THE GRAPH $\Gamma_2(R)$ OVER A RING $R$

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Abstract: For a ring $R$, we define a simple undirected graph $\Gamma_2(R)$ with all the non-zero elements of $R$ as vertices, and two vertices $a, b$ are adjacent if and only if either $ab = 0$ or $ba = 0$ or $a + b$ is a zero-divisor (including 0). We first consider its connectedness. Looking at $\mathbb{Z}_n$, we determine the condition for connectedness of $\Gamma_2(\mathbb{Z}_n)$ and also discuss its structure. We then consider connectedness, 2-connectedness and other properties of $\Gamma_2(R)$ when $R$ is a direct product of rings. Giving particular attention to $\Gamma_2(\mathbb{Z}_n)$, we find out the degree patterns and consider girth, Eulerianity and planarity. Then we look at the non-commutative case of $\Gamma_2$ graph over the matrix rings and the infinite case of $\Gamma_2(\mathbb{Z})$ and $\Gamma_2(\mathbb{Z} \times R)$, where $R$ is any ring $\neq \{0\}$.

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1. Introduction

Among graphs associated to a ring, Zero-divisor graph holds a special place due to the fact that it has revealed that the set of Zero-divisors of a ring, which algebraically does not possess a nice structure, has significant and interesting graph-theoretic structures. After the concept of zero-divisor graph was introduced by Beck [3], it was defined by D.F. Anderson and Livingston [2] in the following way: Let $R$ be a commutative ring with 1. Then the zero-divisor graph of $R$, denoted as $\Gamma(R)$, is the simple undirected graph with all the non-
zero zero-divisors of $R$ as the vertices, and two vertices $x, y$ are adjacent if and only if $xy = 0$. Though it was at first defined for commutative rings, Redmond [7] modified the definition in order to involve non-commutative rings also.

Significant results obtained from structure of $\Gamma(R)$ motivated us to define newer types of graphs over rings, involving both the operations of a ring, viz. addition and multiplication, in the adjacency conditions (The addition operation involving the zero-divisors has been used to define the "total graph" over a commutative ring by Anderson and Badawi [1]). We have earlier defined a graph $\Gamma_1(R)$ over a ring $R$ with unity in the following way:

**Definition 1.1.** [8] Let $R$ be a ring with unity. Let $G = (V, E)$ be an undirected graph in which $V = R - \{0\}$ and for any $a, b \in V$, $ab \in E$ if and only if $a \neq b$ and either $a.b = 0$ or $b.a = 0$ or $a + b$ is a unit. We denote this graph $G$ by $\Gamma_1(R)$.

Here, a second type of graph $\Gamma_2(R)$ for a ring $R$ is introduced, taking the same vertex set and changing the last part of the adjacency condition of $\Gamma_1(R)$ graphs.

In this paper, the symbol $a \leftrightarrow b$ denotes that vertices $a, b$ are adjacent. For graphs $G$ and $H$, $G \cup H$ denotes the disjoint union of $G$ and $H$. $\phi(n)$ is the Euler’s phi function. For usual graph-theoretic terms and definitions, one can look at [9].

2. The Graph $\Gamma_2(R)$

**Definition 2.1.** Let $R$ be a ring. Let $G = (V, E)$ be an undirected graph in which $V = R - \{0\}$ and for any $a, b \in V$, $ab \in E$ if and only if $a \neq b$ and either $a.b = 0$ or $b.a = 0$ or $a + b$ is a zero-divisor (including 0). Clearly $G$ is simple. We denote this simple undirected graph $G$ by $\Gamma_2(R)$.

**Remark 2.2.** (i) From the definition, it is clear that the Zero-divisor graph $\Gamma(R)$ (or $\overline{\Gamma}(R)$, in the non-commutative case [7]) is a subgraph of $\Gamma_2(R)$ for any ring $R$.

(ii) Note that the last part of the adjacency condition of $\Gamma_2(R)$ was used as the only adjacency condition for defining The total graph over a commutative ring in [1]. In total graph, all the elements of the ring are taken as vertices, i.e., 0 is not excluded, unlike our definition.

(iii) Since in a finite ring, every nonzero element is either a zero-divisor or a unit, it is clear that for a finite field $F$, $\overline{\Gamma_1(F)} \cong \Gamma_2(F)$.

Regarding the connectedness of the $\Gamma_2(R)$ graph, we start with the case
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when $R = \mathbb{Z}_n = \{0, \bar{1}, 2, \ldots, n-1\}$. For convenience, we write $m$ instead of $\bar{m}$ for elements in $\mathbb{Z}_n$.

