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A NOTE ON $\mathtt{F}_{\mathtt{a}}^{\mathtt{b}}$ TRANSFORMATION OF GENERALIZED FUNCTIONS

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Abstract: In this article we consider a new transformation named as $\mathtt{F}_{\mathtt{a}}^{\mathtt{b}}$ transformation. The distributional $\mathtt{F}_{\mathtt{a}}^{\mathtt{b}}$ transformation of compact support is defined as an analytic distribution. Two spaces of boehmians are also constructed. The generalized $\mathtt{F}_{\mathtt{a}}^{\mathtt{b}}$ transformation of a Boehmian is viewed as another Boehmian. Various properties of this transform are obtained in the generalized sense.

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1. Introduction

The general equation of an integral transform is usually defined by the integral equation

$$g(\xi) = \int_{\mathtt{a}}^{\mathtt{b}} f(y)\,\mathbb{K}(\xi, y)\,dy, \tag{1}$$

where $\mathbb{K}(\xi, t)$ is called the integral kernel of the transform.

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In this note, we consider an integral transform given by

$$\mathtt{F}_{\mathtt{a}}^{\mathtt{b}} f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathtt{R}} f(y) (\mathtt{a} \cos \xi y + \mathtt{b} \sin \xi y)\, dy, \tag{2}$$

where $\mathtt{a}$ and $\mathtt{b}$ are constants.

The inversion formula of (2) from which can be recovered, giving

$$f(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathtt{R}} \mathtt{F}_{\mathtt{a}}^{\mathtt{b}} f(\xi) (\mathtt{a} \cos \xi y + \mathtt{b} \sin \xi y)\, d\xi. \tag{3}$$

Putting $\mathtt{a} = 1$ and $\mathtt{b} = i$ in (2) and (3), we respectively arrive at the Fourier transformation pair [6-8]

$$\mathtt{F} f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathtt{R}} f(y) (\cos \xi y + i \sin \xi y)\, dy \tag{4}$$

and

$$f(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathtt{R}} \mathtt{F}(\xi) (\cos y\xi - i \sin y\xi)\, d\xi. \tag{5}$$

For $\mathtt{a} = 1$ and $\mathtt{b} = 1$, we obtain the Hartley transform pair [17, 1,3,11,13-14]

$$\mathtt{H} f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathtt{R}} f(y) (\cos \xi y + \sin \xi y)\, dy \tag{6}$$

and

$$f(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathtt{R}} \mathtt{H} f(\xi) (\cos \xi y + \sin \xi y)\, d\xi. \tag{7}$$

Similarly, choice of $\mathtt{a} = 0$ and $\mathtt{b} = 1$, leads to the Fourier sine transform pairs [5-8]

$$\mathtt{F}_{\mathtt{s}} f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathtt{R}} f(y) \sin \xi y dy. \tag{8}$$

and

$$f(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathtt{R}} (\mathtt{F}_{\mathtt{s}} f)(\xi) \sin \xi y d\xi. \tag{9}$$

and, that of $\mathtt{a} = 1$ and $\mathtt{b} = 0$ leads to the Fourier cosine transform

$$\mathtt{F}_{\mathtt{c}} f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathtt{R}} f(y) \cos \xi y dy \tag{10}$$

with the inversion formula

$$f(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathtt{R}} \mathtt{F}_{\mathtt{c}} f(\xi) \cos \xi y d\xi. \tag{11}$$

In notations, above equations are summarized to mean:

Remark 1. Let f be $\mathtt{L}^1$function then

$$\mathtt{F}_1^0 f(\xi) = \mathtt{F}_{\mathtt{c}} f(\xi), \mathtt{F}_0^1 f(\xi) = \mathtt{F}_{\mathtt{s}} f(\xi), \mathtt{F}_1^1 f(\xi) = \mathtt{H} f(\xi)$$

and

$$\mathtt{F}_1^i f(\xi) = \mathtt{F} f(\xi)$$

where $\xi \in \mathtt{R}$.

2. $\mathtt{F}_{\mathtt{a}}^{\mathtt{b}}$ of Distributions

Denote by $\mathtt{E}$ the space of smooth functions over $\mathtt{R}$ and $\acute{\mathtt{E}}$ be the conjugate (dual) space of $\mathtt{E}$ of distributions of bounded support.

For real numbers $\mathtt{a}$ and $\mathtt{b}$ in $\mathtt{R}$ we indeed see that

$$\mathtt{a} \cos \xi y, \mathtt{b} \sin \xi y \in \mathtt{E}(\mathtt{R}). \tag{12}$$

Hence, with this conclusion, (12), we define the distributional $\mathtt{F}_{\mathtt{a}}^{\mathtt{b}}$ transform of $f \in \acute{\mathtt{E}}(\mathtt{R})$ of compact support by means of the kernel method to be

$$\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f(\xi) = \mathtt{a} \langle f(y), \cos \xi y \rangle + \mathtt{b} \langle f(y), \sin \xi y \rangle. \tag{13}$$

Following are true.

Theorem 2. *Let $f \in \acute{\mathtt{E}}(\mathtt{R})$ then we have*

(i) $\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f$ is linear .

(ii) $\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f$ is continuous .

(iii) $\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f$ is one to one .

Proof of (i) : Let $f, g \in \acute{\mathtt{E}}(\mathtt{R})$ and $\gamma \in \mathtt{R}$ then

$$\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} (f+g)(\xi) = \mathtt{a} \langle (f+g)(y), \cos \xi y \rangle + \mathtt{b} \langle (f+g)(y), \sin \xi y \rangle.$$

Properties of distributions [12] and (13) give $\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} (f+g)(\xi) = \widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f(\xi) + \widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} g(\xi)$. Further, $\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} (\gamma f)(\xi) = \mathtt{a} \langle \gamma f(y), \cos \xi y \rangle + \mathtt{b} \langle \gamma f(y), \sin \xi y \rangle = \gamma \widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f(\xi)$.

