$\eta[g] :]0, 1] \longrightarrow \mathbb{R}_+$$

is called a modulus of continuity of g in $VMO(\Omega)$ if

$$[g]_{BMO(\Omega, \tau)} \leq \eta[g](\tau) \quad \forall \tau \in]0, 1], \quad \lim_{\tau \rightarrow 0^+} \eta[g](\tau) = 0.$$

For $\lambda \in [0, n[$, $q \in [1, +\infty[$, the space of Morrey type $M^{q, \lambda}(\Omega, \tau)$ ($\tau \in \mathbb{R}_+$) is the set of all functions g in $L^q_{loc}(\overline{\Omega})$ such that

$$\|g\|_{M^{q, \lambda}(\Omega, \tau)} = \sup_{\substack{r \in]0, \tau] \\ x \in \Omega}} r^{-\lambda/q} \|g\|_{L^q(\Omega(x, r))} < +\infty, \tag{3.2}$$

endowed with the norm defined in (3.2). These spaces generalize to unbounded domains the classical notion of Morrey spaces and were introduced in [28] (see also [10], for further characterizations).

Since for any $\tau_1, \tau_2 \in \mathbb{R}_+$, a function g belongs to $M^{q, \lambda}(\Omega, \tau_1)$ if and only if it belongs to $M^{q, \lambda}(\Omega, \tau_2)$ and the norms of g in these two spaces are equivalent, we can restrict our attention to the space $M^{q, \lambda}(\Omega) = M^{q, \lambda}(\Omega, 1)$.

We recall that one has

$$M^{q, \lambda}(\Omega) \subseteq M^{q_0, \lambda_0}(\Omega) \quad \text{if } q_0 \leq q \text{ and } \frac{\lambda_0 - n}{q_0} \leq \frac{\lambda - n}{q}.$$

We denote by $M^{q, \lambda}_\circ(\Omega)$ the closure of $C^\infty_\circ(\Omega)$ in $M^{q, \lambda}(\Omega)$ and put $M^q(\Omega) = M^{q, 0}(\Omega)$ and $M^q_\circ(\Omega) = M^{q, 0}_\circ(\Omega)$.

For reader's convenience let us mention that, as observed in [27], for $q \in [1, +\infty[$, one has

$$f \in L^q_{loc}(\overline{\Omega}), \quad \lim_{|x| \rightarrow +\infty} f(x) = 0 \Rightarrow f \in M^q_\circ(\Omega). \tag{3.3}$$

In order to define the modulus of continuity of functions belonging to $M_\sigma^{q,\lambda}(\Omega)$, for $h \in \mathbb{R}_+$ and $g \in M^{q,\lambda}(\Omega)$, we set

$$F[g](h) = \sup_{\substack{E \in \Sigma(\Omega) \\ \sup_{x \in \Omega} |E(x,1)| \leq \frac{1}{h}}} \|g \chi_E\|_{M^{q,\lambda}(\Omega)}.$$

Then, we recall that given a function $g \in M^{q,\lambda}(\Omega)$ the following characterization holds:

$$g \in M_\sigma^{q,\lambda}(\Omega) \iff \lim_{h \rightarrow +\infty} (F[g](h) + \|(1 - \zeta_h)g\|_{M^{q,\lambda}(\Omega)}) = 0,$$

where ζ_h denotes a function of class $C_o^\infty(\mathbb{R}^n)$ such that

$$0 \leq \zeta_h \leq 1, \quad \zeta_h|_{\overline{B(0,h)}} = 1, \quad \text{supp } \zeta_h \subset B(0, 2h).$$

Thus, if g is a function in $M_\sigma^{q,\lambda}(\Omega)$ a modulus of continuity of g in $M_\sigma^{q,\lambda}(\Omega)$ is an application $\sigma_\sigma^{q,\lambda}[g] : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$F[g](h) + \|(1 - \zeta_h)g\|_{M^{q,\lambda}(\Omega)} \leq \sigma_\sigma^{q,\lambda}[g](h),$$

$$\lim_{h \rightarrow +\infty} \sigma_\sigma^{q,\lambda}[g](h) = 0.$$

In the sequel a modulus of continuity of a function g in $M_\sigma^q(\Omega)$ will be denoted by $\sigma_\sigma^q[g]$.