**Theorem 2.3.** $\Gamma_2(\mathbb{Z}_n)$ is never a complete graph unless $n = 2$ or 3.

**Proof.** $\Gamma_2(\mathbb{Z}_3)$ is complete graph ($K_2$). $\Gamma_2(\mathbb{Z}_2)$ has only one vertex. For all $n > 3$, $\mathbb{Z}_n$ has two units 1 and $-1$. Now in these cases 1 is not adjacent to $-2$. So $\Gamma_2(\mathbb{Z}_n)$ cannot be complete. $\square$

Now we try to find out the values of $n$ for which $\Gamma_2(\mathbb{Z}_n)$ is connected. We have the following result:

**Theorem 2.4.** $\Gamma_2(\mathbb{Z}_n)$ is connected if and only if either $n \leq 3$ or the prime-power factorization of $n$ has more than one prime factor and in the latter case $\text{diam}(\Gamma_2(\mathbb{Z}_n)) = 2$.

**Proof.** (i) $\Gamma_2(\mathbb{Z}_2)$ and $\Gamma_2(\mathbb{Z}_3)$ being respectively a singleton graph and a $K_2$, are connected. Now suppose $n > 3$ and let $n$ have only one prime factor $p$. Here the zero-divisors are precisely the multiples of $p$ and hence sum of a unit and a zero-divisor cannot be a multiple of $p$, i.e., a zero-divisor (note that it is clearly $\neq 0$). So there is no edge between a zero-divisor and a unit and hence the subgraphs induced respectively by zero-divisors and units are mutually disjoint in this case. So $\Gamma_2(\mathbb{Z}_n)$ is not connected in this case.

(ii) Now consider the case when $n = p_1^{q_1}p_2^{q_2} \ldots p_r^{q_r}$ where $r > 1$.

Case I : Let $z_1, z_2$ be two zero-divisors. If they are not adjacent, then they cannot have a common prime factor $p_i$ among the prime factors of $n$. So we have the path $z_1 - (n/z_1) - z_2$.

Case II : Let $z$ be a zero-divisor and $u$ be a unit. Now suppose they are not adjacent. Let $z$ be not a multiple of some $p_i$ among the prime factors of $n$. Then there exist integers $x, y$ such that $xz + yp_i = 1$. So $uxz + uyp_i = u \Rightarrow u - uxz = uyp_i$. So $u$ is adjacent to $(-uxz)$. So we have the path $u - (-uxz) - z$. Again, if $z$ is a multiple of all the prime factors of $n$, then considering $p_1$ and $p_2$, there exist integers $x, y$ such that $xp_1 + yp_2 = 1 \Rightarrow u - uxp_1 = uyp_2$. So we have the path $u - (-uxp_1) - z$.

Case III : Let $u_1, u_2$ be two units. Suppose that they are not adjacent. If $u_1 - u_2$ is a zero divisor, then we have the path $u_1 - (-u_2) - u_2$. Now let $u_1 - u_2 = v$ be a unit. Now there exist integers $x, y$ such that $xp_1 + yp_2 = 1$. So $v - vxp_1 = vyp_2 \Rightarrow u_1 - u_2 - vxp_1 = vyp_2$. So we have the path $u_1 - (-u_2 - vxp_1) - u_2$. 

From the above three cases it is clear that between any two vertices there is a path of length at most 2. From Theorem 2.3, it is clear that diameter is 1 if and only if \( n = 3 \) and hence when \( n \) has more than one prime factor, \( \Gamma_2(\mathbb{Z}_n) \) is connected with diameter 2.

**Remark 2.5.** Proof of Theorem 2.4 shows that the subgraph of \( \Gamma_2(\mathbb{Z}_n) \) induced by the zero-divisors is also of diameter ≤ 2.

Now we consider the the number of components of \( \Gamma_2(\mathbb{Z}_n) \) when \( n \) has only one prime factor. We know that a natural number can be expressed as sum of two natural numbers in several ways. For an odd prime \( p \), the number of ways is \( (p-1)/2 \) viz. \( \{1, p-1\}, \{2, p-2\}, \ldots, \{(p-1)/2, (p+1)/2\} \). With respect to the operation addition modulo \( p \), these can be represented as \( [p \pm 1], [p \pm 2], [p \pm 3], \ldots, [p \pm (p-1)/2] \) respectively. It can be shown that in the subgraph of \( (\Gamma_2(\mathbb{Z}_p)) \) induced by the units, there are \( (p-1)/2 \) components, each corresponding to those \( (p-1)/2 \) modulo classes. We give the following result whose proof in fact follows from Theorem 2.2 of [1]:

**Theorem 2.6.** Let \( n = p^r \), where \( p \) is an odd prime and \( r \in \mathbb{N} \). Then \( \Gamma_2(\mathbb{Z}_n) \) has \( (p+1)/2 \) components, one \( K_{p^r-1} \) consisting of the zero-divisors, and \( (p-1)/2 \) copies of \( K_{p^r-1,p^r-1} \) for the units.