This simply justifies Part (i) of the theorem.

To prove Part (ii) Let $(f_n) \in \acute{\mathtt{E}}(\mathtt{R})$ be such that $f_n \to 0$ as $n \to \infty$ then $\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f_n(\xi) = \mathtt{a}\langle f_n(y), \cos\xi y\rangle + \mathtt{b}\langle f_n(y), \sin\xi y\rangle \to 0$ as $n \to \infty$.

Proof of Part (iii) follows from similar technique.

The theorem is therefore completely proved.

Theorem 3. *Let* $f \in \acute{\mathtt{E}}(\mathtt{R})$ *then* $\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f$ *is analytic and*

$$\mathcal{D}_{\xi}^{k} \widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f(\xi) = \left\langle f(y), \mathcal{D}_{\xi}^{k}(\mathtt{a}\cos\xi y + \mathtt{b}\sin\xi y)\right\rangle. \tag{14}$$

See [12] for similar proof.

Similarly, in particular case, for $\mathtt{a} = 0$ and $\mathtt{b} = 1$, we reach to corresponding formulae of Fourier sine and Fourier cosine transforms as ,

$$\widehat{\mathtt{F}}_{\mathtt{s}} f(\xi) := \langle f(y), \sin\xi y\rangle \text{ and } \widehat{\mathtt{F}}_{\mathtt{c}} f(\xi) := \langle f(y), \cos\xi y\rangle \tag{15}$$

where $\mathtt{a} = 1$ and $\mathtt{b} = 0$.

Corrolary 4. *Let* $f \in \acute{\mathtt{E}}(\mathtt{R})$ *then*

(i) $\widehat{\mathtt{F}}_{\mathtt{s}} f$ *is analytic and* $\mathcal{D}_{\xi}^{k} \widehat{\mathtt{F}}_{\mathtt{s}} f(\xi) = \left\langle f(y), \mathcal{D}_{\xi}^{k} \sin\xi y\right\rangle$.

(ii) $\widehat{\mathtt{F}}_{\mathtt{c}} f$ *is analytic and* $\mathcal{D}_{\xi}^{k} \widehat{\mathtt{F}}_{\mathtt{c}} f(\xi) = \left\langle f(y), \mathcal{D}_{\xi}^{k} \cos\xi y\right\rangle$.

(iii) $\widehat{\mathtt{F}}_{\mathtt{c}}, \widehat{\mathtt{F}}_{\mathtt{s}}$ *are linear, continuous and one to one.*

3. Boehmian Spaces

In this section we extend the $\mathtt{F}_{\mathtt{a}}^{\mathtt{b}}$ transform to a context of Boehmian spaces. We see that the $\mathtt{F}_{\mathtt{a}}^{\mathtt{b}}$ transform of a Boehmian is not closed on that space but, on the other hand, it belongs to certain image space. With this information, it is needful to describe the space where all images lie. To this end, we devide this section into four subsections.

3.1. General Construction

Later, in 1983, boehmians are objects obtained by an abstract algebraic construction to generalized distributions [20]. The original construction was motivated by regular operators [4] . boehmians are subclass of Mikusinski operators, that are defined as equivalence classes of convolution quotients of functions.

Since boehmians were introduced, the framework of Boehmian has be used to define a variety of spaces of generalized functions and generalized integral transforms on those spaces.

One of the most youngest generalization of functions, and more particularly of distributions, is the theory of boehmians. The name Boehmian space is given to all objects defined by an abstract construction similar to that of field of quotients. The construction applied to function spaces yields various spaces of generalized functions.

For linear space $\mathtt{Y}$ and a subspace $\mathtt{X}$ of $\mathtt{Y}$, assume, to all pair $(f,\phi),(g,\psi)$ of elements, $f,g\in\mathtt{Y},\phi,\psi\in\mathtt{X}$, is assigned the products $f\star\phi, g\star\psi$ such that the following conditions are satisfied:

(1) $\phi\star\psi\in\mathtt{X}$ *and* $\phi\star\psi=\psi\star\phi$.

(2) $(f\star\phi)\star\psi=f\star(\phi\star\psi)$.

(3) $(f+g)\star\phi=f\star\phi+g\star\phi$

(4) $k(f\star\phi)=(kf)\star\phi=f\star(k\phi), k\in\mathtt{R}$.

Let Δ be a family of sequences from $\mathtt{X}$ such that for $f,g\in\mathtt{Y}$ then :

(5)*If* $(\epsilon_n)\in\Delta$ *and* $f\star\epsilon_n=g\star\epsilon_n$,$n=1,2,...,$ *then* $f=g$.

(6) $(\epsilon_n),(\tau_n)\in\Delta\Rightarrow(\epsilon_n\star\tau_n)\in\Delta$.

Elements of Δ are called *delta* sequences.

Consider the class $\mathtt{A}$ of pairs of sequences defined by

$$\mathtt{A}=\left\{((f_n),(\epsilon_n)):(f_n)\subseteq\mathtt{Y}^{\mathtt{N}},(\epsilon_n)\in\Delta\right\},$$

for each $n\in\mathtt{N}$.

