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JACOBIAN-FREE ACCELERATING CONVERGENCE METHOD FOR NONLINEAR SYSTEM OF EQUATIONS

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Abstract: In this paper, we present a Jacobian free accelerating convergence of the successive approximation method for solving nonlinear system of equations. The set of parameters is introduced for successive approximation method as consequence accelerating convergence is achieved. The convergence condition and rate of convergence are derived analytically. Some numerical illustrations are given and we compared these methods with Newton's method and successive approximation method.

AMS Subject Classification: 65H10
Key Words: non-linear equations, Newton-Raphson method, successive approximation method

1. Introduction

The real world problems when modeled mathematically yield complex dynamical systems. Their mathematical descriptions reduce to systems of algebraic, transcendental, functional and differential equations (may be linear or nonlinear).

It is well known that in recent years there has been a considerable interest in various higher order methods for solving nonlinear system of equations.

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Narasimham., K.V achieved the Accelerating Convergence for Newton's Method with third order, successive approximation method for solving nonlinear scalar equations in [7] and [8].

We have considered the nonlinear system of equations of the following

$$F(X) = 0 \tag{1}$$

where $F(X) = (F_1X), \ldots, F_n(X))^T$ and $X = (x_1, \ldots, x_n)^T$

For the successive approximation method the system (1) can be written as

$$x_i = f_i(x_1, \ldots, x_n),\ i = 1, 2 \ldots, n \tag{2}$$

The system (2) converges if the following condition is satisfied [3]

$$\sum_{j=1}^{n} \left| \frac{\partial \mathrm{f_i}}{\partial x_\mathrm{j}} \right| < 1,\ i = 1, 2 \ldots, n \tag{3}$$

The Multi Variable Newton-Raphson Method [5]and[6]for the solution of (1) is given by

$$X^{(k+1)} = X^{(k)} - [J^{(k)}]^{-1} F(X^{(k)}) \tag{4}$$

where $J^{(k)} = \left(\frac{\partial F_i}{\partial x_j}\right)$for each i and j. The system (4) converges if $[J^{(k)}]^{-1}$ exists.

2. Preliminary Results

Definition 1. [1] Let $X^* \in \mathbb{R}^m, X_n \in \mathbb{R}^m$,$n = 0, 1 \ldots$ then the sequence $\{X_n\}$ is said to converge to X^* if $\lim\limits_{n\to\infty} \|X_n - \mathrm{X}^*\| = 0$. If, in addition there exists a constant $c = 0$, an integer $n_0 = 0$, and $\rho = 0$ such that for all $n > n_0, \|X_{n+1} - \mathrm{X}^*\| \leq c \|X_n - \mathrm{X}^*\|^\rho$, then $\{X_n\}$is said to converge to X^* with q-order at least ρ. If ρ is 2 or 3, the convergence is said to be q-quadratic or q-cubic respectively. When $E_n = X_n - X^*$ is the error in the n^{th} iteration, the relation $E_{n+1} = cE_n^\rho + O(E_n^{\rho+1})$ is called the error equation. By substituting $E_n = X_n - X^*$ for all n in any iteration method and simplifying, we obtain the error equation for that method. The value of ρ thus obtained is called the order of this method.

Definition 2. [10] Let X^*be a solution of the system $F(x) = 0$ and suppose that X_{n+1}, X_n and X_{n-1} are three consecutive iteration closes to the root X^*, then the Computational Order of Convergence (COC) $\overline{\rho}$ can be approximated using the formula

$$\overline{\rho} = \frac{\ln\left[\|X_{n+1} - \mathrm{X}^*)\| \,/\, \|X_n - \mathrm{X}^*\|\right]}{\ln\left[\|X_n - \mathrm{X}^*\| \,/\, \|X_{n-1} - \mathrm{X}^*\|\right]}.$$

