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A NOTE ON TWO POINT TAYLOR EXPANSION II

Kazuaki Kitahara[1 §], Takako Yamada[2], Kazuki Fujiwara[3]

[1,3]School of Science and Technology
Kwansei Gakuin University
Sanda, 669-1337, JAPAN

[2]School of Policy Studies
Kwansei Gakuin University
Sanda, 669-1337, JAPAN

Abstract: Let f be a continuous function on $[-r, r]$ $(r > 1 + \sqrt{2})$, α the function f restricted to the subinterval $[0, r]$ and β the function f restricted to the subinterval $[-r, 0]$. If α (resp.β) is expressed as the Taylor expansion of α (resp.β) about 1 (resp.-1), then we show that f is expressed as the two point Taylor expansion about $-1, 1$ on the interval $(-\sqrt{2}, \sqrt{2})$. Furthermore, the k-th order derivatives of f on $(-\sqrt{2}, 0) \cup (0, \sqrt{2})$ are expressed as the termwise k times differentiation of the two point Taylor expansion about $-1, 1$.

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1. Introduction

As is well known, polynomial approximation has a long history and lays the foundation of approximation theory. Especially, interpolations by polynomials play an important role of polynomial approximation and have been furnishing a lot of challenging topics. Before stating the purpose of this note, we briefly review Hermite interpolation by polynomials.

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§Correspondence author

Let I be an infinite subset of $\mathbf{R}$ and let f be a real-valued function on I. For any given $(n+1)$ distinct points $X : x_0, \ldots, x_n$ in the interior of I and for any sequence of positive integers $k_0, \ldots, k_n$, if f is sufficiently differentiable at $x_0, \ldots, x_n$, then there exists a unique approximating polynomial $p_{f,X(k_0,\ldots,k_n)}(x)$ to f which is of degree at most $m(= k_0 + \cdots + k_n - 1)$ and satisfies that

$$p^{(j)}_{f,X(k_0,\ldots,k_n)}(x_i) = f^{(j)}(x_i), \quad 0 \leqq i \leqq n,\ 0 \leqq j \leqq k_i - 1.$$

The points $x_0, \ldots, x_n$ and the polynomial $p_{f,X(k_0,\ldots,k_n)}$ are called *nodes* and the *Hermite interpolating polynomial to f at $x_0, \ldots, x_n$ with multiplicities* $k_0, \ldots, k_n$, respectively. It is well known that for one node $X : x_0$ with multiplicity n, the Hermite interpolating polynomial $p_{f,X(n)}$ to f is the Taylor polynomial of f about x_0,that is,

$$p_{f,X(n)}(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n-1)}(x_0)}{(n-1)!}(x - x_0)^{n-1}.$$

Furthermore, if f is infinitely differentiable at x_0 and if

$$f(x) = \lim_{n\to\infty} p_{f,X(n)}(x) \quad \text{for all } x \in (x_0 - \rho, x_0 + \rho) \ \ (\rho \text{ is some positive number}),$$

then f has the Taylor expansion of f about x_0 on $(x_0 - \rho, x_0 + \rho)$. By this fact, we make the following definition.

Definition 1. Let f be a real-valued function on a subset I of the real line. If there exists a list X consisting of m distinct nodes $x_0, \ldots, x_{m-1}$ in the interior of I such that f is infinitely differentiable at $x_0, \ldots, x_{m-1}$ and

$$\lim_{n\to\infty} p_{f,X(n,\ldots,n)}(x) = f(x) \quad \text{for all } x \in I,$$

then it is said that f has the *m point Taylor expansion about $x_0, \ldots, x_{m-1}$ on I*. And the set of all functions to have m point Taylor expansion on I is denoted by $\mathcal{T}_m[I]$.

The notion of two point or m point Taylor expansion is not new. One can see some representations of $p_{f,X(n,\ldots,n)}(x)$ in Davis[1; p.37] and the theory of m point Taylor expansion in the complex plane in Walsh[6; chap.3]. López and Temme[4, 5] stated how m point Taylor expansion in the complex plane can be used in deriving uniform asymptotic expansions of integrals. Furthermore, Kitahara, Chiyonobu and Tsukamoto[3] shows the following result about functions which belong to $\mathcal{T}_2[(-\sqrt{2}, 0) \cup (0, \sqrt{2})]$ or $\mathcal{T}_2[(-\sqrt{2}, \sqrt{2})]$:

Theorem 1. *Let f be a function on* $\mathbf{R}$, *which is expressed as*

$$f(x) = \begin{cases} \alpha(x) & x \in [0, \infty) \\ \beta(x) & x \in (-\infty, 0) \end{cases} ,$$

where α and β are polynomials of degree at most m. Let $P_\ell, \ell \in \mathbf{N}$ be the Hermite interpolating polynomials to f at $-1, 1$ with multiplicities ℓ, ℓ. Then, the following assertions hold:

(1) *f has the two point Taylor expansion about $-1, 1$ on $(-\sqrt{2}, 0) \cup (0, \sqrt{2})$, that is,*

$$\lim_{\ell \to \infty} P_\ell(x) = f(x) , \quad \text{for all } x \in (-\sqrt{2}, 0) \cup (0, \sqrt{2}).$$

(2) *Moreover, if $\alpha(0) = \beta(-0)$, then f has the two point Taylor expansion about $-1, 1$ on $(-\sqrt{2}, \sqrt{2})$, that is,*

$$\lim_{\ell \to \infty} P_\ell(x) = f(x) , \quad \text{for all } x \in (-\sqrt{2}, \sqrt{2}).$$

In this note, we will show the following results of two point Taylor expansions which are related to Theorem 1.