The pair $((f_n),(\epsilon_n))\in\mathtt{A}$ is said to be quotient of sequences, denoted by $\dfrac{f_n}{\epsilon_n}$, if

$$f_n\star\epsilon_m=f_m\star\epsilon_n,\forall n,m\in\mathtt{N}.$$

Two quotients of sequences $\dfrac{f_n}{\epsilon_n}$ and $\dfrac{g_n}{\tau_n}$ are said to be equivalent, $\dfrac{f_n}{\epsilon_n}\sim\dfrac{g_n}{\tau_n}$, if

$$f_n\star\epsilon_m=g_m\star\tau_n,\forall n,m\in\mathtt{N}. \tag{16}$$

The relation $\sim$ is an equivalent relation on $\mathtt{A}$ and hence, splits $\mathtt{A}$ into equivalence classes. The equivalence class containing $\dfrac{f_n}{\epsilon_n}$ is denoted by $\left[\dfrac{f_n}{\epsilon_n}\right]$. These equivalence classes are called *boehmians* and the space of all boehmians is denoted by $\mathtt{B}(\mathtt{Y},\mathtt{X},\Delta,\star)$.

The sum and multiplication by a scalar of two boehmians can be defined in a natural way

$$\left[\frac{f_n}{\epsilon_n}\right]+\left[\frac{g_n}{\tau_n}\right]=\left[\frac{f_n\star\tau_n+g_n\star\epsilon_n}{\epsilon_n\star\tau_n}\right]$$

and

$$a\left[\frac{f_n}{\epsilon_n}\right]=\left[\frac{af_n}{\epsilon_n}\right], a \text{ being complex number.}$$

The operation $\star$ and differentiation are defined by $\left[\frac{f_n}{\epsilon_n}\right]\star\left[\frac{g_n}{\tau_n}\right]=\left[\frac{f_n\star g_n}{\epsilon_n\star\tau_n}\right]$ and $\mathcal{D}^\alpha\left[\frac{f_n}{\epsilon_n}\right]=\left[\frac{\mathcal{D}^\alpha f_n}{\epsilon_n}\right]$. Many a time, $\mathtt{Y}$ is equipped with a notion of convergence. The intrinsic relationship between the notion of convergence and the product $\star$ are given by:

(1) *If* $f_n\to f$ *as* $n\to\infty$ *in* $\mathtt{Y}$ *and,* $\phi\in\mathtt{X}$ *is any fixed element, then*

$$f_n\star\phi\to f\star\phi \text{ in } \mathtt{Y} \text{ as } n\to\infty.$$

(2) *If* $f_n\to f$ *as* $n\to\infty$ *in* $\mathtt{Y}$ *and* $(\epsilon_n)\in\Delta$*, then* $f_n\star\epsilon_n\to f$ in $\mathtt{Y}$ as $n\to\infty$.

The operation $\star$ is extended to $\mathtt{B}(\mathtt{Y},\mathtt{X},\Delta,\star)\times\mathtt{X}$ by :

If $\left[\frac{f_n}{\epsilon_n}\right]\in\mathtt{B}(\mathtt{Y},\mathtt{X},\Delta,\star)$ *and* $\phi\in\mathtt{X}$*, then* $\left[\frac{f_n}{\epsilon_n}\right]\star\phi=\left[\frac{f_n\star\phi}{\epsilon_n}\right]$.

In $\mathtt{B}(\mathtt{Y},\mathtt{X},\Delta,\star)$, two types of convergence, δ and Δ convergence, are defined as follows:

A sequence of boehmians (β_n) *in* $\mathtt{B}(\mathtt{Y},\mathtt{X},\Delta,\star)$ *is said to be* $\delta-$*convergent to a Boehmian* β *in* $\mathtt{B}(\mathtt{Y},\mathtt{X},\Delta,\star)$*, denoted by* $\beta_n\overset{\delta}{\to}\beta$*, if there exists a delta sequence* (ϵ_n) *such that* $(\beta_n\star\epsilon_n),(\beta\star\epsilon_n)\in\mathtt{Y},\forall k,n\in\mathtt{N}$*,and*

$$(\beta_n\star\epsilon_k)\to(\beta\star\epsilon_k) \text{ as } n\to\infty, \text{ in } \mathtt{Y}, \text{ for every } k\in\mathtt{N}.$$

The following is equivalent for the statement of $\delta-$*convergence*

The sequence $\beta_n\overset{\delta}{\to}\beta$ $(n\to\infty)$ *in* $\mathtt{B}(\mathtt{Y},\mathtt{X},\Delta,\star)$ *if and only if there is* $f_{n,k},f_k\in\mathtt{Y}$ *and* $\epsilon_k\in\Delta$ *such that* $\beta_n=\left[\frac{f_{n,k}}{\epsilon_k}\right],\beta=\left[\frac{f_k}{\epsilon_k}\right]$ *and for each* $k\in\mathtt{N}$,

$$f_{n,k}\to f_k \text{ as } n\to\infty \text{ in } \mathtt{Y}.$$

A sequence of boehmians (β_n) *in* $\mathtt{B}(\mathtt{Y},\mathtt{X},\Delta,\star)$ *is said to be* $\Delta-$*convergent to a Boehmian* β *in* $\mathtt{B}(\mathtt{Y},\mathtt{X},\Delta,\star)$*, denoted by* $\beta_n\overset{\Delta}{\to}\beta$*, if there exists a* $(\epsilon_n)\in\Delta$ *such that* $(\beta_n-\beta)\star\epsilon_n\in\mathtt{Y},\forall n\in\mathtt{N}$*, and* $(\beta_n-\beta)\star\epsilon_n\to 0$ *as* $n\to\infty$ *in* $\mathtt{Y}$.

See $[1,2,7,9,15,18,19,20]$ for more analysis.