Theorem 3.1 of [15], adapted to our framework, guarantees the following embedding result:

Theorem 3.1. *Let $p > 1$ and assume that k and t are numbers such that:*

$$k \in \mathbb{N}, \quad t \geq p, \quad t \geq \frac{n}{k}, \quad \text{with } t > p \text{ if } p = \frac{n}{k}.$$

If Ω has the cone property and $g \in M^t(\Omega)$, then for any $u \in W_s^{k,p}(\Omega)$ we have $gu \in L_s^p(\Omega)$ and

$$\|gu\|_{L_s^p(\Omega)} \leq c \|g\|_{M^t(\Omega)} \|u\|_{W_s^{k,p}(\Omega)}, \tag{3.4}$$

with c positive constant depending only on Ω, n, k, p and t .

As a straightforward consequence, we can prove the corollary below:

Corollary 3.2. *Let $p > 1$ and assume that k and t are numbers such that:*

$$k \in \mathbb{N}, \quad t \geq p, \quad t \geq \frac{n}{k}, \quad \text{with } t > p \text{ if } p = \frac{n}{k}.$$

If Ω has the cone property and $g \in M^t_\circ(\Omega)$, then for any $\varepsilon \in \mathbb{R}_+$ there exist a positive constant $c(\varepsilon) \in \mathbb{R}_+$ and a bounded open subset $\Omega_\varepsilon \subset\subset \Omega$ with the cone property such that

$$\|gu\|_{L^p_s(\Omega)} \leq \varepsilon \|u\|_{W^{k,p}_s(\Omega)} + c(\varepsilon) \|u\|_{L^p(\Omega_\varepsilon)} \quad \forall u \in W^{k,p}_s(\Omega),$$

with $c(\varepsilon)$ and Ω_ε depending only on $\varepsilon, \Omega, n, k, p, \rho, s, t, \sigma^t_\circ[g]$.

Proof. Fix $\varepsilon > 0$ and let c be the constant in (3.4). Since $g \in M^t_\circ(\Omega)$, there exists $g_\varepsilon \in C^\infty_\circ(\Omega)$ such that $\|g - g_\varepsilon\|_{M^t(\Omega)} < \frac{\varepsilon}{c}$. Let Ω_ε be a bounded open subset of Ω , with the cone property, such that $\text{supp } g_\varepsilon \subset \Omega_\varepsilon$, by Theorem 3.1 and taking into account that $\rho \in C^2(\bar{\Omega})$ one has

$$\begin{aligned} \|gu\|_{L^p_s(\Omega)} &\leq c \|g - g_\varepsilon\|_{M^t(\Omega)} \|u\|_{W^{k,p}_s(\Omega)} + \|g_\varepsilon u\|_{L^p_s(\Omega_\varepsilon)} \\ &\leq \varepsilon \|u\|_{W^{k,p}_s(\Omega)} + \|g_\varepsilon \rho^s\|_{L^\infty(\Omega_\varepsilon)} \|u\|_{L^p(\Omega_\varepsilon)} \end{aligned}$$

for any u in $W^{k,p}_s(\Omega)$. This concludes the proof. □

4. An a Priori Bound

From now on, let $p > 1$ and $s \in \mathbb{R}$ and suppose that

(h_0) Ω is an unbounded open subset of \mathbb{R}^n , $n \geq 2$, with the uniform $C^{1,1}$ -regularity property.

