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A NEW NON-UNIFORM BOUND ON THE POISSON-BINOMIAL RELATIVE ERROR

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Abstract: The Stein-Chen method is used to determine new non-uniform bounds on two forms of the relative error between the binomial and Poisson cumulative distribution functions. The bounds obtained in this study are sharper than those reported in Teerapabolarn [6].

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1. Introduction

Let X be the binomial random variable with parameters $n \in \mathbb{N}$ and $p \in (0,1)$. Its distribution is a well-known discrete distribution that can be applied in topics related to probability and statistics. The probability function of X is of the form

$$p_X(x) = \binom{n}{x} p^x q^{n-x},\ x = 0, 1, ..., n, \tag{1.1}$$

where $q = 1 - p$ and its mean and variance are $E(X) = np$ and $Var(X) = npq$, respectively. It is well-known that if $n \to \infty$ and $p \to 0$ while $\lambda = np$ remains a constant ($0 < \lambda < \infty$) then $\binom{n}{x} p^x q^{n-x} \to \frac{e^{-\lambda}\lambda^x}{x!}$ for every $x =$

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$0, 1, ..., n$. Therefore, the Poisson distribution with mean $\lambda = np$ can be used as an approximation of the binomial distribution with parameters n and p when n is large and p is small. In the past, there has been some research on topics related to approximation relations between the Poisson and binomial distributions. For example, in the case of pointwise approximation, Anderson and Samuels [1] gave some inequalities of binomial and Poisson probabilities and in terms of the relative error of two such probabilities can be found in [2]. In the case of cumulative probability approximation, Anderson and Samuels [1] gave inequality of the error as follows:

$$\mathbb{P}_\lambda(x_0) - \mathbb{B}_{n,p}(x_0) \begin{cases} > 0 & \text{if } x_0 \leq \frac{\lambda n}{n+1}, \\ < 0 & \text{if } x_0 \geq \lambda, \end{cases} \tag{1.2}$$

where $\mathbb{P}_\lambda(x_0) = \sum_{k=0}^{x_0} \frac{e^{-\lambda}\lambda^k}{k!}$ and $\mathbb{B}_{n,p}(x_0) = \sum_{k=0}^{x_0} \binom{n}{k} p^k q^{n-k}$ are the Poisson and binomial cumulative distribution functions at $x_0 \in \{0, 1, ..., n\}$, respectively. Because the approximate relation (1.2) does not give any conditions in order to have a good Poisson approximation. So, Teerapabolarn [6] used the Stein-chen method to give a non-uniform bound on the relative error of the binomial and Poisson cumulative distributions functions, which is a criteria for measuring the accuracy of the approximation as follows.

$$\left| \frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)} - 1 \right| \leq \frac{(e^\lambda - 1)p\Delta(x_0)}{x_0 + 1}, \ x_0 = 0, 1, ..., n, \tag{1.3}$$

where

$$\Delta(x_0) = \begin{cases} e^{-\lambda} q^{-n} & \text{if } x_0 < \lambda, \\ 1 & \text{if } x_0 \geq \lambda, \end{cases} \tag{1.4}$$

and he also gave a non-uniform bound for the another form of the relative error of two such cumulative distribution functions,

$$\left| \frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)} - 1 \right| \leq \frac{(e^\lambda - 1)p}{x_0 + 1}, \ x_0 = 0, 1, ..., n. \tag{1.5}$$

For uniform bounds, Teerapabolarn [7] gave the results in two forms of the relative error as follows:

$$\sup_{0 \leq x_0 \leq n} \left| \frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)} - 1 \right| \leq \frac{(1 - e^{-\lambda})(1 - q^n)}{nq^n} \tag{1.6}$$

and

$$\sup_{0\leq x_0\leq n}\left|\frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)}-1\right|\leq\frac{(e^\lambda-1)(1-q^n)}{n}. \tag{1.7}$$

These bounds later improved by the same author in [8], which say that

$$\sup_{0\leq x_0\leq n}\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)}-1\right| \leq \max\left\{e^{-\lambda}q^{-n}-1,\frac{1-(1+\lambda)e^{-\lambda}}{nq^n}\min\left(1,\frac{2(1-q^n)}{\lambda}\right)\right\} \tag{1.8}$$

and

$$\sup_{0\leq x_0\leq n}\left|\frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)}-1\right| \leq \max\left\{1-e^{\lambda}q^n,\frac{e^\lambda-\lambda-1}{n}\min\left(1,\frac{2(1-q^n)}{\lambda}\right)\right\}. \tag{1.9}$$

The aim of this article, we are interested to improve the bounds in (1.3) and (1.5) sharper than ever by using the Stein-Chen method, which is described and used to determine the desired results in Sections 2 and 3, respectively. Concluding remarks are presented in the last section.

2. Method

The classical Stein's method was first introduced by Stein [4]. The version appropriate for the Poisson case was first developed by Chen [3], which is referred to as the Stein-Chen method.