3.2. Convolution Theorem of $\mathtt{F}_{\mathtt{a}}^{\mathtt{b}}$ (Theorem 5)

Let f and g be $\mathtt{L}^1$ functions defined on $\mathtt{R}$ and $*$ denote the usual convolution product

$$(f * g)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathtt{R}} f(y)\, g(t-y)\, dy. \tag{17}$$

then

$$\mathtt{F}_{\mathtt{a}}^{\mathtt{b}}(f * g)(\xi) = \mathtt{F}_{\mathtt{a}}^{\mathtt{b}} g(\xi)\, \mathtt{F}_{\mathtt{c}} f(\xi) + \mathtt{F}_{-\mathtt{a}}^{\mathtt{b}} g(\xi)\, \mathtt{F}_{\mathtt{s}} f(\xi).$$

Proof : Assume the hypothesis of the theorem is satisfied then using (10) for (17) and Fubinitz theorem imply

$$\begin{aligned}
\mathtt{F}_{\mathtt{a}}^{\mathtt{b}}(f * g)(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathtt{R}} (f * g)(t)\, (\mathtt{a} \cos \xi t + \mathtt{b} \sin \xi t)\, dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathtt{R}} \int_{\mathtt{R}} f(y)\, g(t-y)\, dy\, (\mathtt{a} \cos \xi t + \mathtt{b} \sin \xi t)\, dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathtt{R}} f(y) \int_{\mathtt{R}} g(t-y)\, (\mathtt{a} \cos \xi t + \mathtt{b} \sin \xi t)\, dt dy. \qquad (18)
\end{aligned}$$

The substitution $\eta = t - y$ changes (18), producing

$$\mathtt{F}_{\mathtt{a}}^{\mathtt{b}}(f * g)(x) = \int_{\mathtt{R}} f(y) \int_{\mathtt{R}} g(\eta)\, (\mathtt{a} \cos x(\eta + y) + \mathtt{b} \sin x(\eta + y))\, d\eta dy. \tag{19}$$

Invoking the identities $\cos \xi (\eta + y) = \cos \xi\eta \cos \xi y - \sin \xi\eta \sin \xi y$ and $\sin \xi (\eta + y)$ $= \sin \xi\eta \cos \xi y + \sin \xi y \cos x\eta$ in (19) implies

$$\begin{aligned}
\mathtt{F}_{\mathtt{a}}^{\mathtt{b}}(f * g)(\xi) &= \int_{\mathtt{R}} f(y) \int_{\mathtt{R}} g(\eta) \begin{pmatrix} \cos \xi y\, (\mathtt{a} \cos \xi\eta + \mathtt{b} \sin \xi\eta) \\ + \sin \xi y\, (\mathtt{b} \cos \xi\eta - \mathtt{a} \sin \xi\eta) \end{pmatrix} d\eta dy \\
&= \int_{\mathtt{R}} g(\eta)\, (\mathtt{a} \cos \xi\eta + \mathtt{b} \sin \xi\eta)\, d\eta \int_{\mathtt{R}} f(y) \cos \xi y dy \\
&\quad + \int_{\mathtt{R}} g(\eta)\, (\mathtt{b} \cos \xi\eta - \mathtt{a} \sin \xi\eta)\, d\eta \int_{\mathtt{R}} f(y) \sin \xi y dy \\
&= \mathtt{F}_{\mathtt{a}}^{\mathtt{b}} g(\xi)\, \mathtt{F}_{\mathtt{c}} f(\xi) + \mathtt{F}_{-\mathtt{a}}^{\mathtt{b}} g(\xi)\, \mathtt{F}_{\mathtt{s}} f(\xi).
\end{aligned}$$

Hence the theorem is proved.

Nevertheless, our transform does not possess a factorization property of Fourier convolution type, we, in next investigation, still pay attention to extending the cited transform to further space of boehmians.

3.3. The Boehmian Space $\mathtt{B}\left(\acute{\mathtt{E}}, \mathtt{D}, \boldsymbol{\Delta_1}, *\right)$

Denote by Δ_1 the collection of all sequenes (δ_n) from $\mathtt{D}$ with properties :

(1) $\int_{\mathtt{R}} \delta_n(y)\, dy = 1$;

(2) $\int_{\mathtt{R}} |\delta_n(y)|\, dy < \mathtt{M}, \mathtt{M}$ positive real number ; and

(3) $\sup \delta_n(y) \to 0$ as $n \to \infty$.

Such sequences, $(\delta_n) \in \Delta_1$, are called delta sequences or approximating identities which indeed correspond to the dirac delta function.

From literature, $\mathtt{D} \subset \mathtt{E} \subset \acute{\mathtt{E}}$, (In fact, $\mathtt{D}$ dense in $\mathtt{E}$ and $\mathtt{E}$ is dense in $\acute{\mathtt{E}}$), see [12].

Let $\mathtt{B}(\acute{\mathtt{E}}, \mathtt{D}, \Delta, *)$ be the Boehmian space with $\mathtt{Y} = \acute{\mathtt{E}}$ as a group, $\mathtt{X} = \mathtt{D}$ as a subgroup of $\acute{\mathtt{E}}$ and, $*$ being the usual convolution product as an operation for $\acute{\mathtt{E}}$ and $\mathtt{D}$.