We consider in Ω the differential operator

$$L = - \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} + a, \tag{4.1}$$

with the following conditions on the leading coefficients:

$$(h_1) \quad \begin{cases} a_{ij} = a_{ji} \in L^\infty(\Omega) \cap VMO_{loc}(\bar{\Omega}), & i, j = 1, \dots, n, \\ \exists \nu > 0 : \sum_{i,j=1}^n a_{ij} \xi_i \xi_j \geq \nu |\xi|^2 & \text{a.e. in } \Omega, \forall \xi \in \mathbb{R}^n. \end{cases}$$

Furthermore, assume that there exist some functions e_{ij} , $i, j = 1, \dots, n$,

and g , and a positive real constant μ such that

$$(h_2) \quad \begin{cases} e_{ij} = e_{ji} \in L^\infty(\Omega), (e_{ij})_{x_h} \in M_\circ^{t,\lambda}(\Omega), \quad i, j, h = 1, \dots, n, \\ \text{with } t > 2 \text{ if } p \leq 2, t = p \text{ if } p > 2 \text{ and } \lambda = 0 \text{ for } n = 2, \\ t \in [2, n] \quad \text{and } \lambda = n - t \text{ for } n > 2, \\ \sum_{i,j=1}^n e_{ij} \xi_i \xi_j \geq \mu |\xi|^2 \quad \text{a.e. in } \Omega, \quad \forall \xi \in \mathbb{R}^n, \\ \lim_{r \rightarrow +\infty} \sum_{i,j=1}^n \|e_{ij} - g a_{ij}\|_{L^\infty(\Omega \setminus B_r)} = 0, \\ \text{with } g \in L^\infty(\Omega), g_0 = \text{ess inf}_\Omega g > 0. \end{cases}$$

For the lower order term coefficients assume that

$$(h_3) \quad \begin{cases} a_i \in M_\circ^{t_1}(\Omega), \quad i = 1, \dots, n, \\ a = a' + a'', a' \in M_\circ^{t_2}(\Omega), a'' \in L^\infty(\Omega), a''_0 = \text{ess inf}_\Omega a'' > 0, \\ \text{with } t_1 = t \text{ and } t_2 = p \text{ for } n = 2, \\ \text{and with } t_1 \geq p, \quad t_1 \geq n, \quad t_1 > p \text{ if } p = n, \\ t_2 \geq p, \quad t_2 \geq n/2, \quad t_2 > p \text{ if } p = n/2 \text{ for } n > 2. \end{cases}$$

We remark that under assumptions $(h_0) - (h_3)$, in view of Theorem 3.1, the operator $L : W_s^{2,p}(\Omega) \rightarrow L_s^p(\Omega)$ is bounded.

Set

$$L_0 = - \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j},$$

the following preliminary bound holds true:

Lemma 4.1. *Suppose that assumptions $(h_0) - (h_2)$ hold and that a'' is as in (h_3) . Then, there are two real positive constants c and r_0 and a bounded open set $\Omega_0 \subset\subset \Omega$, with the cone property, such that*

$$\|u\|_{W^{2,p}(\Omega)} \leq c \left(\|(L_0 + a'')u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega_0)} \right) \quad \forall u \in W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega),$$

where c and Ω_0 depend on $n, p, \Omega, \nu, \mu, g_0, a''_0, t, \|a_{ij}\|_{L^\infty(\Omega)}, \|e_{ij}\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}, \|a''\|_{L^\infty(\Omega)}, \eta[\zeta_{2r_0} a_{ij}], \sigma_0^{t,\lambda}[(e_{ij})_x]$ and r_0 depends on $n, p, \Omega, \mu, g_0, a''_0, t, \|e_{ij}\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}, \|a''\|_{L^\infty(\Omega)}, \sigma_0^{t,\lambda}[(e_{ij})_x]$.

Proof. For $n \geq 3$ the thesis can be obtained applying Theorem 3.3 of [9] to the operator $L_0 + a''$.

For $n = 2$ one can proceed as in the proof of Theorem 3.3 of [9] replacing the applications of Theorem 5.1 of [11] with those of Lemma 3.1 of [12]. \square

Now we can prove our main a priori estimate.