Theorem 2. (An Extension of Theorem 1) *Let f be a real-valued function on $[-r, r] (r > 1 + \sqrt{2})$, which is expressed as*

$$f(x) = \begin{cases} \alpha(x) & x \in [0, r] \\ \beta(x) & x \in [-r, 0) \end{cases} ,$$

where α (resp. β) is expressed as the Taylor expansion of α $(resp. \beta)$ about 1 $(resp. -1)$. Let $P_\ell, \ell \in \mathbf{N}$ be the Hermite interpolating polynomials to f at $-1, 1$ with multiplicities ℓ, ℓ. Then, the following assertions hold:

(1) *f has the two point Taylor expansion about $-1, 1$ on $(-\sqrt{2}, 0) \cup (0, \sqrt{2})$, that is,*

$$\lim_{\ell \to \infty} P_\ell(x) = f(x) \quad \text{for all } x \in (-\sqrt{2}, 0) \cup (0, \sqrt{2}).$$

(2) *Moreover, if $\alpha(0) = \beta(-0)$, then f has the two point Taylor expansion about $-1, 1$ on $(-\sqrt{2}, \sqrt{2})$, that is,*

$$\lim_{\ell \to \infty} P_\ell(x) = f(x) \quad \text{for all } x \in (-\sqrt{2}, \sqrt{2}).$$

Theorem 3. (Evaluation of $\lim_{\ell\to\infty} P_\ell(0)$) *Let f be a real-valued function on $[-r,r](r > 1+\sqrt{2})$ which satisfies the same condition as in Theorem 2. Let $P_\ell, \ell \in \mathbf{N}$ be the Hermite interpolating polynomials to f at $-1,1$ with multiplicities ℓ,ℓ. Then*

$$\lim_{\ell\to\infty} P_\ell(0) = \frac{\alpha(0)+\beta(-0)}{2}$$

holds.

Theorem 4. (Termwise Differentiation) *Let f be a real-valued function on $[-r,r](r > 1+\sqrt{2})$ which satisfies the same condition as in Theorem 2. Let $P_\ell, \ell \in \mathbf{N}$ be the Hermite interpolating polynomials to f at $-1,1$ with multiplicities ℓ,ℓ. It holds that, for any given positive integer k*

$$\lim_{\ell\to\infty} P_\ell^{(k)}(x) = f^{(k)}(x) \quad \text{for all } x \in (-\sqrt{2},0)\cup(0,\sqrt{2}).$$

2. Preliminaries

First we begin with a proposition which states the existence of Hermite interpolating polynomials.

Proposition 5. (see p. 365 in Kincaid and Cheney [2]) *Let $x_0 \leqq x_1 \leqq \cdots \leqq x_n$ be a list of nodes. In the list of nodes, only distinct nodes $z_0,\ldots,z_p$ appear and each node $z_i, i=0,\ldots,p$ is just appeared k_i times. Let f be sufficiently differentiable at $z_0,\ldots,z_p$. Then, there exists a unique polynomial p of degree at most n satisfying that*

$$p^{(j)}(z_i) = f^{(j)}(z_i), \quad j=0,\ldots,k_i-1, \quad i=0,\ldots,p. \tag{2.1}$$

In Proposition 5, we call each positive integer $k_i, i=0,\ldots,p$ *the multiplicity at x_i*. Divided differences of functions can be defined by this proposition.

Definition 2. Let $x_0 \leqq x_1 \leqq \cdots \leqq x_n$ be a list of nodes and let f be sufficiently differentiable at $x_0,\ldots,x_n$. Then the coefficient of x^n of the polynomial p with the property (2.1) stated above is called *the n-th order divided difference of f at $x_0,\ldots,x_n$* and is denoted by $f[x_0,\ldots,x_n]$.

By Definition 2, it is easily seen that the divided difference $f[x_0]$ of a function f at a point x_0 is equal to $f(x_0)$. The following proposition of a recursive formula and a divided difference table are of much use to calculate divided differences of functions.

Proposition 6. (see p. 372 in Kincaid and Cheney [2]) *Let $x_0 \leqq \cdots \leqq x_n$ be a list of nodes and let f be sufficiently differentiable at $x_0, \ldots, x_n$. Then the divided differences obey this recursive formula:*

$$f[x_0, \ldots, x_n] = \begin{cases} \dfrac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0} & (x_0 \neq x_n) \\ \dfrac{f^{(n)}(x_0)}{n!} & (x_0 = x_n) \end{cases}$$

If data points $(x_i, f(x_i)), i = 0, \ldots, n$ are given, then we can construct the following divided difference table $T[f, x_0, \ldots, x_n]$ from them. By Proposition 6, the $(i+1)$-th order divided differences in the table are calculated from the i-th order divided differences.

$$\begin{array}{ccccc}
x_0 & f[x_0] & & & \\
 & & f[x_0, x_1] & & \\
x_1 & f[x_1] & & & \\
 & & f[x_1, x_2] & & \\
x_2 & f[x_2] & & & \\
 & & & & \\
\vdots & \vdots & \vdots & \cdots & f[x_0, x_1, \cdots, x_{n-1}, x_n] \\
 & & & & \\
x_{n-2} & f[x_{n-2}] & & & \\
 & & f[x_{n-2}, x_{n-1}] & & \\
x_{n-1} & f[x_{n-1}] & & & \\
 & & f[x_{n-1}, x_n] & & \\
x_n & f[x_n] & & &
\end{array}$$

Divided difference table $T[f, x_0, \ldots, x_n]$

In the divided difference table stated above, we call the column vector consisting of the i-th order divided differences *the i-th order column vector* for convenience.

