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ROBUST DENSITY OF PERIODIC SINKS AND SOURCES FOR ITERATED FUNCTION SYSTEMS

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Abstract: Given any compact m-dimensional manifold M, we describe C^1-open sets of iterated function systems admitting infinite number of attracting and repelling periodic orbits. We show that attracting periodic orbits are dense in the ambient manifold M. Also, the same property holds for repelling periodic orbits.

Moreover, the step skew product maps corresponding to these iterated function systems are topologically transitive.

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1. Introduction

In this paper, we will study the robust existence of infinite number of periodic sinks and sources for iterated function systems defined on a compact m-dimensional manifold M. We show that attracting periodic orbits are dense

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in the ambient manifold for minimal iterated function systems.

Gorodetskii and Ilyashenko [2,3] provide an example of an iterated function system generated by two circle diffeomorphisms which is robustly minimal in the C^1-topology. In [1], the first author with Homburg and Sarizade generalize this example to iterated function systems on any m-dimensional compact manifold (see also [4]).

Here, we construct an iterated function system $\mathcal{F}$ on m-dimensional compact manifold M which is C^1-robustly minimal and attracting periodic orbits of $\mathcal{F}$ are dense in M. Also, the same property holds for the repelling periodic orbits. Finally, we prove that the corresponding step skew product map is topologically transitive.

First we collect definitions and basic properties about iterated function systems. Suppose that M is an m-dimensional compact manifold. Consider diffeomorphisms $f_i,\ i = 1, \dots, k$, defined on M. The *iterated function system* $\mathcal{F}(M; f_1, \dots, f_k)$ is the semigroup generated by $f_1, \dots, f_k$, i.e. , the set of all maps $f_{t_j} \circ \dots \circ f_{t_1}$, where $t_j, \dots, t_1 \in \{1, \dots, k\}$. The concatenation $w = t_1 \dots t_l$ is called a *word* of length l on the alphabet $\{1, \dots, k\}$. Also,we denote $f_{[w]} = f_{t_l} \circ \dots \circ f_{t_1}$.

The $\mathcal{F}$-*orbit* of $x \in M$ is the set of points $f_{t_i} \circ \dots \circ f_{t_1}(x),\ t_j \geq 0$.

An iterated function system $\mathcal{F}(M; f_1, \dots, f_k)$ is called *minimal* if each closed subset $A \subset M$ such that $f_i(A) \subset A$ for all i is empty or coincides with M. This means that $\mathcal{F}$-orbit of each $x \in M$ is dense in M.

Let σ be the Bernoulli shift map on the symbol space $\Sigma^k = \{1, \dots, k\}^{\mathbb{Z}}$. For the iterated function system $\mathcal{F}(M; f_1, \dots, f_k)$ on M, we define the *corresponding step skew product*

$$F : \Sigma^k \times M \to \Sigma^k \times M,$$

over σ as $(\omega, x) \mapsto (\sigma\omega, f_{\omega_0}(x))$, where ω_0 is the zeroth symbol of the sequence ω and the fiber map $f_{\omega_0} := f_i$, for $\omega_0 = i$.

Consider the iterations of step skew product map F. It is obvious that, for $n > 0$

$$F^n(\omega, x) = (\sigma^n\omega, \overline{f}_n[\omega](x)),$$

where

$$\overline{f}_n[\omega] = f_{\omega_{n-1}} \circ \dots \circ f_{\omega_0},$$

and

$$F^{-n}(\omega, x) = (\sigma^{-n}\omega, \overline{f}_{-n}[\omega](x)),$$

where

$$\overline{f}_{-n}[\omega] = f^{-1}_{\omega_{-n}} \circ \dots \circ f^{-1}_{\omega_{-1}}.$$

Also, we set $\overline{f}_0[\omega] := id$.

Let us note that the action orbit of an iterated function system $\mathcal{F}(M; f_1, \ldots, f_k)$ coincide with the projections of positive semitrajectories of the corresponding step skew product map F onto the fiber M along the base Σ^k.

We say that F is *topologically transitive* on $\Sigma^k \times M$, if it admits a dense orbit.

Our main result is as follows.

Theorem 1.1. *Suppose that M is an m-dimensional compact manifold. There exist open sets $\mathcal{U}_i \subset Diff^1(M)$, $i = 1, \ldots, m+3$, such that for any $f_i \in \mathcal{U}_i$, the iterated function system $\mathcal{F}(M; f_1, \ldots, f_{m+3})$ possesses the following properties.*

i) The iterated function systems $\mathcal{F}(M; f_1, \ldots, f_{m+3})$ and $\mathcal{F}(M; {f_1}^{-1}, \ldots, {f_{m+3}}^{-1})$ are C^1-robustly minimal.

ii) Attracting periodic orbits of $\mathcal{F}(M; f_1, \ldots, f_{m+3})$ are dense in M. Moreover, the same property holds for the repelling periodic orbits.

iii) If F is the corresponding step skew product of the iterated function system $\mathcal{F}$, then F is topologically transitive on $\Sigma^{m+3} \times M$.