Theorem 4.2. *Suppose that assumptions $(h_0) - (h_3)$ hold and that the weight function ρ satisfies (2.3). Then, there are two real positive constants c and r_0 and a bounded open set $\Omega_1 \subset\subset \Omega$, with the cone property, such that*

$$\|u\|_{W_s^{2,p}(\Omega)} \leq c \left(\|Lu\|_{L_s^p(\Omega)} + \|u\|_{L^p(\Omega_1)} \right) \quad \forall u \in W_s^{2,p}(\Omega) \cap \mathring{W}_s^{1,p}(\Omega),$$

where c and Ω_1 depend on $n, p, \Omega, \nu, \mu, g_0, a''_0, t, t_1, t_2, \rho, s, \|a_{ij}\|_{L^\infty(\Omega)}, \|e_{ij}\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}, \|a''\|_{L^\infty(\Omega)}, \eta[\zeta_{2r_0} a_{ij}], \sigma_0^{t,\lambda}[(e_{ij})_x], \sigma_0^{t_1}[a_i], \sigma_0^{t_2}[a']$ and r_0 depends on $n, p, \Omega, \mu, g_0, a''_0, t, \|e_{ij}\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}, \|a''\|_{L^\infty(\Omega)}, \sigma_0^{t,\lambda}[(e_{ij})_x]$.

Proof. Let $u \in W_s^{2,p}(\Omega) \cap \mathring{W}_s^{1,p}(\Omega)$. In view of Lemma 2.1

$$\rho^s u \in W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega).$$

Therefore, applying Lemma 4.1 we have that there exist two real numbers $c_0, r_0 \in \mathbb{R}_+$ and an open bounded subset $\Omega_0 \subset\subset \Omega$ with the cone property such that

$$\|\rho^s u\|_{W^{2,p}(\Omega)} \leq c_0 \left(\|(L_0 + a'')(\rho^s u)\|_{L^p(\Omega)} + \|\rho^s u\|_{L^p(\Omega_0)} \right),$$

where c_0 and Ω_0 depend on $n, p, \Omega, \nu, \mu, g_0, a''_0, t, \|a_{ij}\|_{L^\infty(\Omega)}, \|e_{ij}\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}, \|a''\|_{L^\infty(\Omega)}, \eta[\zeta_{2r_0} a_{ij}], \sigma_0^{t,\lambda}[(e_{ij})_x]$, and r_0 depends on $n, p, \Omega, \mu, g_0, a''_0, t, \|e_{ij}\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}, \|a''\|_{L^\infty(\Omega)}, \sigma_0^{t,\lambda}[(e_{ij})_x]$.

If we observe that

$$\begin{aligned} (L_0 + a'')(\rho^s u) &= \rho^s Lu - s(s-1)\rho^{s-2} \sum_{i,j=1}^n a_{ij} \rho_{x_i} \rho_{x_j} u - \\ & 2s\rho^{s-1} \sum_{i,j=1}^n a_{ij} \rho_{x_i} u_{x_j} - s\rho^{s-1} \sum_{i,j=1}^n a_{ij} \rho_{x_i x_j} u - \rho^s \sum_{i=1}^n a_i u_{x_i} - \rho^s a' u, \end{aligned}$$

we obtain

$$\|u\|_{W_s^{2,p}(\Omega)} \leq c_1 \left(\|Lu\|_{L_s^p(\Omega)} + \|u\|_{L^p(\Omega_0)} + \sum_{i,j=1}^n \|\rho^{s-2} \rho_{x_i} \rho_{x_j} u\|_{L^p(\Omega)} + \right.$$

$$\sum_{i,j=1}^n \|\rho^{s-1} \rho_{x_i} u_{x_j}\|_{L^p(\Omega)} + \sum_{i,j=1}^n \|\rho^{s-1} \rho_{x_i x_j} u\|_{L^p(\Omega)} + \quad (4.2)$$

$$\sum_{i=1}^n \|a_i u_{x_i}\|_{L_s^p(\Omega)} + \|a' u\|_{L_s^p(\Omega)},$$

where c_1 depends on the same parameters as c_0 and on ρ, s .