Notation. Let $x_0 \leqq x_1 \leqq \cdots \leqq x_n$ be a list of nodes and let f be sufficiently differentiable at $x_0, \ldots, x_n$. In the list of nodes, only distinct points $z_0, \ldots, z_p$

appear and each point $z_i, i = 0, \dots, p$ is just appeared k_i times. To make sure of multiplicities, we write

$$f[z_0, \dots, z_p; k_0, \dots, k_p]$$

for the divided difference $f[x_0, \dots, x_n]$. And the divided difference table $T[f, x_0, \dots, x_n]$ is also denoted by $T[f; z_0, \dots, z_p; k_0, \dots, k_p]$.

Let $x_0 \leqq x_1 \leqq \cdots \leqq x_n$ be a list of nodes and let f be sufficiently differentiable at $x_0, \dots, x_n$. It is well known that the Hermite interpolating polynomial p to f at $x_0, \dots, x_n$, which p satisfies (2.1), is expressed as

$$p(x) = \sum_{j=0}^{n} f[x_0, \dots, x_j] \Pi_{i=0}^{j-1} (x - x_i),$$

where $\Pi_{i=0}^{-1}(x - x_i) = 1$ (see p.370 in Kincaid and Cheney [1]).

The following proposition is a basic statement, but it is a key result to prove our theorems.

Propostion 7. *Let $(a \leqq) x_0 \leqq x_1 \leqq \cdots \leqq x_n (\leqq b)$ be a list of nodes and let f be a real-valued function on an interval $[a, b]$ which is sufficiently differentiable at $x_0, \dots, x_n$. If p is the the Hermite interpolating polynomial to f at $x_0, \dots, x_n$, then*

$$f(x) - p(x) = f[x, x_0, \dots, x_n](x - x_0)(x - x_1) \cdots (x - x_n), \quad x \in [a, b].$$

3. Proofs of Theorems

We show the following two lemmas before proving Theorem 2.

Lemma 8. *Let f be a real-valued function on $[-r, r] (r > 1 + \sqrt{2})$, which is expressed as*

$$f(x) = \begin{cases} \alpha(x) & x \in [0, r] \\ \beta(x) & x \in [-r, 0) \end{cases},$$

where α (resp. β) is expressed as the Taylor expansion of α (resp.β) about 1 (resp. -1). Then, the following assertions hold:

(1) $\sup_{i,j \in \mathbf{N} \cup \{0\}, i^2 + j^2 \neq 0} |f[-1, 1; i, j]| < +\infty$

(2) $\sup_{i,j \in \mathbf{N} \cup \{0\}, i^2 + j^2 \neq 0} |f[-1, x, 1; i, 1, j]| < +\infty$ *for each* $x \in (-\sqrt{2}, \sqrt{2})$

Proof. (1) It is sufficient to show the existence of a positive number M such that

$$\max\{|b| : b \in T[f; -1, 1; n, n]\} \leqq M \text{ for all } n \in \mathbf{N}. \tag{3.1}$$

Because any $f[-1, 1; i, j], i, j \in \mathbf{N} \cup \{0\}, i^2 + j^2 \neq 0$ belongs to some $T[f; -1, 1; n, n]$.

By the condition of f, $\sum_{k=0}^{\infty} \left|\frac{\alpha^{(k)}(1)}{k!}\right|$ and $\sum_{k=0}^{\infty} \left|\frac{\beta^{(k)}(-1)}{k!}\right|$ converge. So we have

$$A = \sup\left\{\left|\frac{\alpha^{(k)}(1)}{k!}\right| \ : \ k \in \mathbf{N} \cup \{0\}\right\} < \infty,$$

$$B = \sup\left\{\left|\frac{\beta^{(k)}(-1)}{k!}\right| \ : \ k \in \mathbf{N} \cup \{0\}\right\} < \infty.$$

Since every divided difference in $T[f; -1, 1; n, n]$ is a real number expressed as $\frac{\alpha^{(k)}(1)}{k!}$, $\frac{\beta^{(k)}(1)}{k!}$ or $\frac{b_1 - b_2}{2}$, where b_1, b_2 are divided differences in the same column vector of $T[f; -1, 1; n, n]$, if we put $M = \max\{A, B\}$, then from the recursive formula of the divided differences, (3.1) immediately follows.

(2) In case $x = 1$ or -1, the assertion (2) is reduced to the assertion (1). Hence, without loss of generality, we assume that $0 \leq x < \sqrt{2}$ and $x \neq 1$.