3.4. The Boehmian Space $\mathtt{B}\left(\mathtt{G}_{\mathtt{a}}^{\mathtt{b}}, \mathtt{G}_{\mathtt{c}}, \mathtt{F}_{\mathtt{c}} \boldsymbol{\Delta_1}, \vee\right)$

Let us now consider another space of boehmians (generalized functions). Let $\mathtt{G}_{\mathtt{c}}$ be the set of Fourier cosine transforms $\mathtt{F}_{\mathtt{c}}$ of functions from $\mathtt{D}$ and $\mathtt{G}_{\mathtt{a}}^{\mathtt{b}}$ the set of $\mathtt{F}_{\mathtt{a}}^{\mathtt{b}}$ transforms of functions from $\acute{\mathtt{E}}$ and, define a mapping $\vee : \mathtt{G}_{\mathtt{a}}^{\mathtt{b}} \times \mathtt{G}_{\mathtt{c}} \to \mathtt{G}_{\mathtt{a}}^{\mathtt{b}}$ by

$$f \vee g = fg + \left(\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f_1\right)(\mathtt{F}_{\mathtt{c}} g_1) \tag{20}$$

where, $f_1 \in \acute{\mathtt{E}}$ and $g_1 \in \mathtt{D}$ are such that $\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f_1 = f$ and $\mathtt{F}_{\mathtt{c}} g_1 = g$.

With this operation (20), following theorem is justified.

Theorem 6. *Let $f_1 \in \acute{\mathtt{E}}$ and $g_1 \in \mathtt{D}$ be such that $\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f_1 = f$ and $\mathtt{F}_{\mathtt{c}} g_1 = g$ then*

$$f \vee g = \widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} (f_1 * g_1).$$

Detailed proof is as follows : Assume the requirements of the theorem, for some f_1 and g_1, are fulfilled then, by (20) and Theorem 5, we have

$$f \vee g = fg + \widehat{\mathtt{F}}_{-\mathtt{a}}^{\mathtt{b}} f_1 \mathtt{F}_{\mathtt{s}} g_1 = \widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f_1 \mathtt{F}_{\mathtt{c}} g_1 + \widehat{\mathtt{F}}_{-\mathtt{a}}^{\mathtt{b}} f_1 \mathtt{F}_{\mathtt{s}} g_1 = \widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} (f * g).$$

Hence the theorem.

Theorem 7. *Let $f_1 \in \acute{\mathtt{E}}$ and $g_1, h_1 \in \mathtt{D}$ where $\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f_1 = f, \mathtt{F}_{\mathtt{c}} g_1 = g, \mathtt{F}_{\mathtt{c}} h_1 = h$ then*

$$(f \vee g) \vee h = f \vee (g \vee h).$$

Proof. By (20) and Theorem 6 we write

$$\begin{aligned}(f \vee g) \vee h &= \widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} (f_1 * g_1) \vee h \\ &= \widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} ((f_1 * g_1) * h_1) \\ &= \widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} (f_1 * (g_1 * h_1)) \\ &= f \vee \widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} (g_1 * h_1) \\ &= f \vee (g \vee h)\,.\end{aligned}$$

Hence the proof is completed.

Theorem 8. *Let* $f_1, q_1 \in \acute{\mathtt{E}}, g_1 \in \mathtt{D}$ *and* $\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f_1 = f, \widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} q_1 = q \left(\in \mathtt{G}_{\mathtt{a}}^{\mathtt{b}}\right)$ *and* $\mathtt{F}_{\mathtt{c}} g_1 = g\,(\in \mathtt{G}_{\mathtt{c}})$ *then*

$$(f + q) \vee g = f \vee g + q \vee g.$$

Proof. With the hypothesis of the theorem and linearity of $\mathtt{F}_{\mathtt{a}}^{\mathtt{b}}$ transform yields

$$\begin{aligned}(f + q) \vee g &= \left(\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f_1 + \widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} q_1\right) \vee g \\ &= \widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} (f_1 + q_1) \vee g.\end{aligned}$$

(20) then gives

$$\begin{aligned}(f + q) \vee g &= \widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} ((f_1 + q_1) * g_1) \\ &= \widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} (f_1 * g_1 + q_1 * g_1)\end{aligned}$$

Once again, linearity of $\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}}$ suggests $(f + q) \vee g = \widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} (f_1 * g_1) + \widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} (q_1 * g_1)$.
Employing Theorem 6 for above equation yields

$$(f + q) \vee g = f \vee g + q \vee g.$$

Theorem 9 *Let* $g_1, h_1 \in \mathtt{D}$ *with* $\mathtt{F}_{\mathtt{c}} g_1 = g, \mathtt{F}_{\mathtt{c}} h_1 = h \in \mathtt{G}_{\mathtt{a}}^{\mathtt{b}}$ *then*

$$g \vee h = h \vee g.$$

Proof. By commutitivity of $*$,we have

$$g \vee h = \mathtt{F}_{\mathtt{a}}^{\mathtt{b}} (g_1 * h_1) = \mathtt{F}_{\mathtt{a}}^{\mathtt{b}} (h_1 * g_1) = h \vee g.$$

The theorem is therefore proved.

As a prior step it was needed to varify that, $f \vee h \in \mathtt{G}_{\mathtt{a}}^{\mathtt{b}}$ for every $f = \mathtt{F}_{\mathtt{c}} f_1 \in \mathtt{G}_{\mathtt{c}}, g = \widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} g_1 \in \mathtt{G}_{\mathtt{a}}^{\mathtt{b}}$. Indeed, this is trivial from the fact $f \vee g = \widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} (f_1 * g_1) \in \mathtt{G}_{\mathtt{a}}^{\mathtt{b}}$ since $f_1 * g_1 \in \mathtt{E} \subset \acute{\mathtt{E}}$.

Denote by

$$\mathtt{F}_{\mathtt{c}} \Delta_1 := \Delta := \{\mathtt{F}_{\mathtt{c}} \delta_{1n} : \delta_{1n} \in \Delta_1\}$$

then we prove the following:

Theorem 10. *Let* $f \in \mathtt{G}_{\mathtt{a}}^{\mathtt{b}}, f = \widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f_1, f_1 \in \acute{\mathtt{E}}, (\delta_{1n}) \in \Delta_1, \delta_n = \mathtt{F}_{\mathtt{c}} \delta_{1n}$ *then if* $f \vee \delta_n \to 0$ as $n \to \infty$ *then* $f = 0$ a.e.