The regularity of ρ and the second hypothesis in (2.3) put us in a position to exploit (3.3) in order to deduce from Corollary 3.2 that for any $\varepsilon \in \mathbb{R}_+$ and $i, j = 1, \dots, n$ there exist $c_1(\varepsilon), c_2(\varepsilon), c_3(\varepsilon) \in \mathbb{R}_+$ and some bounded open subsets $\Omega_1(\varepsilon) \subset\subset \Omega, \Omega_2(\varepsilon) \subset\subset \Omega, \Omega_3(\varepsilon) \subset\subset \Omega$ with the cone property such that

$$\|\rho^{-2} \rho_{x_i} \rho_{x_j} u\|_{L_s^p(\Omega)} \leq \varepsilon \|u\|_{W_s^{2,p}(\Omega)} + c_1(\varepsilon) \|u\|_{L^p(\Omega_1(\varepsilon))}, \quad (4.3)$$

$$\|\rho^{-1} \rho_{x_i} u_{x_j}\|_{L_s^p(\Omega)} \leq \varepsilon \|u\|_{W_s^{2,p}(\Omega)} + c_2(\varepsilon) \|u_{x_j}\|_{L^p(\Omega_2(\varepsilon))}, \quad (4.4)$$

$$\|\rho^{-1} \rho_{x_i x_j} u\|_{L_s^p(\Omega)} \leq \varepsilon \|u\|_{W_s^{2,p}(\Omega)} + c_3(\varepsilon) \|u\|_{L^p(\Omega_3(\varepsilon))}, \quad (4.5)$$

where $c_1(\varepsilon), c_2(\varepsilon), c_3(\varepsilon), \Omega_1(\varepsilon), \Omega_2(\varepsilon), \Omega_3(\varepsilon)$ are dependent on $\varepsilon, \Omega, n, p, \rho, s$.

Moreover, using again Corollary 3.2 and Theorem 4.7 of [1] we have that there exist $c_4(\varepsilon), c_5(\varepsilon) \in \mathbb{R}_+$ and two bounded open sets $\Omega_4(\varepsilon) \subset\subset \Omega, \Omega_5(\varepsilon) \subset\subset \Omega$ with the cone property such that:

$$\|a_i u_{x_i}\|_{L_s^p(\Omega)} \leq \varepsilon \|u\|_{W_s^{2,p}(\Omega)} + c_4(\varepsilon) \|u_{x_i}\|_{L^p(\Omega_4(\varepsilon))} \leq \quad (4.6)$$

$$\varepsilon \|u\|_{W_s^{2,p}(\Omega)} + c_4(\varepsilon) \left(\|u_{xx}\|_{L^p(\Omega_4(\varepsilon))}^{\frac{1}{2}} \|u\|_{L^p(\Omega_4(\varepsilon))}^{\frac{1}{2}} + \|u\|_{L^p(\Omega_4(\varepsilon))} \right),$$

$$\|a' u\|_{L_s^p(\Omega)} \leq \varepsilon \|u\|_{W_s^{2,p}(\Omega)} + c_5(\varepsilon) \|u\|_{L^p(\Omega_5(\varepsilon))}, \quad (4.7)$$

where $c_4(\varepsilon)$ and $\Omega_4(\varepsilon)$ depend on $\varepsilon, \Omega, n, p, \rho, s, t_1, \sigma_0^{t_1}[a_i]$ and $c_5(\varepsilon)$ and $\Omega_5(\varepsilon)$ depend on $\varepsilon, \Omega, n, p, \rho, s, t_2, \sigma_0^{t_2}[a']$.

From (4.2)-(4.7) we have

$$\|u\|_{W_s^{2,p}(\Omega)} \leq c_2 \left(\|Lu\|_{L_s^p(\Omega)} + \|u\|_{L^p(\Omega_0)} + \varepsilon \|u\|_{W_s^{2,p}(\Omega)} + c_6(\varepsilon) \left(\|u_{xx}\|_{L^p(\Omega_6(\varepsilon))}^{\frac{1}{2}} \|u\|_{L^p(\Omega_6(\varepsilon))}^{\frac{1}{2}} + \|u\|_{L^p(\Omega_6(\varepsilon))} \right) \right),$$