Let x be a given nonnegative number which is less than $\sqrt{2}$. As in the analogous way to the proof of (1), it is sufficient to show the existence of a postive number L such that

$$\max\{|b| : b \in T[f; -1, x, 1; n, 1, n]\} \leqq L \text{ for all } n \in \mathbf{N}. \tag{3.2}$$

Because any $f[-1, x, 1; i, 1, j]$ belongs to some $T[f; -1, x, 1; n, 1, n]$. The 0-th and the first column vectors of $T[f; -1, x, 1; n, 1, n]$ are as follows:

$$
\begin{array}{cc}
f(-1) = \beta(-1) & \\
 & \dfrac{\beta'(-1)}{1!} \\
f(-1) = \beta(-1) & \\
 & \vdots \\
\vdots & \\
 & \dfrac{\beta'(-1)}{1!} \\
f(-1) = \beta(-1) & \\
 & \dfrac{\alpha(x) - \beta(-1)}{x+1} \\
f(x) = \alpha(x) & \\
 & \dfrac{\alpha(x) - \alpha(1)}{x-1} \\
f(1) = \alpha(1) & \\
 & \dfrac{\alpha'(1)}{1!} \\
f(1) = \alpha(1) & \\
 & \dfrac{\alpha'(1)}{1!} \\
\vdots & \\
 & \vdots \\
 & \dfrac{\alpha'(1)}{1!} \\
f(1) = \alpha(1) & \\
 & \\
\text{the 0-th column vector} & \text{the 1-st column vector}
\end{array}
$$

The $k(2 \leq k \leq n-1)$-th column vector of the divided difference table is as follows:

$$\frac{\beta^{(k)}(-1)}{k!}$$
$$\frac{\beta^{(k)}(-1)}{k!}$$
$$\vdots$$
$$\frac{\beta^{(k)}(-1)}{k!}$$
$$\frac{1}{(x+1)^k}\left\{\alpha(x)-\left(\beta(-1)+\frac{\beta'(-1)}{1!}(x+1)+\cdots+\frac{\beta^{(k-1)}(-1)}{(k-1)!}(x+1)^{k-1}\right)\right\}$$
$$\frac{1}{2}\{f[-1,x,1;k-2,1,1]-f[-1,x;k-1,1]\}=f[-1,x,1;k-1,1,1]$$
$$\frac{1}{2}\{f[-1,x,1;k-3,1,2]-f[-1,x,1;k-2,1,1]\}=f[-1,x,1;k-2,1,2]$$
$$\vdots$$
$$\frac{1}{2}\{f[x,1;1,k-1]-f[-1,x,1;1,1,k-2]\}=f[-1,x,1;1,1,k-1]$$
$$\frac{1}{(x-1)^k}\left\{\alpha(x)-\left(\alpha(1)+\frac{\alpha'(1)}{1!}(x-1)+\cdots+\frac{\alpha^{(k-1)}(1)}{(k-1)!}(x-1)^{k-1}\right)\right\}$$
$$\frac{\alpha^{(k)}(1)}{k!}$$
$$\vdots$$
$$\frac{\alpha^{(k)}(1)}{k!}$$

And the n-th and the $n+1$-th column vectors of the divided difference table is as follows:

$$\frac{1}{(x+1)^n}\left\{\alpha(x)-\left(\beta(-1)+\frac{\beta'(-1)}{1!}(x+1)+\cdots+\frac{\beta^{(n-1)}(-1)}{(n-1)!}(x+1)^{n-1}\right)\right\}$$

$$\frac{1}{2}\{f[-1,x,1;n-2,1,1]-f[-1,x;n-1,1]\}=f[-1,x,1;n-1,1,1]$$

$$\frac{1}{2}\{f[-1,x,1;n-3,1,2]-f[-1,x,1;n-2,1,1]\}=f[-1,x,1;n-2,1,2]$$

$$\vdots$$

$$\frac{1}{2}\{f[x,1;1,n-1]-f[-1,x,1;1,1,n-2]\}=f[-1,x,1;1,1,n-1]$$

$$\frac{1}{(x-1)^n}\left\{\alpha(x)-\left(\alpha(1)+\frac{\alpha'(1)}{1!}(x-1)+\cdots+\frac{\alpha^{(n-1)}(1)}{(n-1)!}(x-1)^{n-1}\right)\right\}$$

and

$$\frac{1}{2}\{f[-1,x,1;n-1,1,1]-f[-1,x;n,1]\}=f[-1,x,1;n,1,1]$$

$$\frac{1}{2}\{f[-1,x,1;n-2,1,2]-f[-1,x,1;n-1,1,1]\}=f[-1,x,1;n-1,1,2]$$

$$\vdots$$

$$\frac{1}{2}\{f[x,1;1,n]-f[-1,x,1;1,1,n-1]\}=f[-1,x,1;1,1,n]$$

We see that every divided difference in $T[f;-1,x,1;n,1,n]$ is a real number expressed as one of the following four types:

(a) $A_k=\dfrac{\alpha^{(k)}(1)}{k!}$, $B_k=\dfrac{\beta^{(k)}(-1)}{k!}$

(b) $$C_k=\frac{1}{(x+1)^k}\left\{\alpha(x)-\left(\beta(-1)+\frac{\beta'(-1)}{1!}(x+1)+\cdots+\frac{\beta^{(k-1)}(-1)}{(k-1)!}(x+1)^{k-1}\right)\right\}$$