Definition 3. [2]and[4] Let X^* be a solution of the system $F(X) = 0$ and suppose that X_{n+1}, X_n and X_{n-1} are three consecutive iteration closes to the root X^*, then the Approximated computational order of convergence (ACOC) $\widehat{\rho}$ can be approximated using the formula

$$\widehat{\rho} = \frac{\ln[\|X_{n+1} - X_n\|]/\ln[\|X_n - X_{n-1}\|]}{\ln[\|X_n - X_{n-1}\|]/\ln[\|X_{n-1} - X_{n-2}\|}.$$

Stopping Criteria. We have to accept an approximate solution rather than the exact root, depending on the precision (ϵ) of the computer so, we use the following stopping criteria for computer programs:(i) $\|X_{n+1} - \mathrm{X}^*\| < \varepsilon$ (ii) $\|F(X_{n+1})\| < \varepsilon$

3. Jacobian-Free Accelerating Convergence Method for Nonlinear System of Equations

Let $X^* = (x_1^*, \ldots, x_n^*)$ be a solution of the system (2), and in the region $R : (x_i^* - h, x_i^* + h), i = 1, 2, \ldots, n$ such that

$$\sum_{j=1}^{n} \left| \frac{\partial \mathrm{f_i}}{\partial x_\mathrm{j}} \right| < \gamma < 1, i = 1, 2, \ldots, n.$$

Let us consider a set of non-negative parameters $\alpha_i \in (0, 1/\lambda_i]$, such that

$$\lambda_i = \sum_{j=1, j\neq i}^{n} \left| \frac{\partial \mathrm{f_i}}{\partial x_\mathrm{j}} \right| (\neq 0),\ i = 1, 2, \ldots, n. \tag{5}$$

and

$$x_i^{(k+1)} = (1 - \alpha_i^{(k)}) x_i^{(k)} + \alpha_i^{(k)} x_i,\ i = 1, 2, \ldots, n. \tag{6}$$

From (2) and (6), we have

$$x_i^{(k+1)} = (1 - \alpha_i^{(k)}) x_i^{(k)} + \alpha_i^{(k)} f_i(x_1^{(k)}, \ldots, x_n^{(k)}),\ i = 1, 2, \ldots, n. \tag{7}$$

Let $\phi_i(x_1^{(k)}, \ldots, x_n^{(k)}) = (1 - \alpha_i^{(k)}) x_i{}^{(k)} + \alpha_i^{(k)} f_i(x_1^{(k)}, \ldots, x_n^{(k)})$.

Now, $x_i^{(k+1)} - x_i^* = \phi_i(x_1^{(k)}, \ldots, x_n^{(k)}) - \phi_i(x_1^*, \ldots, x_n^*)$ and applying the law of mean for a function of n variables, we have

$$x_i^{(k+1)} - x_i^* = \left(\frac{\partial \phi_i}{\partial x_1} \right)_{\xi^k} (x_1^{(k)} - x_1^*) + \cdots + \left(\frac{\partial \phi_i}{\partial x_n} \right)_{\xi^k} (x_n{}^{(k)} - x_n^*),$$

where $\xi^k = (\xi_1^k, \ldots, \xi_n^k), \xi_i^k$ lies between ${x_i}^{(k)}$ and x_i^*, so that $\xi^k = (\xi_1^k, ..., \xi_n^k)$ is in $\mathbb{R}^n$. Then, representing the largest of the quantities, $|x_i^{(k)} - x_i^*|$, by $M^{(k)}$, we have