where c_2 depends on the same parameters as $c_1, c_6(\varepsilon)$ is the maximum between the constants $c_1(\varepsilon), c_2(\varepsilon), c_3(\varepsilon), c_4(\varepsilon), c_5(\varepsilon)$, and $\Omega_6(\varepsilon)$ is a bounded open subset

of Ω with the cone property such that $\Omega_6(\varepsilon) \subset\subset \Omega$ and $\Omega_6(\varepsilon) \supset \Omega_1(\varepsilon) \cup \Omega_2(\varepsilon) \cup \Omega_3(\varepsilon) \cup \Omega_4(\varepsilon) \cup \Omega_5(\varepsilon)$. If we choose $\varepsilon = \frac{1}{2c_2}$, from the last inequality we deduce

$$\begin{aligned} \|u\|_{W_s^{2,p}(\Omega)} &\leq c_3 \left(\|Lu\|_{L_s^p(\Omega)} + \|u\|_{L_p(\Omega_0)} + \right. \\ &\left. \|u_{xx}\|_{L^p(\Omega_6(\varepsilon))}^{\frac{1}{2}} \|u\|_{L^p(\Omega_6(\varepsilon))}^{\frac{1}{2}} + \|u\|_{L^p(\Omega_6(\varepsilon))} \right), \end{aligned} \tag{4.8}$$

where c_3 depends on the same parameters as c_2 and on $t_1, t_2, \sigma_0^{t_1}[a_i], \sigma_0^{t_2}[a']$. Now, using Young's inequality, (4.8) can be rewritten as

$$\begin{aligned} \|u\|_{W_s^{2,p}(\Omega)} &\leq c_3 \left(\|Lu\|_{L_s^p(\Omega)} + \|u\|_{L_p(\Omega_0)} + \right. \\ &\left. \varepsilon' \|u_{xx}\|_{L^p(\Omega_6(\varepsilon))} + c(\varepsilon') \|u\|_{L^p(\Omega_6(\varepsilon))} + \|u\|_{L^p(\Omega_6(\varepsilon))} \right), \end{aligned} \tag{4.9}$$

where $c(\varepsilon')$ is a positive constant depending only on ε' .

From (4.9), choosing $\varepsilon' = \frac{1}{2c_3}$ and denoting by Ω_1 a bounded open subset of Ω with the cone property such that $\Omega_1 \subset\subset \Omega$ and $\Omega_1 \supset \Omega_6(\varepsilon) \cup \Omega_0$, we have the result. □

5. Uniqueness and Existence Results

In this section we want to obtain an existence and uniqueness result for a Dirichlet problem associated to the operator (4.1). A preliminary result, whose proof can be found in [23], is needed.

Lemma 5.1. *If hypothesis (h_0) is satisfied, then the Dirichlet problem*

$$\begin{cases} u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}_s^{1,p}(\Omega), \\ -\Delta u + bu = f, \quad f \in L_s^p(\Omega), \end{cases} \tag{5.1}$$

is uniquely solvable, with

$$b = 1 + \left| -s(s+1) \sum_{i=1}^n \frac{\sigma_{x_i}^2}{\sigma^2} + s \sum_{i=1}^n \frac{\sigma_{x_i x_i}}{\sigma} \right|, \tag{5.2}$$

if $\rho \rightarrow +\infty$ for $|x| \rightarrow +\infty$, or

$$b = 1 + \left| -s(s-1) \sum_{i=1}^n \frac{\sigma_{x_i}^2}{\sigma^2} - s \sum_{i=1}^n \frac{\sigma_{x_i x_i}}{\sigma} \right|, \tag{5.3}$$

if $\rho \rightarrow 0$ for $|x| \rightarrow +\infty$.