(c) $$D_k=\frac{1}{(x-1)^k}\left\{\alpha(x)-\left(\alpha(1)+\frac{\alpha'(1)}{1!}(x-1)+\cdots+\frac{\alpha^{(k-1)}(1)}{(k-1)!}(x-1)^{k-1}\right)\right\}$$

(d) $\dfrac{d_1 - d_2}{2}$, where d_1, d_2 are divided differences in the same column vector of $T[f; -1, x, 1; n, 1, n]$

Hence, if we put $A = \sup\{|A_k| : k \in \mathbf{N} \cup \{0\}\}$, $B = \sup\{|B_k| : k \in \mathbf{N} \cup \{0\}\}$, $C = \sup\{|C_k| : k \in \mathbf{N}\}$, $D = \sup\{|D_k| : k \in \mathbf{N}\}$ and $L = \max\{A, B, C, D\}$, then from the condition of f, L is finite. Moreover, from the recursive formula of the divided differences, (3.2) immediately follows.

Lemma 9. *Let f be a real-valued function on $[-r, r](r > 1 + \sqrt{2})$, which is expressed as*

$$f(x) = \begin{cases} \alpha(x) & x \in [0, r] \\ \beta(x) & x \in [-r, 0) \end{cases},$$

where α (resp. β) is expressed as the Taylor expansion of α (resp.β) about 1 (resp. -1). Suppose that $\alpha(0) = \beta(-0)$. Then, it holds that

$$\lim_{\ell \to \infty} f[-1, 0, 1; \ell, 1, \ell] = 0.$$

Proof. We will show that for any given positive number ε, there exists a positive integer ℓ_0 such that $|f[-1, 0, 1; \ell, 1, \ell]| < 2\varepsilon$ for all $\ell \geq \ell_0$.

Suppose that by Lemma 8, M is a positive number with

$$\sup\{f[-1, 0, 1; i, 1, j] : i, j \in \mathbf{N} \cup \{0\}, i^2 + j^2 \neq 0\} \leq M. \tag{3.3}$$

From the proof of Lemma 8, we can find a positive integer $m(\geqq 2)$ satisfying

$$\max\{|A_k|, |B_k|, |C_k|, |D_k|\} < \varepsilon \tag{3.4}$$

for all $k \geq m$, where A_k, B_k, C_k, D_k denote the numbers in the proof of Lemma 8.

Assume that a positive integer ℓ is much larger than m. Then, from (3.3) and (3.4), we observe that the m-th column vector $\mathbf{a}^{(m)} = (a_i^{(m)})_{1 \leq i \leq 2\ell+1-m} \in \mathbf{R}^{2\ell+1-m}$ of $T[f; -1, 0, 1; \ell, 1, \ell]$ satisfies

$$\begin{aligned} &|a_i^{(m)}| < \varepsilon, \quad 1 \leq i \leq \ell - m + 1, \quad \ell + 1 \leq i \leq 2\ell + 1 - m \\ &|a_i^{(m)}| \leq M, \quad \ell - m + 2 \leq i \leq \ell. \end{aligned} \tag{3.5}$$

On the other hand, we introduce a recursive relation by which a column vector $\mathbf{c}' = (c_i')_{1 \leq i \leq n}$ is obtained from a column vector $\mathbf{c} = (c_i)_{1 \leq i \leq n+1}$ such that

$$c_i' = \frac{c_i + c_{i+1}}{2}, \quad i = 1, 2, \ldots, n. \tag{3.6}$$

Let $\mathbf{c}^{(1)} \in \mathbf{R}^r$ be an initial column vector and let $\mathbf{c}^{(k)}$, $2 \leq k \leq n$ be the k-th column vector which is obtained by the recursive relation (3.6). It is easily seen that for an initial column vector $\mathbf{c}^{(1)} = (c_i^{(1)})_{1 \leq i \leq 2\ell+1-m} \in \mathbf{R}^{2\ell+1-m}$ with

$$c_i^{(1)} = \begin{cases} \varepsilon & , \quad 1 \leq i \leq \ell - m + 1, \ \ell + 1 \leq i \leq 2\ell + 1 - m \\ 0 & , \quad \ell - m + 2 \leq i \leq \ell. \end{cases}$$

As an estimation of the absolute value of $\mathbf{c}^{(2\ell+1-m)} \in \mathbf{R}$ obtained by (3.6), we have

$$|\mathbf{c}^{(2\ell+1-m)}| < \varepsilon. \tag{3.7}$$

Let us consider another initial column vector $\mathbf{d}^{(1)} = (d_i^{(1)})_{1 \leq i \leq 2\ell+1-m} \in \mathbf{R}^{2\ell+1-m}$ with

$$d_i^{(1)} = \begin{cases} 0 & , \quad 1 \leq i \leq \ell - m + 1, \ \ell + 1 \leq i \leq 2\ell + 1 - m \\ M & , \quad \ell - m + 2 \leq i \leq \ell. \end{cases}$$