Proof : Let $(\delta_{1n}) \in \Delta_1$ then $f \vee \delta_n = \widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} (f_1 * \delta_{1n}) \to \widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f_1 \to 0$ as $n \to \infty$, because $f_1 * \delta_{1n} \to f_1$ as $n \to \infty$.

Hence

$$f \vee \delta_n \to \widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f_1 \to 0$$

as $n \to \infty$.

This complete the proof of the theorem.

Theorem 11. *Let* $\delta_n, \Psi_n \in \mathtt{F}_{\mathtt{c}} \Delta_1$ *be such that* $\delta_n = \mathtt{F}_{\mathtt{c}} \delta_{1n}$ *and* $\Psi_n = \mathtt{F}_{\mathtt{c}} \Psi_{1n}$ *then*

$$\delta_n \vee \Psi_n \in \mathtt{F}_{\mathtt{c}} \Delta_1.$$

Proof. By Theorem 6, $\delta_n \vee \Psi_n = \mathtt{F}_{\mathtt{a}}^{\mathtt{b}} (\delta_{1n} * \Psi_{1n}) \in \mathtt{F}_{\mathtt{c}} \Delta_1$, by the fact that $\delta_{1n} * \Psi_{1n} \in \Delta_1$.

The Boehmian space

$$\mathtt{B}\left(\mathtt{G}_{\mathtt{a}}^{\mathtt{b}}, \mathtt{G}_{\mathtt{c}}, \mathtt{F}_{\mathtt{c}} \Delta_1, \vee\right)$$

is constructed.

Sum and multiplication by scalars of two boehmians in $\mathtt{B}\left(\mathtt{G}_{\mathtt{a}}^{\mathtt{b}}, \mathtt{G}_{\mathtt{c}}, \mathtt{F}_{\mathtt{c}} \Delta_1, \vee\right)$ is defined as

$$\left[\frac{f_n}{\epsilon_n}\right] + \left[\frac{g_n}{\tau_n}\right] = \left[\frac{f_n \vee \tau_n + g_n \vee \epsilon_n}{\epsilon_n \vee \tau_n}\right]$$

and

$$a \left[\frac{f_n}{\epsilon_n}\right] = \left[\frac{a f_n}{\epsilon_n}\right], a \text{ being complex number.}$$

The operations $\vee$ and $\mathcal{D}^\alpha$ are defined on $\mathtt{B}\left(\mathtt{G}_{\mathtt{a}}^{\mathtt{b}}, \mathtt{G}_{\mathtt{c}}, \mathtt{F}_{\mathtt{c}} \Delta_1, \vee\right)$ as

$$\left[\frac{f_n}{\epsilon_n}\right] \vee \left[\frac{g_n}{\tau_n}\right] = \left[\frac{f_n \vee g_n}{\epsilon_n \vee \tau_n}\right] \text{ and } \mathcal{D}^\alpha \left[\frac{f_n}{\epsilon_n}\right] = \left[\frac{\mathcal{D}^\alpha f_n}{\epsilon_n}\right].$$

The operation $\vee$ is extended to $\mathtt{B}\left(\mathtt{G}_{\mathtt{a}}^{\mathtt{b}}, \mathtt{G}_{\mathtt{c}}, \mathtt{F}_{\mathtt{c}} \Delta_1, \vee\right) \times \mathtt{G}_{\mathtt{c}}$ as :

$$\left[\frac{f_n}{\epsilon_n}\right] \vee \phi = \left[\frac{f_n \vee \phi}{\epsilon_n}\right].$$

A sequence of boehmians (β_n) *in* $\mathtt{B}\left(\mathtt{G}_{\mathtt{a}}^{\mathtt{b}}, \mathtt{G}_{\mathtt{c}}, \mathtt{F}_{\mathtt{c}}\Delta_1, \vee\right)$ *is be* $\delta-$*convergent to* β *in* $\mathtt{B}\left(\mathtt{G}_{\mathtt{a}}^{\mathtt{b}}, \mathtt{G}_{\mathtt{c}}, \mathtt{F}_{\mathtt{c}}\Delta_1, \vee\right)$ *if there exists a delta sequence* $(\epsilon_n) \in \mathtt{F}_{\mathtt{c}}\Delta_1$ *such that* $(\beta_n \vee \epsilon_n), (\beta \vee \epsilon_n) \in \mathtt{Y}, \forall k, n \in \mathtt{N}$,*and*

$$(\beta_n \vee \epsilon_k) \to (\beta \vee \epsilon_k) \text{ as } n \to \infty, \text{ for every } k \in \mathtt{N}.$$

Or, equivalently,

The sequence $\beta_n \xrightarrow{\delta} \beta \ (n \to \infty)$ *in* $\mathtt{B}\left(\mathtt{G}_{\mathtt{a}}^{\mathtt{b}}, \mathtt{G}_{\mathtt{c}}, \mathtt{F}_{\mathtt{c}}\Delta_1, \vee\right)$ *if there is* $f_{n,k}, f_k \in \mathtt{G}_{\mathtt{a}}^{\mathtt{b}}$ *and* $\epsilon_k \in \mathtt{F}_{\mathtt{c}}\Delta_1$ *such that* $\beta_n = \left[\frac{f_{n,k}}{\epsilon_k}\right], \beta = \left[\frac{f_k}{\epsilon_k}\right]$ *and for each* $k \in \mathtt{N}$,

$$f_{n,k} \to f_k \text{ as } n \to \infty \text{ in } \mathtt{G}_{\mathtt{a}}^{\mathtt{b}}.$$