Following [6], Stein's equation of the Poisson cumulative distribution function with parameter $\lambda>0$ is of the form

$$h_{x_0}(x)-\mathbb{P}_\lambda(x_0)=\lambda f_{x_0}(x+1)-xf_{x_0}(x) \tag{2.1}$$

for $x_0,x\in\mathbb{N}\cup\{0\}$, where function $h_{x_0}:\mathbb{N}\cup\{0\}\to\mathbb{R}$ is defined by

$$h_{x_0}(x)=\begin{cases}1 & \text{if } x\leq x_0,\\ 0 & \text{if } x>x_0\end{cases}$$

and

$$f_{x_0}(x) = \begin{cases} (x-1)!\lambda^{-x}e^{\lambda}[\mathbb{P}_\lambda(x-1)[1-\mathbb{P}_\lambda(x_0)]] & \text{if } x \le x_0, \\ (x-1)!\lambda^{-x}e^{\lambda}[\mathbb{P}_\lambda(x_0)[1-\mathbb{P}_\lambda(x-1)]] & \text{if } x > x_0, \\ 0 & \text{if } x = 0. \end{cases} \tag{2.2}$$

Lemma 2.1. *For $x_0, x \in \mathbb{N}$, we have the following inequalities:*

$$\sup_{x\ge 1} |\Delta f_0(x)| \le \frac{\lambda^{-1}\{(e^{\lambda}-1) - \lambda^{-1}(e^{\lambda}-1-\lambda)\}e^{-\lambda}}{x} \tag{2.3}$$

and

$$\sup_{x\ge 1} |\Delta f_{x_0}(x)| \le \frac{2\lambda^{-2}(e^{\lambda}-\lambda-1)\mathbb{P}_\lambda(x_0)}{x_0+1}. \tag{2.4}$$

Proof. The inequality (2.3) is directly obtained from Lemma 2.1 in [9]. Next, we have to show that the inequality (2.4) holds.

For $x \le x_0$, because Δf_{x_0} is an increasing function for $x \le x_0$ (Lemma 2.1 in [5]), we obtain

$$\begin{aligned}
\Delta f_{x_0}(x) &\le \Delta f_{x_0}(x_0) \\
&= (x_0-1)!e^{-\lambda}\sum_{k=0}^{x_0}(x_0-k)\frac{\lambda^k}{k!}\sum_{j=x_0+1}^{\infty}\frac{\lambda^{j-(x_0+1)}}{j!} \\
&\le \mathbb{P}_\lambda(x_0)x_0!\sum_{j=x_0+1}^{\infty}\frac{\lambda^{j-(x_0+1)}}{j!} \\
&= \frac{\mathbb{P}_\lambda(x_0)}{x_0+1}\left\{1 + \frac{\lambda}{x_0+2} + \frac{\lambda^2}{(x_0+2)(x_0+3)} + \cdots\right\} \\
&\le \frac{\mathbb{P}_\lambda(x_0)\lambda^{-2}}{x_0+1}\left\{\lambda^2 + \frac{\lambda^3}{3} + \frac{\lambda^4}{12} + \cdots\right\} \\
&= \frac{2\lambda^{-2}(e^{\lambda}-\lambda-1)\mathbb{P}_\lambda(x_0)}{x_0+1},
\end{aligned}$$

and for $x > x_0$, because Δf_{x_0} is a negative function for this case (Lemma 2.1 in [6]), we have

$$\begin{aligned}
0 &< -\Delta f_{x_0}(x) \\
&= (x-1)!e^{-\lambda}\sum_{j=0}^{x_0}\frac{\lambda^j}{j!}\sum_{k=x+1}^{\infty}(k-x)\frac{\lambda^{k-(x+1)}}{k!}
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{P}_\lambda(x_0)(x-1)! \left\{ \frac{1}{(x+1)!} + \frac{2\lambda}{(x+2)!} + \frac{3\lambda^2}{(x+3)!} + \cdots \right\} \\
&= \mathbb{P}_\lambda(x_0) \frac{(x-1)!}{x!} \left\{ \frac{1}{x+1} + \frac{2\lambda}{(x+1)(x+2)} + \frac{3\lambda^2}{(x+1)(x+2)(x+3)} + \cdots \right\} \\
&\le \frac{\mathbb{P}_\lambda(x_0)}{x_0+1} \lambda^{-2} \left\{ \frac{\lambda^2}{3} + \frac{\lambda^3}{6} + \frac{\lambda^4}{20} + \cdots \right\} \\
&\le \frac{2\lambda^{-2}(e^\lambda - \lambda - 1)\mathbb{P}_\lambda(x_0)}{x_0+1}.
\end{aligned}
$$

Therefore, the inequality (2.4) holds. □

Lemma 2.2. *For $\lambda = np$, we have the following:*

$$
\left\{ (e^\lambda - 1) - \frac{e^\lambda - 1 - \lambda}{\lambda} \right\} (1 - q^n) \le 2(e^\lambda - 1 - \lambda). \tag{2.5}
$$