2. Robust Minimal IFS's with Infinite Number of Periodic Sinks and Sources

Consider Morse-Smale diffeomorphisms $f_1, \ldots, f_{m+3}$, each of which admitting a unique attracting hyperbolic fixed point p_i; a unique repelling hyperbolic fixed point q_i and a finite number of hyperbolic saddles. We choose the diffeomorphisms f_i, $i = 1, \ldots, m+1$, such that the subsets $\{p_i,\ i = 1, \ldots, m+1\}$ and $\{q_i,\ i = 1, \ldots, m+1\}$ are affine independents, i.e. they are the vertices of m-dimensional simplices. Moreover, they are contained in domains of two disjoint local charts. By Lemma 2.1 and Lemma 2.2 of [1], there exist compact sets Δ and Δ' of M with nonempty interiors such that the acting of iterated function systems $\mathcal{F}(f_1, \ldots, f_{m+1})$ and $\mathcal{F}({f_1}^{-1}, \ldots, {f_{m+1}}^{-1})$ are minimal on Δ and Δ', respectively.

Let T be the time-1 map of a gradient Morse-Smale vector field with a unique hyperbolic repelling equilibrium $q \in int(\Delta)$ and a unique hyperbolic attracting equilibrium $p \in int(\Delta')$. We set $f_{m+2} := T$ and $f_{m+3} := T^{-1}$.

By Theorem 1.1 of [1], the iterated function systems $\mathcal{F}(M; f_1, \ldots, f_{m+3})$ and $\mathcal{F}(M; {f_1}^{-1}, \ldots, {f_{m+3}}^{-1})$ are C^1-robustly minimal.

The rest of this section is devoted to prove the main result.

Lemma 2.1. *Consider the iterated function system $\mathcal{F}(M; f_1, \ldots, f_{m+3})$ as above. For every nonempty open set $U \subset M$ there exists $k \leq k_0 \in \mathbb{N}$ and $\rho = \rho(U) > 0$ such that for every ball $B \subset M$ of radius ρ, there exists a word $w = t_1 \ldots t_{t_k}$ on the alphabet $\{1, \ldots, m+3\}$ of length $k \leq k_0$ such that $f_{[w]}(B) \subset U$.*

Proof. Let $U \subset M$ be an open subset. Since the acting $\mathcal{F}$ on M is minimal, for each $x \in M$ there exists a word $w(x)$ on alphabet $\{1, \ldots, m+3\}$ such that $f_{[w(x)]}(x) \in U$.

By continuity, there is a neighborhood V_x of x such that $f_{[w]}(V_x) \subset U$. Since M is compact, we can cover M by finitely many sets V_{x_i}. We take k_0 the maximum of the lengths of the words $w(x_i)$ and $\rho > 0$ the Lebesgue number of this covering.

Then, every ball $B \subset M$ of radius less than ρ is contained in some V_{x_i}, so there exists a word $w = t_1 \ldots t_{t_k}$ on the alphabet $\{1, \ldots, m+3\}$ of length $k \leq k_0$ such that $f_{[w]}(B) \subset U$. $\square$

Let $\mathcal{F}(M; f_1, \ldots, f_k)$ be an iterated function system with the corresponding step skew product map F. In the following, we will also use the notation

$$C_{\overline{\alpha}} = \{\omega \in \Sigma^k | \omega_j = \alpha_j, \ -n \leq j \leq n-1\},$$

where $\overline{\alpha} = \alpha_{-n} \ldots \alpha_0 \ldots \alpha_{n-1}$ is a segment of $\{1, \ldots, k\}$.

Proposition 2.2. *If an iterated function system $\mathcal{F}(M; f_1, \ldots, f_k)$ is minimal, then the corresponding step skew product map F is topologically transitive on $\Sigma^k \times M$.*

Proof. First, we note that open sets $C_{\overline{\alpha}} \times J$, where J is an open ball of M and $C_{\overline{\alpha}}$ is a cylinder set of Σ^k, form a base of the topology of the space $\Sigma^k \times M$.

Consider two open sets $C_{\overline{\alpha}} \times J$ and $C_{\overline{\beta}} \times J'$ of $\Sigma^k \times M$, where $\overline{\alpha} = \alpha_{-n} \ldots \alpha_0 \ldots \alpha_{n-1}$ and $\overline{\beta} = \beta_{-m} \ldots \beta_0 \ldots \beta_{m-1}$ are two segments of $1, \ldots, k$. By Birkhoff Transitivity theorem, it is sufficient to prove that

$$F^t(C_{\overline{\alpha}} \times J) \cap C_{\overline{\beta}} \times J' \neq \emptyset, \ F^{-l}(C_{\overline{\alpha}} \times J) \cap C_{\overline{\beta}} \times J' \neq \emptyset,$$

for some positive integers t and l.