A sequence of boehmians (β_n) *in* $\mathtt{B}\left(\mathtt{G}_{\mathtt{a}}^{\mathtt{b}}, \mathtt{G}_{\mathtt{c}}, \mathtt{F}_{\mathtt{c}}\Delta_1, \vee\right)$ *is* $\Delta-$*convergent to a* β *in* $\mathtt{B}\left(\mathtt{G}_{\mathtt{a}}^{\mathtt{b}}, \mathtt{G}_{\mathtt{c}}, \mathtt{F}_{\mathtt{c}}\Delta_1, \vee\right)$ *if there exists a* $(\epsilon_n) \in \mathtt{F}_{\mathtt{c}}\Delta_1$ *such that* $(\beta_n - \beta) \vee \epsilon_n \in \mathtt{G}_{\mathtt{a}}^{\mathtt{b}}, \forall n \in \mathtt{N}$, *and* $(\beta_n - \beta) \vee \epsilon_n \to 0$ *as* $n \to \infty$ *in* $\mathtt{G}_{\mathtt{a}}^{\mathtt{b}}$.

From the context it is clear that $\mathtt{G}_{\mathtt{c}} \subseteq \mathtt{G}_{\mathtt{a}}^{\mathtt{b}}$ since $\mathtt{F}_{\mathtt{c}}f$ is particular case of $\mathtt{F}_{\mathtt{a}}^{\mathtt{b}}f$, for $\mathtt{a} = 1, \mathtt{b} = 0$, and $\mathtt{D}$ is dense in $\acute{\mathtt{E}}$.

4. $\mathtt{F}_{\mathtt{a}}^{\mathtt{b}}$ of Boehmians

Let $\beta = \left[\frac{f_{1n}}{\Psi_{1n}}\right]$ be in $\mathtt{B}(\acute{\mathtt{E}}, \mathtt{D}, \Delta_1, *)$. We define the extended $\mathtt{F}_{\mathtt{a}}^{\mathtt{b}}$ of β as the mapping in $\mathtt{B}\left(\mathtt{G}_{\mathtt{a}}^{\mathtt{b}}, \mathtt{G}_{\mathtt{c}}, \mathtt{F}_{\mathtt{c}}\Delta_1, \vee\right)$ given as

$$\widetilde{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}}\beta = \left[\frac{\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f_{1n}}{\mathtt{F}_{\mathtt{c}}\Psi_{1n}}\right]. \tag{21}$$

Definition in (21) is well-defined by next declaration :

Let $\beta_1 = \left[\frac{f_{1n}}{\Psi_{1n}}\right]$ and $\beta_2 = \left[\frac{g_{1n}}{\epsilon_{1n}}\right]$ be in $\mathtt{B}(\acute{\mathtt{E}}, \mathtt{D}, \Delta_1, *)$ such that $\beta_1 = \beta_2$. Then

$$f_{1n} * \epsilon_{1m} = g_{1m} * \Psi_{1n}. \tag{22}$$

Applying $\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}}$ to (22) yields

$$\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}}(f_{1n} * \epsilon_{1m}) = \widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}}(g_{1m} * \Psi_{1n}).$$

Applying Theorem 6 gives

$$\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f_{1n} \vee \mathtt{F}_{\mathtt{c}}\epsilon_{1m} = \widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} g_{1m} \vee \mathtt{F}_{\mathtt{c}}\Psi_{1n}. \tag{23}$$

From (23) we get

$$\frac{\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f_{1n}}{\mathtt{F}_{\mathtt{c}}\Psi_{1n}} \sim \frac{\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} g_{1m}}{\mathtt{F}_{\mathtt{c}}\epsilon_{1m}}. \tag{24}$$

(24) gives $\widetilde{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}}\beta_1 = \widetilde{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}}\beta_2$ in the sense of $\mathtt{B}\left(\mathtt{G}_{\mathtt{a}}^{\mathtt{b}}, \mathtt{G}_{\mathtt{c}}, \mathtt{F}_{\mathtt{c}}\Delta_1, \vee\right)$.

Theorem 12. $\widetilde{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} : \mathtt{B}\left(\acute{\mathtt{E}}, \mathtt{D}, \Delta_1, *\right) \to \mathtt{B}\left(\mathtt{G}_{\mathtt{a}}^{\mathtt{b}}, \mathtt{G}_{\mathtt{c}}, \mathtt{F}_{\mathtt{c}}\Delta_1, \vee\right)$ *is linear.*

Proof. Let $\beta_1 = \left[\frac{f_{1n}}{\Psi_{1n}}\right], \beta_2 = \left[\frac{g_{1n}}{\epsilon_{1n}}\right] \in \mathtt{B}\left(\acute{\mathtt{E}}, \mathtt{D}, \Delta_1, *\right)$ and $\alpha \in \mathbb{C}$, the set of complex numbers. Then

$$\begin{aligned}
\widetilde{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}}\left(\beta_1+\beta_2\right) &= \widetilde{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}}\left(\left[\frac{f_{1n}*\epsilon_{1n}+g_{1n}*\Psi_{1n}}{\Psi_{1n}*\epsilon_{1n}}\right]\right)\\
&= \left[\frac{\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}}\left(f_{1n}*\epsilon_{1n}+g_{1n}*\Psi_{1n}\right)}{\mathtt{F}_{\mathtt{c}}\left(\Psi_{1n}*\epsilon_{1n}\right)}\right]\\
&= \left[\frac{\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f_{1n}\vee \mathtt{F}_{\mathtt{c}}\epsilon_{1n}+\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} g_{1n}\vee \mathtt{F}_{\mathtt{c}}\Psi_{1n}}{\mathtt{F}_{\mathtt{c}}\Psi_{1n}\vee \mathtt{F}_{\mathtt{c}}\epsilon_{1n}}\right]\\
&= \left[\frac{\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f_{1n}}{\mathtt{F}_{\mathtt{c}}\Psi_{1n}}\right] + \left[\frac{\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} g_{1n}}{\mathtt{F}_{\mathtt{c}}\epsilon_{1n}}\right]
\end{aligned}$$