To obtain the solvability of the Dirichlet problem we are interested in stronger assumptions are required:

$$(h'_3) \quad \begin{cases} a_i \in M_{\circ}^{t_1}(\Omega), \quad i = 1, \dots, n, \\ a = a' + a'', \quad a' \in M_{\circ}^{t_2}(\Omega), \quad a'' \in L^{\infty}(\Omega), \quad a''_0 = \operatorname{ess\,inf}_{\Omega} a'' > 0, \\ \text{with } t_1 = t \text{ and } t_2 = p \text{ for } n = 2, \\ \text{and with } t_1 > n \text{ if } p \leq n, \quad t_1 = p \text{ if } p > n, \\ t_2 > n/2 \text{ if } p \leq n/2, \quad t_2 = p \text{ if } p > n/2 \text{ for } n > 2, \\ g \in Lip(\bar{\Omega}). \end{cases}$$

Theorem 5.2. *Suppose that assumptions $(h_0) - (h_2)$ and (h'_3) hold and that the weight function ρ satisfies (2.3). Then the problem*

$$\begin{cases} u \in W_s^{2,p}(\Omega) \cap \mathring{W}_s^{1,p}(\Omega), \\ Lu = 0, \end{cases} \tag{5.4}$$

admits only the zero solution.

Proof. Fix $u \in W_s^{2,p}(\Omega) \cap \mathring{W}_s^{1,p}(\Omega)$. If $\rho \rightarrow +\infty$ for $|x| \rightarrow +\infty$, $\sigma = \rho$, hence by Lemma 2.1 one has that $\sigma^s u \in W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega)$.

Thus, if we write, $\eta_k = \eta$, for a fixed $k \in \mathbb{N}$, since η and σ are equivalent, one also has that $\eta^s u \in W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega)$. Therefore, Theorem 5.2 of [12], for $n = 2$, and Theorem 4.3 of [9], for $n \geq 3$, together with the bounded inverse theorem (see Theorem 3.8 in [26]) give that there exists $c_0 \in \mathbb{R}_+$, independent of u and k , such that

$$\|\eta^s u\|_{W^{2,p}(\Omega)} \leq c_0 \|L(\eta^s u)\|_{L^p(\Omega)}. \tag{5.5}$$

Simple computations give then

$$\begin{aligned} L(\eta^s u) &= \eta^s Lu - s \sum_{i,j=1}^n a_{ij} \left((s-1)\eta^{s-2}\eta_{x_i}\eta_{x_j}u + \eta^{s-1}\eta_{x_i x_j}u + \right. \\ &\quad \left. 2\eta^{s-1}\eta_{x_i}u_{x_j} \right) + s \sum_{i=1}^n a_i \eta^{s-1}\eta_{x_i}u. \end{aligned} \tag{5.6}$$

Using (5.5) and (5.6) we deduce that

$$\begin{aligned} \|\eta^s u\|_{W^{2,p}(\Omega)} \leq & c_1 \left(\|\eta^s Lu\|_{L^p(\Omega)} + \sum_{i,j=1}^n (\|\eta^{s-2} \eta_{x_i} \eta_{x_j} u\|_{L^p(\Omega)} + \right. \\ & \left. \|\eta^{s-1} \eta_{x_i x_j} u\|_{L^p(\Omega)} + \|\eta^{s-1} \eta_{x_i} u_{x_j}\|_{L^p(\Omega)}) + \right. \\ & \left. \sum_{i=1}^n \|a_i \eta^{s-1} \eta_{x_i} u\|_{L^p(\Omega)} \right), \end{aligned} \tag{5.7}$$

with c_1 independent of u and k .

On the other hand, from Theorem 3.1 and (2.5) we get

$$\|a_i \eta^{s-1} \eta_{x_i} u\|_{L^p(\Omega)} \leq c_2 \sup_{\Omega \setminus \Omega_k} \frac{\sigma_x}{\sigma} \|a_i\|_{M^{t_1}(\Omega)} \|\eta^s u\|_{W^{1,p}(\Omega)}, \tag{5.8}$$

with c_2 independent of u and k .

Putting together (2.5), (2.6), (5.7) and (5.8), we obtain the bound

$$\begin{aligned} \|\eta^s u\|_{W^{2,p}(\Omega)} \leq & c_3 \left[\|\eta^s Lu\|_{L^p(\Omega)} + \left(\sup_{\Omega \setminus \Omega_k} \frac{\sigma_x^2 + \sigma \sigma_{xx}}{\sigma^2} + \right. \right. \\ & \left. \left. \sup_{\Omega \setminus \Omega_k} \frac{\sigma_x}{\sigma} \right) \|\eta^s u\|_{W^{2,p}(\Omega)} \right], \end{aligned} \tag{5.9}$$

with c_3 independent of u and k .