$$\begin{aligned}
|x_i^{(k+1)} - x_i^*| &< \left|\frac{\partial \phi_i}{\partial x_1} + \cdots + \frac{\partial \phi_i}{\partial x_n}\right| |x_i^{(k)} - x_i^*| \\
&< \left|(1-\alpha_i) + \alpha_i \frac{\partial \mathrm{f_i}}{\partial x_i} + \sum_{j=1, j\neq k}^{n} \alpha_i \frac{\partial \mathrm{f_i}}{\partial x_\mathrm{j}}\right| M^{(k)} \\
&< \left[\left|(1-\alpha_i) + \alpha_i \frac{\partial \mathrm{f_i}}{\partial x_i}\right| + \sum_{j=1, j\neq k}^{n} \left|\alpha_i \frac{\partial \mathrm{f_i}}{\partial x_\mathrm{j}}\right|\right]^2 M^{(k-1)} \ldots \\
&< \left[\left|(1-\alpha_i) + \alpha_i \frac{\partial \mathrm{f_i}}{\partial x_i}\right| + \sum_{j=1, j\neq k}^{n} \left|\alpha_i \frac{\partial \mathrm{f_i}}{\partial x_\mathrm{j}}\right|\right]^k M^{(0)}.
\end{aligned}$$

The convergence follows from these inequalities,

$$\left|(1-\alpha_i) + \alpha_i \frac{\partial \mathrm{f_i}}{\partial x_i}\right| + \sum_{j=1, j\neq k}^{n} \left|\alpha_i \frac{\partial \mathrm{f_i}}{\partial x_\mathrm{j}}\right| < 1, i = 1, 2, \ldots, n. \tag{8}$$

For i = k, the condition (8) reduces to

$$\left|1 - \alpha_k + \alpha_k \frac{\partial \mathrm{f_k}}{\partial x_\mathrm{k}}\right| + \alpha_k \sum_{j=1, j\neq k}^{n} \left|\frac{\partial \mathrm{f_k}}{\partial \mathrm{x_j}}\right| < 1. \tag{9}$$

By rewriting (9), we get

$$1 - \alpha_k \sum_{j=1, j\neq k}^{n} \left|\frac{\partial \mathrm{f_k}}{\partial x_\mathrm{j}}\right| > \left|1 - \alpha_k + \alpha_k \frac{\partial \mathrm{f_k}}{\partial x_\mathrm{k}}\right|. \tag{10}$$

But

$$1 - \alpha_k \sum_{j=1, j\neq k}^{n} \left|\frac{\partial \mathrm{f}_k}{\partial x_\mathrm{j}}\right| \geq 0, \tag{11}$$

From (10) and (11), we get

$$\left|1 - \alpha_k + \alpha_k \frac{\partial \mathrm{f_k}}{\partial x_\mathrm{j}}\right| = 0. \tag{12}$$

By simplifying (12), we have

$$\alpha_k = \left[1 - \frac{\partial f_k}{\partial x_k}\right]^{-1} (\neq 0), k = 1, 2, \ldots, n. \tag{13}$$

Theorem 4. *Let us assume that the system (2) has a solution, then the Accelerating parameters α_i, $i = 1, 2, \ldots, n$, are always non-negative and $\alpha_i \in (0, 1/\lambda_i]$, where*

$$\lambda_i = \sum_{j=1, j\neq i}^{n} \left|\frac{\partial \mathrm{f_i}}{\partial x_\mathrm{j}}\right| (\neq 0).$$

Proof. From (2), we have $\left|\frac{\partial \mathrm{f_i}}{\partial x_\mathrm{i}}\right| < 1$, $i = 1, 2, \ldots, n$.

Case(i) Let $0 < \frac{\partial \mathrm{f_i}}{\partial x_\mathrm{i}} < 1, i = 1, 2, \ldots, n$.

This gives that $\alpha_i = \left[1 - \frac{\partial f_i}{\partial x_i}\right]^{-1}$ is positive and $\alpha_i > 1, i = 1, 2, \ldots, n$. And $1 - \frac{\partial \mathrm{f_i}}{\partial x_\mathrm{i}} \geq 1 - \left|\frac{\partial \mathrm{f_i}}{\partial x_\mathrm{i}}\right|$, $i = 1, 2, \ldots, n$.