By (3.5), we have $abs(\mathbf{a}^{(m)}) \leq \mathbf{c}^{(1)} + \mathbf{d}^{(1)}$, which means that $|a_i^{(m)}| \leq c_i^{(1)} + d_i^{(1)}$, $i = 1, \ldots, 2\ell + 1 - m$. Futhermore, from the recursive relations of divided differences and (3.6), we obtain

$$abs(\mathbf{a}^{(m+k)}) \leq \mathbf{c}^{(1+k)} + \mathbf{d}^{(1+k)}, \quad 0 \leq k \leq 2\ell - m$$

and in particular,

$$|\mathbf{a}^{(2\ell)}| \leq \mathbf{c}^{(2\ell+1-m)} + \mathbf{d}^{(2\ell+1-m)}. \tag{3.8}$$

Moreover, by the proof of Theorem in Kitahara, Chiyonobu and Tsukamoto [3], there exists a positive integer p such that $0 \leq \mathbf{d}^{(2\ell+1-m)} < \varepsilon$ for all $\ell \geq p$. So, if we put $\ell_0 = \max\{m, p\}$, then from (3.7) and (3,8), it follows that $|\mathbf{a}^{(2\ell)}| = |f[-1, 0, 1; \ell, 1, \ell]| < 2\varepsilon$ for all $\ell \geq \ell_0$. This completes the proof.

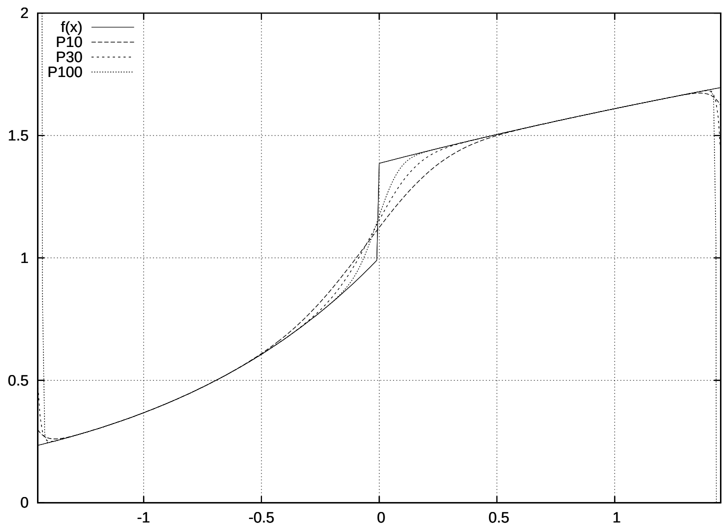
Now we are in position to prove Theorem 2.

Proof of Theorem 2. For each positive integer ℓ, let P_ℓ be the Hermite interpolating polynomial to f at $-\ell, \ell$ with multiplicities ℓ, ℓ. By Proposition 7, we have for each $x \in (-\sqrt{2}, \sqrt{2})$

$$\begin{aligned} |f(x) - p_\ell(x)| &= |(x-1)^\ell (x+1)^\ell f[-1, x, 1; \ell, 1, \ell]| \\ &= |x^2 - 1|^\ell |f[-1, x, 1; \ell, 1, \ell]|. \end{aligned} \tag{3.9}$$

(1) Since $|x^2 - 1| < 1$ for each $x \in (-\sqrt{2}, 0) \cup (0, -\sqrt{2})$, it is sufficient to show

$$\sup_{\ell \geqq 1} |f[-1, x, 1; \ell, 1, \ell]| < +\infty.$$



But this immediately follows from Lemma 8 (2).

(2) Without loss of generality, we assumu that $\alpha(0) = \beta(-0) = 0$. From (3.9), we have

$$|f(0) - P_\ell(0)| = |(0-1)^\ell(0+1)^\ell f[-1,0,1;\ell,1,\ell]| = |f[-1,0,1;\ell,1,\ell]|.$$

By Lemma 9, since $\lim_{n\to\infty} f[-1,0,1;\ell,1,\ell] = 0$, we get $\lim_{n\to\infty} P_\ell(0) = f(0)$. From this and (1), it holds that $\lim_{n\to\infty} P_\ell(x) = f(x)$ for all $x \in (-\sqrt{2}, -\sqrt{2})$.

Example. As a function on $[-3,3]$ which satisfies the condition of Theorem 2, let us consider $f(x) = \begin{cases} \log(4+x) & x \in [0,3] \\ e^x & x \in [-3,0). \end{cases}$ Let $P\ell$ denote the Hermite interpolating polynomials to f at $-1, 1$ with multiplicities ℓ, ℓ for $\ell = 10, 30, 100$, respectively. One can see that as multiplicities get larger, the Hermite interpolating polynomials get closer to $f(x)$ on $(-1.4, 1.4)$.

We prepare the following lemma to prove Theorem 3.

Lemma 10. *Let f be a real-valued function on $\mathbf{R}$ which is expressed as*

$$f(x) = \begin{cases} 1 & x \in (0,\infty) \\ 0 & x = 0 \\ -1 & x \in (-\infty, 0) \end{cases}$$

Let $P_\ell, \ell \in \mathbf{N}$ be the Hermite interpolating polynomials to f at $-1, 1$ with multiplicities ℓ, ℓ. Then, it holds that $P_\ell(0) = 0,\ \ell \in \mathbf{N}$.