Observe that by (2.7) it follows that there exists $k_o \in \mathbb{N}$ such that

$$\left(\sup_{\Omega \setminus \Omega_{k_o}} \frac{\sigma_x^2 + \sigma \sigma_{xx}}{\sigma^2} + \sup_{\Omega \setminus \Omega_{k_o}} \frac{\sigma_x}{\sigma} \right) \leq \frac{1}{2c_3}. \tag{5.10}$$

Therefore, if we still denote by η the function η_{k_o} , combining (5.9) and (5.10) we obtain

$$\|\eta^s u\|_{W^{2,p}(\Omega)} \leq 2c_3 \|\eta^s Lu\|_{L^p(\Omega)}.$$

This, together with (2.4) written for $k = k_o$ and Lemma 2.1 applied considering σ as weight function, gives

$$\sum_{|\alpha| \leq 2} \|\sigma^s \partial^\alpha u\|_{L^p(\Omega)} \leq c_4 \|\sigma^s Lu\|_{L^p(\Omega)}, \tag{5.11}$$

with c_4 independent of u . This means that

$$\|u\|_{W_s^{2,p}(\Omega)} \leq c \|Lu\|_{L_s^p(\Omega)}, \quad \forall u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}_s^{1,p}(\Omega),$$

with c independent of u . Thus, problem (5.4) admits only the zero solution.

If $\rho \rightarrow 0$ for $|x| \rightarrow +\infty$, then $\sigma = \rho^{-1}$, thus, always in view of the isomorphism of Lemma 2.1, one has that $\sigma^{-s}u \in W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega)$. Therefore arguing as to get (5.11) one obtains

$$\sum_{|\alpha| \leq 2} \|\sigma^{-s} \partial^\alpha u\|_{L^p(\Omega)} \leq c_5 \|\sigma^{-s} Lu\|_{L^p(\Omega)}. \tag{5.12}$$

This concludes the proof of Theorem 5.2. □

We are now in a position to establish the solvability of our weighted problem by means of the following existence and uniqueness result:

Theorem 5.3. *Suppose that assumptions $(h_0) - (h_2)$ and (h'_3) hold and that the weight function ρ satisfies (2.3). Then the problem*

$$\begin{cases} u \in W_s^{2,p}(\Omega) \cap \mathring{W}_s^{1,p}(\Omega) \\ Lu = f, \quad f \in L_s^p(\Omega), \end{cases} \tag{5.13}$$

is uniquely solvable.

Proof. For each $\tau \in [0, 1]$ we put

$$L_\tau = \tau L + (1 - \tau)(-\Delta + b),$$

with b defined in (5.2) if $\rho \rightarrow +\infty$ for $|x| \rightarrow +\infty$, or in (5.3) if $\rho \rightarrow 0$ for $|x| \rightarrow +\infty$.

Let us start observing that the function

$$\tau \in [0, 1] \longmapsto L_\tau \in B(W_s^{2,p}(\Omega) \cap \mathring{W}_s^{1,p}(\Omega), L_s^p(\Omega))$$

is continuous, in view of Theorem 3.1. Moreover, by Theorem 4.2 one has that the operator L_τ has closed range and by Theorem 5.2 it also has null kernel. Thus, Lemma 4.1 of [9] applies giving that there exists a positive real number c such that

$$\|u\|_{W_s^{2,p}(\Omega)} \leq c \|L_\tau u\|_{L_s^p(\Omega)} \tag{5.14}$$

$$\forall u \in W_s^{2,p}(\Omega) \cap \mathring{W}_s^{1,p}(\Omega), \quad \forall \tau \in [0, 1].$$

Furthermore, by Lemma 5.1 problem (5.1) is uniquely solvable. Putting together this latter result and estimate (5.14) and using the method of continuity along a parameter, we finally get that problem (5.13) is uniquely solvable too. □

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