Since

$$\sum_{j=1}^{n} \left|\frac{\partial \mathrm{f_i}}{\partial x_\mathrm{j}}\right| < 1, i = 1, 2, \ldots, n,$$

then

$$\sum_{j=1, j\neq i}^{n} \left|\frac{\partial \mathrm{f_i}}{\partial x_\mathrm{j}}\right| + \left|\frac{\partial \mathrm{f_i}}{\partial x_\mathrm{i}}\right| < 1.$$

This implies that,

$$\sum_{j=1, j\neq i}^{n} \left|\frac{\partial \mathrm{f_i}}{\partial x_\mathrm{j}}\right| < 1 - \left|\frac{\partial \mathrm{f_i}}{\partial x_\mathrm{i}}\right|.$$

Hence,

$$\frac{1}{1 - \frac{\partial \mathrm{f_i}}{\partial x_\mathrm{i}}} < \left[\sum_{j=1, j\neq i}^{n} \left|\frac{\partial \mathrm{f_i}}{\partial x_\mathrm{j}}\right|\right]^{-1}, \ i = 1, 2, \ldots, n.$$

This gives $\alpha_i < 1/\lambda_i$, $i = 1, 2, \ldots, n$.

Case (ii) Let $-1 < \frac{\partial \mathrm{f_i}}{\partial x_\mathrm{i}} < 0$, $i = 1, 2, \ldots, n$.

Clearly, $\alpha_i = \left[1 - \frac{\partial f_i}{\partial x_i}\right]^{-1}$ is positive and $\frac{1}{2} < \alpha_i$, $i = 1, 2, \ldots, n$.

Case (iii) Let $\frac{\partial \mathrm{f_i}}{\partial x_\mathrm{i}} = 0$. (i.e.,) $\alpha_i = 1$.
This implies that no acceleration is achieved. □

3.1. Convergence Condition

Theorem 5. *The Jacobian-Free Accelerating Convergence of successive approximation method will always converge for*

$$\alpha_i = \left[1 - \frac{\partial f_i}{\partial x_i}\right]^{-1},$$

for each i, if the system(2) is convergent.

Proof. Let us assume the system (2) is convergent.
Let

$$\varphi_i(X) = (1-\alpha_i^{(k)})x_i^{(k)} + \alpha_i^{(k)} f_i(x_1^{(k+1)}, \dots, x_{i-1}^{(k+1)}, x_i^{(k)}, \dots, x_n^{(k)})$$

for each $i = 1, 2, \dots, n$.
Then the system (7) takes the form

$$x_i^{(k+1)} = \varphi_i(x_1^{(k+1)}, \dots, x_{i-1}^{(k+1)}, x_i^{(k)}, \dots, x_n^{(k)}),\ i = 1, 2, \dots, n.$$

Now,

$$\sum_{j=1}^{n} \left|\frac{\partial \varphi_i}{\partial x_j}\right| = \left|1 - \alpha_i + \alpha_i \frac{\partial f_i}{\partial x_i}\right| + \sum_{j=1, j\neq i}^{n} \left|\alpha_i \frac{\partial f_i}{\partial x_j}\right|,\ i = 1, 2, \dots, n.$$

Hence,

$$\sum_{j=1}^{n} \left|\frac{\partial \varphi_i}{\partial x_j}\right| = \sum_{j=1, j\neq i}^{n} \left|\alpha_i \frac{\partial f_i}{\partial x_j}\right| < 1.$$

□

3.2. Rate of Convergence

Theorem 6. *Let $f_i : D \subseteq R^n \to R$ be continuously differentiable functions such that*

$$\sum_{j=1}^{n} \left|\frac{\partial f_i}{\partial x_j}\right| < \gamma < \frac{1}{3} < 1,$$

for each $i = 1, 2, \dots, n$, then the Accelerated Convergence of successive approximation method defined by (3.3), $\alpha_i^{(k)} = \left[1 - \frac{\partial f_i^*}{\partial x_i}\right]^{-1}$ and $f_i^* = f_i(x_1^*, \dots, x_n^*)$, $x_i^* = f_i(x_1{}^{(k)}, \dots, x_n^{(k)})$ for $k \geq 0$ converges to $X^* = (x_1^*, \dots, x_n^*)^T$.