Proof. Let us consider

$$
\begin{aligned}
2(e^\lambda - 1 - \lambda) - &\left\{ (e^\lambda - 1) - \frac{e^\lambda - 1 - \lambda}{\lambda} \right\} (1 - q^n) \\
&= (1 + q^n)(e^\lambda - 1) + \frac{(1 - q^n)(e^\lambda - 1 - \lambda)}{\lambda} - 2\lambda \\
&\ge (1 - q^n)\frac{\lambda}{2} + (1 + q^n)\frac{\lambda^2}{2} + (1 - q^n)\frac{\lambda^2}{3!} \\
&\ge -\frac{(1 - q^n)}{\lambda}\frac{\lambda^2}{2} + \frac{\lambda^2}{2} \\
&= \left[1 - \frac{(1 - q^n)}{\lambda} \right] \frac{\lambda^2}{2} \\
&\ge 0, \left(\frac{1 - q^n}{\lambda} \le 1 \right)
\end{aligned}
$$

this implies that (2.5) holds. □

3. Results

The results of this study are new non-uniform bounds on two forms of the relative error between the binomial and Poisson cumulative distribution functions.

Theorem 3.1. *For* $x_0 \in \{0, 1, ..., n\}$, *we have the following:*

$$\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)} - 1\right| \leq \frac{2(e^\lambda - \lambda - 1)\Delta(x_0)}{n(x_0+1)}, \tag{3.1}$$

where $\Delta(x_0)$ *is defined as in* (1.4).

Proof. For $1 \leq x_0 \leq n$, Teerapabolarn [8] showed that

$$\begin{aligned}
|\mathbb{B}_{n,p}(x_0) - \mathbb{P}_\lambda(x_0)| &\leq p \sup_{x \geq 1} |\Delta f_{x_0}(x)| \sum_{x=1}^{n} x p_X(x) \\
&\leq \lambda p \frac{2\lambda^{-2}(e^\lambda - \lambda - 1)\mathbb{P}_\lambda(x_0)}{x_0+1} \quad \text{(by (2.4))} \\
&= \frac{2(e^\lambda - \lambda - 1)\mathbb{P}_\lambda(x_0)}{n(x_0+1)}.
\end{aligned} \tag{3.2}$$

For $x_0 = 0$, using the same arguments as mentioned above, we can express

$$\begin{aligned}
|\mathbb{B}_{n,p}(0) - \mathbb{P}_\lambda(0)| &\leq p \sum_{x=1}^{n} \frac{\lambda^{-1}\{(e^\lambda - 1) - \lambda^{-1}(e^\lambda - 1 - \lambda)\}e^{-\lambda}}{x} x p_X(x) \text{ (by (2.3))} \\
&= \frac{\{(e^\lambda - 1) - \lambda^{-1}(e^\lambda - 1 - \lambda)\}(1 - q^n)\mathbb{P}_\lambda(0)}{n} \\
&\leq \frac{2(e^\lambda - 1 - \lambda)\mathbb{P}_\lambda(0)}{n} \quad \text{(by (2.5))}.
\end{aligned} \tag{3.3}$$

From (3.2) and (3.3), it follows that

$$|\mathbb{B}_{n,p}(x_0) - \mathbb{P}_\lambda(x_0)| \leq \frac{2(e^\lambda - \lambda - 1)\mathbb{P}_\lambda(x_0)}{n(x_0+1)} \tag{3.4}$$

for every $x_0 \in \{0, 1, ..., n\}$. Dividing the last equality by $\mathbb{B}_{n,p}(x_0)$ and using Lemma 2.2 in [6], we obtain

$$\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)} - 1\right| \leq \frac{2(e^\lambda - \lambda - 1)\Delta(x_0)}{n(x_0+1)}.$$

Hence, (3.1) is obtained. □

Dividing (3.4) by $\mathbb{P}_\lambda(x_0)$, the following corollary shows the another result.

Corollary 3.1. *Fore* $x_0 \in \{0, 1, ..., n\}$, *we have the following inequality:*

$$\left|\frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)} - 1\right| \leq \frac{2(e^\lambda - \lambda - 1)}{n(x_0+1)}. \tag{3.5}$$

Remark. By comparing between the bounds in (1.3) and (1.5) and the bounds in Theorem 3.1 and Corollary 3.1, because

$$\frac{2(e^{\lambda}-\lambda-1)}{n}=2\lambda^{-1}(e^{\lambda}-\lambda-1)p=\left\{\lambda+\frac{\lambda^2}{3}+\frac{\lambda^3}{12}+\cdots\right\}p<(e^{\lambda}-1)p,$$

the bounds in Theorem 3.1 and Corollary 3.1 are sharper than the bounds in (1.3) and (1.5), respectively.

4. Conclusion

The bounds in Theorem 3.1 and Corollary 3.1, which were improved by the Stein-Chen method, provide new general criteria for measuring the accuracy in approximating the binomial cumulative distribution with parameter n and p by the Poisson cumulative distribution with mean $\lambda = np$. The bounds in this study are sharper than those reported in Teerapabolarn [6].

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