and

$$\widetilde{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}}\left(\alpha\beta_1\right) = \left[\frac{\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}}\left(\alpha f_{1n}\right)}{\mathtt{F}_{\mathtt{c}}\Psi_{1n}}\right] = \alpha\left[\frac{\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f_{1n}}{\mathtt{F}_{\mathtt{c}}\Psi_{1n}}\right] = \alpha\widetilde{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}}\left(\beta_1\right).$$

Hence the theorem.

Theorem 13. $\widetilde{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} : \mathtt{B}\left(\acute{\mathtt{E}}, \mathtt{D}, \Delta_1, *\right) \to \mathtt{B}\left(\mathtt{G}_{\mathtt{a}}^{\mathtt{b}}, \mathtt{G}_{\mathtt{c}}, \mathtt{F}_{\mathtt{c}}\Delta_1, \vee\right)$ *is one to one.*

Proof. Assume that there are $\beta_1 = \left[\frac{f_{1n}}{\Psi_{1n}}\right], \beta_1 = \left[\frac{g_{1n}}{\epsilon_{1n}}\right]$ in $\mathtt{B}\left(\acute{\mathtt{E}}, \mathtt{D}, \Delta_1, *\right)$ such that $\widetilde{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}}\beta_1 = \widetilde{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}}\beta_2$ then

$$\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f_{1n} \vee \mathtt{F}_{\mathtt{c}}\epsilon_{1m} = \widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} g_{1m} \vee \mathtt{F}_{\mathtt{c}}\Psi_{1n}.$$

Theorem 6 implies

$$\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}}\left(f_{1n}*\epsilon_{1m}\right) = \widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}}\left(g_{1m}*\Psi_{1n}\right).$$

Linearity of $\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}}$, Theorem 2, implies

$$f_{1n}*\epsilon_{1m} = g_{1m}*\Psi_{1n}.$$

Hence $\frac{f_{1n}}{\Psi_{1n}} \sim \frac{g_{1n}}{\epsilon_{1n}}$ in the sense of $\mathtt{B}\left(\acute{\mathtt{E}}, \mathtt{D}, \Delta_1, *\right)$. Therefore $\beta_1 = \beta_2$.

The proof is completed.

Theorem 14. $\widetilde{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} : \mathtt{B}\left(\acute{\mathtt{E}}, \mathtt{D}, \Delta_1, *\right) \to \mathtt{B}\left(\mathtt{G}_{\mathtt{a}}^{\mathtt{b}}, \mathtt{G}_{\mathtt{c}}, \mathtt{F}_c\Delta_1, \vee\right)$ *is* δ *convergent.*

Proof. As in [1], given $\beta_n \overset{\Delta}{\to} \beta$ as $n \to \infty$ we can find $f_{1n,k}$ and $f_{1n,k}$ in $\acute{\mathtt{E}}(\mathtt{R})$ such that $\beta_n = \left[\dfrac{f_{1n,k}}{\Psi_{1n}}\right], \beta = \left[\dfrac{f_{1n,k}}{\Psi_{1n,k}}\right]$ and that $f_{1n,k} \to f_{1,k}$ as $n \to \infty$.

Continuity of the distributional transform $\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}}$ implies $\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f_{1n,k} \to \widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f_{1,k}$ as $n \to \infty$.

Hence the proof of the theorem.

On the other hand, as usual for integral transforms, we define the inverse transform of $\widetilde{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}}$ of a Boehmian $\alpha = \left[\dfrac{\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f_{1n,k}}{\mathtt{F}_{\mathtt{c}}\Psi_{1n}}\right] \in \mathtt{B}\left(\mathtt{G}_{\mathtt{a}}^{\mathtt{b}}, \mathtt{G}_{\mathtt{c}}, \mathtt{F}_c\Delta_1, \vee\right)$ to be the Boehmian $\beta = \left[\dfrac{f_{1n}}{\Psi_{1n}}\right] \in \mathtt{B}\left(\acute{\mathtt{E}}, \mathtt{D}, \Delta_1, *\right)$ defined by

$$\left(\widetilde{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}}\right)^{-1}\left(\left[\frac{\widehat{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}} f_{1n}}{\mathtt{F}_{\mathtt{c}}\Psi_{1n}}\right]\right) = \left[\frac{f_{1n}}{\Psi_{1n}}\right].$$

The following result indeed apply for $\left(\widetilde{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}}\right)^{-1}$. Detailed proofs carry routine techniques similar to that of the corresponding ones of [1]. We prefer to omit repeated proofs.

Theorem 15. $\left(\widetilde{\mathtt{F}}_{\mathtt{a}}^{\mathtt{b}}\right)^{-1} : \mathtt{B}\left(\mathtt{G}_{\mathtt{a}}^{\mathtt{b}}, \mathtt{G}_{\mathtt{c}}, \mathtt{F}_{\mathtt{c}}\Delta_1, \vee\right) \to \mathtt{B}\left(\acute{\mathtt{E}}, \mathtt{D}, \Delta_1, *\right)$ *is*

(1) *Well-defined.*

(2) *Linear.*

(3) *One to one.*

(4) *Continuous with respect to* δ *convergence.*

Detailed proofs are avoided.

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