Proof. Since $f_i : D \subseteq R^n \to R$, the sequence $\{X^{(k)}\}_{k=1}^{\infty}$, where $X = (x_1, \ldots, x_n)^T$, is defined for all $k > 0$. Now,

$$\sum_{i=1}^{n} | x_i^{(k+1)} - x_i^* | = \sum_{i=1}^{n} (1 - \alpha_i^{(k)}) x_i^{(k)} + \alpha_i^{(k)} f_i(X^{(k)}) - x_i^*,$$

where $f_i(X^{(k)}) = f_i(x_1^{(k+1)}, \ldots, x_{i-1}^{(k+1)}, x_i^{(k)}, \ldots, x_n^{(k)})$.

This implies

$$\begin{aligned}\sum_{i=1}^{n} | x_i^{(k+1)} - x_i^* | &= \sum_{i=1}^{n} \alpha_i^{(k)} (f_i(X^{(k)}) - x_i^*) + (1 - \alpha_i^{(k)})({x_i}^{(k)} - x_i^*) \\ &= \sum_{i=1}^{n} \left[\alpha_i^{(k)} f_i(X^{(k)}) - x_i^* + 1 - \alpha_i^{(k)} x_i^{(k)} - x_i^*\right] \\ &= \sum_{i=1}^{n} \left[{\alpha_i}^{(k)} f_i(X^{(k)}) - f_i(X^*) + 1 - \alpha_i^{(k)} x_i^{(k)} - x_i^*\right] \\ &= \sum_{i=1}^{n} \alpha^{(k)} x^{(k)} - x^* \sum_{j=1}^{n} \left| \frac{\partial \mathrm{f}_\mathrm{i}^*}{\partial x_\mathrm{j}} \right| + 1 - \alpha^{(k)} x^{(k)} - x^*\end{aligned}$$

This implies that

$$\begin{aligned}\sum_{i=1}^{n} | x_i^{(k+1)} - x_i^* | &< \sum_{i=1}^{n} \left[\gamma \alpha_i^{(k)} x_i^{(k)} - x_i^* + 1 - \alpha_i^{(k)} x_i^{(k)} - x_i^*\right] \\ &< \sum_{i=1}^{n} x_i^{(k)} - x_i^* (\gamma \alpha_i^{(k)} + 1 - \alpha_i^{(k)}) \\ &< \sum_{i=1}^{n} x_i^{(k)} - x_i^* \left[\frac{\gamma + \left| \frac{\partial \mathrm{f}_\mathrm{i}^*}{\partial x_x} \right|}{\left| 1 - \frac{\partial \mathrm{f}_\mathrm{i}^*}{\partial x_x} \right|}\right].\end{aligned}$$

Thus,

$$\sum_{i=1}^{n} | {x_i}^{(k+1)} - {x_i}^* | < \sum_{i=1}^{n} {x_i}^{(k)} - {x_i}^* \left[\frac{2\gamma}{1 - \gamma}\right].$$

Therefore,

$$\sum_{i=1}^{n} | x_i^{(k)} - {x_i}^* | < \left[\frac{2\gamma}{1 - \gamma}\right] \sum_{i=1}^{n} x_i^{(k-1)} - x_i^*$$

$$
\begin{aligned}
&< \left[\frac{2\gamma}{1-\gamma}\right]^2 \sum_{i=1}^{n} \left| x_i^{(k-2)} - x_i^* \right| \\
&< \left[\frac{2\gamma}{1-\gamma}\right]^3 \sum_{i=1}^{n} \left| x_i^{(k-3)} - x_i^* \right| \\
&\ldots \\
&< \left[\frac{2\gamma}{1-\gamma}\right]^k \sum_{i=1}^{n} \left| x_i^{(0)} - x_i^* \right|.
\end{aligned}
$$

Since $0 < \gamma < \frac{1}{3} < 1$, we have $\lim_{k\to\infty} \left[\frac{2\gamma}{1-\gamma}\right]^k \mid x^{(0)} - x^* \mid = 0$.