Proof. By Proposition 7, we have for each $x \in \mathbf{R}$

$$f(x) - P_\ell(x) = (x-1)^\ell (x+1)^\ell f[-1, x, 1; \ell, 1, \ell].$$

Since $f(0)-P_\ell(0) = (-1)^\ell f[-1, 0, 1; \ell, 1, \ell]$, it is sufficient to show $f[-1, x, 1; \ell, 1, \ell] = 0,\ \ell \in \mathbf{N}$. The tranpose of the 0-th and the 1-st column vectors of $T[f : -1, 0, 1; \ell, 1, \ell]$ are $-1, \dots, -1, 0, 1, \dots, 1$ and $0, \dots, 0, 1, 1, 0, \dots, 0$, respectively. Hence, we guess that the transpose of any even numbered colomn vector is expressed as

$$-a_m, -a_{m-1}, \dots, -a_1, 0, a_1, \dots, a_{m-1}, a_m \tag{3.10}$$

and the transpose of any odd numbered colum vector is expressed as

$$b_m, b_{m-1}, \dots, b_1, b_1, \dots b_{m-1}, b_m. \tag{3.11}$$

Clearly, (3.10), (3.11) hold for the 0-th and the 1-st column vector. If an even (odd) numbered column vector is expressed as (3.10)((3.11)), we easily see that the next odd (even) numbered colomn vector is expressed as (3.11)((3.10)). Hence, any column vector of $T[-1, 0, 1; \ell, 1, \ell]$ is expressed as (3.10) or (3.11).

Consequently, we obtain the 2ℓ-th column vector $f[-1, 0, 1; \ell, 1, \ell] = 0$.

Corollary 11. *Let f be a real-valued function on $\mathbf{R}$ which is expressed as*

$$f(x) = \begin{cases} C_1 & x \in [0, \infty) \\ C_2 & x \in (-\infty, 0), \end{cases}$$

where C_1 and C_2 are real numbers. Let $P_\ell, \ell \in \mathbf{N}$ be the Hermite interpolating polynomials to f of $-1, 1$ with multiplicities ℓ, ℓ. Then, it holds that $P_\ell(0) = \dfrac{C_1 + C_2}{2}, \ell \in \mathbf{N}$.

Now we show a proof of Theorem 3.

Proof of Theorem 3. First we consider two functions $g(x)$, $h(x)$ such that

$$g(x) = \begin{cases} \alpha(x) & x \in [0, \sqrt{2}) \\ \beta(x) - \beta(-0) + \alpha(0) & x \in (-\sqrt{2}, 0). \end{cases}$$

and

$$h(x) = \begin{cases} 0 & x \in [0, \sqrt{2}) \\ \beta(-0) - \alpha(0) & x \in (-\sqrt{2}, 0). \end{cases}$$

$g(x), h(x)$ are real-valued functions on $(-\sqrt{2}, \sqrt{2})$ which satisfy the condition of Theorem 2 and $g(x)$ is continuous on $(-\sqrt{2}, \sqrt{2})$. Let $P_\ell, Q_\ell, R_\ell, \ell \in \mathbf{N}$ be the Hermite interpolating polynomials to f, g, h at $-1, 1$ with multiplicities ℓ, ℓ, respectively. Since $f(x) = g(x) + h(x), x \in (-\sqrt{2}, \sqrt{2})$, by Theorem 2 (2) and Corollary 11, we have

$$\lim_{\ell\to\infty} P_\ell(0) = \lim_{\ell\to\infty} (Q_\ell(0) + R_\ell(0)) = \alpha(0) + \frac{\beta(-0) - \alpha(0)}{2} = \frac{\alpha(0) + \beta(-0)}{2}.$$

Finally, we turn to prove Theorem 4.

Proof of Theorem 4. Let x be any given number in $(-\sqrt{2}, 0) \cup (0, \sqrt{2})$ and let k be any given positive integer. Suppose that a positive integer ℓ is lager than k. For a given real-valued function f on $[-r, r] (r > 1 + \sqrt{2})$ which satisfies the same condition as in Theorem 2, we put the Hermite interpolating polynomials P_ℓ to f at $-1, 1$ with multiplicities ℓ, ℓ

$$\begin{aligned} P_\ell(x) = f[-1,1;\ell,\ell](x+1)^\ell(x-1)^{\ell-1} + f[-1,1;\ell,\ell-1](x+1)^{\ell-1}(x-1)^{\ell-1} + \cdots \\ + f[-1,1;1,0] = U_{2\ell-1}(x) + U_{2\ell-2}(x) + \cdots + U_0(x), \quad (3.12) \end{aligned}$$

where $U_{2k-1}(x) = f[-1,1;k,k](x+1)^k(x-1)^{k-1}$ and $U_{2k-2}(x) = f[-1,1;k,k-1](x+1)^{k-1}(x-1)^{k-1}, 1 \le k \le \ell$. By Lemma 8 (1), we put

$$M = \sup_{i,j \in \mathbf{N}\cup\{0\}, i^2+j^2 \neq 0} |f[-1,1;i,j]|.$$

Let $R_{\ell-k}$ be the Hermite interpolating polynomials to $f^{(k)}$ at $-1, 1$ with multiplicities $\ell-k, \ell-k$. Since $R_{\ell-k}$ and the k-th order derivative $P_\ell^{(k)}$ of P_ℓ satisfy

$$(P_\ell^{(k)})^{(i)}(\pm 1) = (R_{\ell-k})^{(i)}(\pm 1) = f^{(k+i)}(\pm 1), \quad 0 \le i \le \ell - k - 1,$$

we have

$$\begin{aligned} P_\ell^{(k)}(x) - R_{\ell-k}(x) &= (x+1)^{\ell-k}(x-1)^{\ell-k}(a_{k-1,\ell}x^{k-1} + a_{k-2,\ell}x^{k-2} + \cdots + a_{0,\ell}) \\ &= c_{2\ell-k-1}x^{2\ell-k-1} + \cdots + c_{2\ell-2k}x^{2\ell-2k} + \cdots + c_0. \quad (3.13) \end{aligned}$$