Let $x \in J$. Since the acting $\mathcal{F}$ on M is minimal, there exists a word $\gamma = \gamma_{t-m-1} \ldots \gamma_0$ on the alphabet $\{1, \ldots, k\}$ such that $\gamma_0 = \alpha_0, \ldots, \gamma_{n-1} = \alpha_{n-1}$ and

$$f_{\gamma_{t-m-1}} \circ \ldots \circ f_{\gamma_n}(f_{\gamma_{n-1}} \circ \ldots \circ f_{\gamma_0}(x)) \in f^{-1}_{\beta_{-m}} \circ \ldots \circ f^{-1}_{\beta_{-1}}(J').$$

Then

$$f_{\beta_{-1}} \circ \ldots \circ f_{\beta_{-m}} \circ f_{\gamma_{t-m-1}} \circ \ldots \circ f_{\gamma_0}(x) \in J'.$$

Take a sequence $\omega \in \Sigma^k$ such that

$$\omega_{-n} = \alpha_{-n}, \ldots, \omega_0 = \alpha_0, \ldots, \omega_{n-1} = \alpha_{n-1},$$

$$\omega_t = \beta_0, \omega_{t-1} = \beta_{-1}, \ldots, \omega_{t-m} = \beta_{-m},$$

$$\omega_n = \gamma_n, \ldots, \omega_{t-m-1} = \gamma_{t-m-1}.$$

Then $\omega \in C_{\overline{\alpha}}$ and $F^t(\omega, x) \in C_{\overline{\beta}} \times J'$.

Analogously, for $x' \in J$, we can choose a sequence $\omega' \in \Sigma^k \cap C_{\overline{\alpha}}$ such that

$$F^{-l}(\omega', x') \in C_{\overline{\beta}} \times J'.$$

This terminates the proof of the proposition. □

Now let us prove the main result of the paper.

Proof of the Main Result. The last statement of the main theorem follows from Proposition 2.2.

Now, it is sufficient to show that the attracting periodic orbits are dense in M. Let $x \in M$ and $\mathcal{W}$ be an open subset of M containing x. Let $U_1 \subset int(\Delta)$ be an open subset containing fixed point p_1 of f_1.

By applying Lemma 2.1 for U_1, there exist $\rho_1 = \rho_1(U_1) > 0$ and $k_1 = k_1(U_1) \in \mathbb{N}$ such that for every ball $B \subset M$ of radius ρ_1, there exists a word $w = t_1 \ldots t_k$ on the alphabet $\{1, \ldots, m+1\}$ of length $k \leq k_1$ such that $f_{[w]}(B) \subset U_1$.

Let $U(x)$ be the open ball of center x and radius ρ_1 which is contained in $\mathcal{W}$. Therefore, there exists a word $w(x)$ of length $k \leq k_1$ such that $f_{[w(x)]}(U(x)) \subset U_1$.

Now, we can apply Lemma 2.1 for $U(x)$, there exist $\rho_2 = \rho_2(U(x)) > 0$ and $k_2 = k_2(U(x)) \in \mathbb{N}$ such that for every ball $B \subset M$ of radius ρ_2, there exist a word $w = s_1 \ldots s_l$ of length $l \leq k_2$ such that $f_{[w]}(B) \subset U(x)$.

Let $U(p_1)$ be the open ball of center p_1 and radius ρ_2 which is contained in U_1. So there exists a word $w(p_1)$ of length $l \leq K_2$ such that $f_{[w(p_1)]}(U(p_1)) \subset U(x)$.

Let λ_1 be the minimum rate of contraction of $Df(p_1)$, i.e. $\lambda_1 = m(Df(p_1))$, where $m(Df(p_1)) = inf\{\|Df(p_1)(v)\| : \|v\| = 1\}$. We set

$$L := max\{\|Df_i(x)\| : \ x \in M, \ i = 1, \ldots, m+3\}.$$

Let us choose a positive integer r such that $\lambda_1^r L^{k_1+k_2} < 1$ and $f_1^r(U_1) \subset U(p_1)$. Also, we take

$$w := t_1 \dots t_k \underbrace{1 \dots 1}_{r-times} s_1 \dots s_l,$$

where $w(x) = t_1 \dots t_k$ and $w(p_1) = s_1 \dots s_l$. Then

$$f_{[w]}(U(x)) = f_{[w(p_1)]} \circ f_1^r \circ f_{[w(x)]}(U_x) \subset f_{[w(p_1)]}(f_1^r(U_1)) \subset f_{[w(p_1)]}(U(p_1)) \subset U(x).$$

Moreover, the choice of r shows that $\| f_{[w]} \| < 1$ on $U(x)$. These facts imply the existence an attracting fixed point for $f_{[w]}$ on $U(x) \subset \mathcal{W}$, which is an attracting periodic point for the iterated function system $\mathcal{F}$.

References

[1] F.H. Ghane, A.J. Homburg, A. Sarizadeh, C^1 robustly minimal iterated function systems, *Stochastic Dynam.*, **10** (2010), 155-160.

[2] A. Gorodetsky, Y.S. Ilyashenko, Some new robust properties of invariant sets and attractors of dynamical systems, *Funksional. Anals. Prizlohen*, **33** (1999), 16-30.

[3] A. Gorodetsky, Y.S. Ilyashenko, Some properties of skew products over a horseshoe and solenoid, *Proc. Steklov Inst.Math*, **231** (2000), 96-118.

[4] A.J. Homburg, Robustly minimal iterated function systems on compact manifolds generated by two diffeomorphisms, *Preprint* (2011).