Hence $\left\{X^{(k)}\right\}_{k=1}^{\infty}$ converges to X^*. □

Algorithm 1.

The Jacobian-Free Accelerating convergence method is

$$
x_i^{(k+1)} = (1 - \alpha_i^{(k)}) x_i^{(k)} + \alpha_i^{(k)} f_i(x_1^{(k+1)}, \ldots, x_{i-1}^{(k+1)}, x_i^{(k)}, \ldots, x_n^{(k)}),
$$

where $\alpha_i = \left[1 - \frac{\partial f_i^*}{\partial x_i}\right]^{-1}$ and $f_i^* = f_i(x_1^*, \ldots, x_n^*), x_i^* = f_i(x_1, \ldots, x_n)$.

4. Numerical Problems

In this Section, we employ the new method, The Jacobian-Free Accelerating Convergence Method (JFACM) to solve nonlinear systems of equations and compare it with Newton-Raphson Method (NM), and successive approximation methods (SAM). All Numerical experiments were done in MATLAB with 15 digits precision.

Table 1 gives the comparison of the number of iterations (IT), functional evaluations (NFE), Computational Order of Convergence (COC), and Approximated Computational Order of Convergence (ACOC)of Test Problem 1.The solution and initial guess of Test Problem 2 are listed in Table 2.The computational results show that the new methods require less IT and NFE than NM and SAM as far as the numerical results are concerned. Therefore, the new methods are of practical interest.

Test Problem 1. We consider the non-linear system of three equations in three unknowns[11],

$$
F_1(x_1, x_2, x_3) = 3x_1 - Cos(x_2 x_3) - 0.5 = 0,
$$

Test Problem-1			
Parameter	**Method**	$X_0 = (0, 0, 0)$	$X_0 = (1, 1, 0)$
IT	JFACM	3	3
	NM	∞	10
	SAM	20	20
NFE	JFACM	18	18
	NM	∞	120
	SAM	60	48
COC	JFACM	1.5570	1.5570
	NM	∞	1.1154
	SAM	0.9884	0.9881
ACOC	JFACM	0.5818	0.5818
	NM	∞	1.0136
	SAM	1.0111	0.9893

Table 1

$$F_2(x_1, x_2, x_3) = x_1^2 - 625x_2^2 = 0,$$
$$F_3(x_1, x_2, x_3) = e^{-x_1x_2} + 20x_3 + 9 = 0.$$

The solution of Test Problem 1 is

$$x_1 = 0.499983367726846,$$
$$x_2 = 0.019999334709074$$
$$x_3 = -0.499502524620480.$$

Test Problem 2. Consider the integral equation [9]

$$u(t) = 1 + \frac{1}{4} t u(t) \int_0^1 \frac{u(s)}{s+t} ds \tag{14}$$

This equation is a special case of the H-equations arising in the study of radiative transfer. We approximate this integral equation applying the trapezoidal rule at the $n + 1$ equidistant nodes ih, where $h = 1/n$. Making use of $u(0) = 1$, we obtain the nonlinear systems of equations.

$$F_i(X) = x_i \left(c - \frac{h}{4} \sum_{j=1}^{n} w_{ij} x_j \right) - 1, i = 1, 2, \ldots, n \tag{15}$$