Then, it is sufficient to show the existence of a polynomial $S(\ell)$ with

$$|a_{k-1,\ell}x^{k-1} + a_{k-2,\ell}x^{k-2} + \cdots + a_{0,\ell}| < S(\ell), x \in (-\sqrt{2}, 0) \cup (0, \sqrt{2}). \quad (3.14)$$

Because, if (3.14) holds, since $|x^2 - 1| < 1$,

$$\lim_{\ell\to\infty} |P_\ell^{(k)}(x) - R_{\ell-k}(x)| = \lim_{\ell\to\infty} |x^2 - 1|^{\ell-k} S(\ell) = 0.$$

We put

$$(x+1)^{\ell-k}(x-1)^{\ell-k} = x^{2\ell-2k}+b_{2\ell-2k-1}x^{2\ell-2k-1}+\cdots+b_{2\ell-3k+1}x^{2\ell-3k+1}+\cdots+b_0.$$

Since the absolute value of each coefficient $b_i, 2\ell-3k+1 \leq i \leq 2\ell-2k-1$, is smaller than the coefficient of $x^{2\ell-k}$ of $(x+1)^{2\ell}$, we have

$$|b_i| < \binom{2\ell}{k} < (2\ell)^k. \tag{3.15}$$

Furthermore, each $c_{2\ell-2k+t}, 0 \leq t \leq k-1$ satisfies

$$c_{2\ell-2k+t} = a_{t,\ell} + a_{t+1,\ell}b_{2\ell-2k-1} + \cdots + a_{k-1,\ell}b_{2\ell-3k+t+1}. \tag{3.16}$$

Noting that $R_{\ell-k}(x)$ is a polynomial of degree at most $2\ell-2k-1$, from (3.12) and (3.13), we obtain, for $0 \leq t \leq k-1$

$$\begin{aligned}|c_{2\ell-2k+t}| \leq{}& ((\text{the absolute value of the coefficient of } x^{2\ell-k+t} \text{ of } U_{2\ell-1})\\ &+\cdots+(\text{the absolute value of the coefficient of } x^{2\ell-k+t}\\ &\text{of } U_{2\ell-k+t}))(2\ell-k+t)\cdots(2\ell-2k+t+1)\\ \leq{}& M\binom{2\ell}{k}(k-t)(2\ell-k+1)\cdots(2\ell-2k+t+1) < M(2\ell)^k k(2\ell)^k = 4^k kM\ell^{2k}\\ &= L\cdot\ell^{2k}, \end{aligned} \tag{3.17}$$

where $L = 4^k kM$.

Now we will give estimations of the absolute values of $a_{k-1,\ell},\ldots,a_{0,\ell}$. For $|a_{k-1,\ell}|$, by (3.16) and (3.17) we have

$$|a_{k-1,\ell}| = |c_{2\ell-k-1}| < L\ell^{2k} = S_1(\ell).$$

$S_1(\ell)$ is a polynomial of degree $2k$. For $|a_{k-2,\ell}|$, since, by (3.16) and (3.17),

$$|c_{2\ell-k-2}| = |a_{k-2,\ell} + a_{k-1,\ell}b_{2\ell-2k-1}| < L\ell^{2k},$$

by the triangle inequality and (3.15), we get

$$|a_{k-2,\ell}| < L\ell^{2k} + S_1(\ell)(2\ell)^k = S_2(\ell).$$

$S_2(\ell)$ is a polynomial of degree $3k$. Repeating this procedure, we see that

$$|a_{k-i,\ell}| < S_i(\ell), \quad 1 \leq i \leq k,$$

where each $S_i(\ell)$ is a polynomial of degree $(i+1)k$. Hence, if we put $S(\ell) = 2^k(S_1(\ell)+S_2(\ell)+\cdots+S_k(\ell))$, then $S(\ell)$ is a polynomial of degree $k(k+1)$ and we easily have

$$\begin{aligned}|a_{k-1,\ell}x^{k-1} + a_{k-2,\ell}x^{k-2} + \cdots + |a_{0,\ell}| &\leq 2^k(|a_{k-1,\ell}| + |a_{k-2,\ell}| + \cdots + |a_{0,\ell}|)\\ &< 2^k(S_1(\ell) + S_2(\ell) + \cdots + S_k(\ell)) = S(\ell).\end{aligned}$$

This completes the proof.

In this note, we show a second step to the problem "What functions does $\mathcal{T}_m[I]$ consist of ?" Finally, we give problems which lead to a third step to this problem.

Problems. (1) Consider cases corresponding to Theorem 2 of $m(\geq 3)$ point Taylor expansions.

(2) Find other types of functions which belong to $\mathcal{T}_m[I](m \geq 2)$.

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