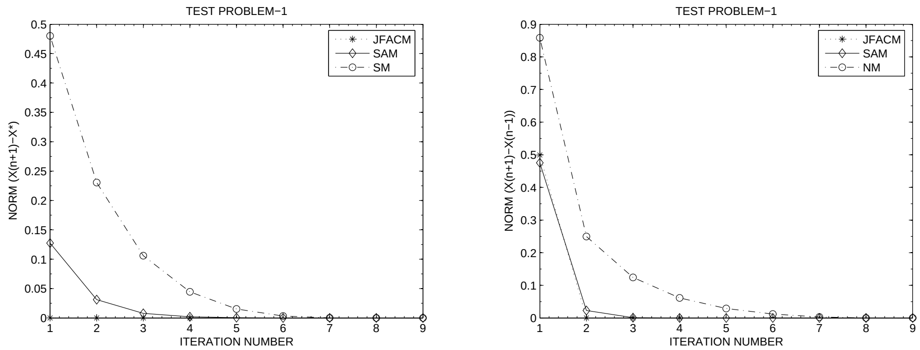

Figure 1: (a) & (b)

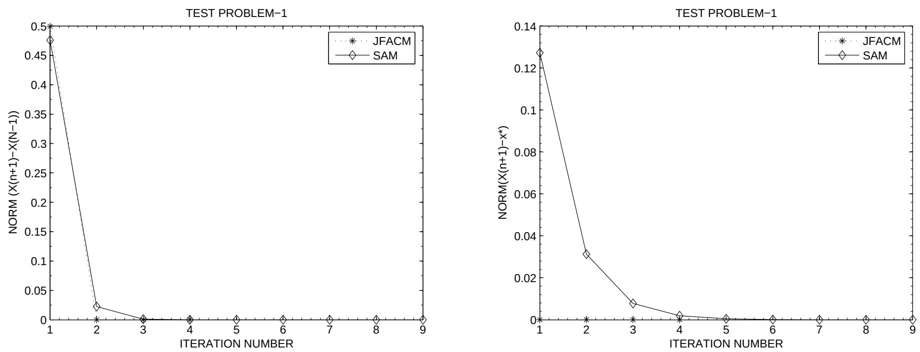

Figure 2: (a) & (b)

with $c = \left(1 - \frac{h}{8}\right), w_{in} = \frac{i}{2(i+n)}$ and $w_{ij} = \frac{i}{i+j}$ for $j = 1, 2, \ldots, n-1$. Herein x_i is an approximation of $u(ih)$. The above system (15) can be written as,

$$f_i = x_i = \frac{\left[1 + \frac{h}{4} x_i \sum_{j=1}^{n} w_{ij} x_j\right]}{c}, i = 1, 2, \ldots, n. \tag{16}$$

The calculation was done on computer using MATLAB program of 15 decimal digits with $n = 300$. The iterative values of $x_{300,}$ corresponding to $u(1)$,was tabulated in Table 2

Remark. JFACM requires n number of partial derivatives where as Classical Newton's Method (CNM) required n^2 number of partial derivatives for solving nonlinear systems of equations with n variables,n equations.

In Test Problem 2, 300 partial derivatives are used in JFACM and $90,000$ partial derivatives are required for CNM. So, CNM is quite difficult to im-

	JFACM	SAM
Initial Guess	(1,1,...,1)	(1,1,...,1)
Solution x_{300}	1.251259561665	1.251259561665
IT	9	19

Table 2

plement for Test Problem 2 whereas JFACM is much suitable for solving the integral equation (14). Further we note that the inverse of the Jacobian matrix of order 300×300 is required for implementation of CNM but inverse matrix is not required for JFACM.

5. Conclusion

By computational analysis of Test Problem and results in Section 3, we observed that the introduction of acceleration parameters α_i forced the Acceleration of convergence of SAM for solving nonlinear systems of equations. Further, JFACM for Test Problem 1 is also applicable when classical NM fails or divergence cases, which is shown in Table 1. The introduction of appropriate relaxation parameters forced the convergence for the divergence causes of Test Problem 1. We conclude that the acceleration parameters α_i played a vital role in the Accelerating Convergence of SAM. These computational analyses show that the new Algorithm is more efficient and it performs better than classical Newton's method and some other methods.

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