**Theorem 2.7.** \( \Gamma_2(\mathbb{Z}_{2^r}) \), where \( r \in \mathbb{N} \), has two components consisting of zero-divisors and units of \( \mathbb{Z}_{2^r} \) respectively. The first is a \( K_{2^r-1} \) and the other is a \( K_{2^r-1} \).

**Proof.** Here zero-divisors are precisely the even numbers and units are precisely the odd numbers. So the result follows since sum of two odd or two even is a even number, but sum of an odd and an even is odd.

So \( \Gamma_2(R) \) is not always a connected graph. However, it is connected when \( R \) is a direct product of rings.

**Theorem 2.8.** Let \( R = R_1 \times R_2 \times \ldots \times R_n \), where \( R_i \)'s are rings with unity and \( n > 1, n \in \mathbb{N} \). Then \( \Gamma_2(R) \) is connected with \( \text{diam}(\Gamma_2(R)) \leq 2 \) and \( \text{girth}(\Gamma_2(R)) = 3 \).

**Proof.** (i) From Theorem 2.3 in [2], there is a path between any two vertices \( a, b \), corresponding to two zero-divisors of \( R \). So if we can show that each vertex which corresponds to an element which is not a zero-divisor of \( R \), is adjacent to at least one zero-divisor, then it is assured that there will be a path between any two vertices. Now consider a vertex \( u = (a_1, a_2, \ldots, a_n) \) such that \( u \) is not a zero-divisor in \( R \). So \( a_i \neq 0 \) for \( i = 1, 2, \ldots, n \). Now \( u \) is adjacent to \( (-a_1, 0, 0, \ldots, 0) \),
the latter being a zero-divisor. So any vertex \( u \) which is not a zero-divisor in \( R \), has a zero-divisor adjacent with it. So as we argued, \( \Gamma_2(R) \) is connected.

(ii) Consider two vertices \((a_1, a_2, \ldots, a_n)\) and \((b_1, b_2, \ldots, b_n)\). If they are not adjacent, then we must have \( a_i + b_i \neq 0 \) for \( i = 1, 2, \ldots, n \). Now if at least one of \( a_1 \) and \( b_1 \) is non-zero, we have \((a_1, a_2, \ldots, a_n) \leftrightarrow (-a_1, 0, 0, \ldots, 0, -b_n) \leftrightarrow (b_1, b_2, \ldots, b_n)\). Note that \((-a_1, 0, 0, \ldots, 0, -b_n)\) is equal to neither \((a_1, a_2, \ldots, a_n)\) nor \((b_1, b_2, \ldots, b_n)\) since \( a_1 + b_1 \neq 0 \) and \( a_n + b_n \neq 0 \). Again, let \( a_1 = b_n = 0 \). Then if \( n > 2 \), we have \((a_1, a_2, \ldots, a_n) \leftrightarrow (0, 1, 0, \ldots, 0) \leftrightarrow (b_1, b_2, \ldots, b_n)\) and if \( n = 2 \), then \((a_1, a_2) \leftrightarrow (b_1, b_2)\). So diameter is \( \leq 2 \).

(iii) First, let \( R_i \neq \mathbb{Z}_2 \) for at least one \( i \). Then that \( R_i \) has an element, say \( x \), apart from 0, 1. Let \( e_i^x \) be the \( n \)-tuple whose \( i \)-th component is \( x \) and others are 0. Then for any \( j \neq i \), we have the 3-cycle \( e_i^x - e_j^1 - e_i^{-1} \). Again, let \( R_i \cong \mathbb{Z}_2 \) for \( i = 1, 2, \ldots, n \). Then we have the 3-cycle \( e_1^1 - e_n^1 - (1, 1, \ldots, 1) - e_1^1 \). So girth of \( \Gamma_2(R) \) is 3. Hence the result. \( \square \)

**Example 2.9.** The graph \( \Gamma_2(\mathbb{Z}_3 \times \mathbb{Z}_3) \) is a 5-regular graph. The two subgraphs induced by the zero-divisors and the units are both \( K_4 \) (i.e., the complete graph with 4 vertices).

**Remark 2.10.** Regarding \( diam(\Gamma_2(R_1 \times R_2 \cdots \times R_n)) \) where \( R_i \)'s are finite rings with unity, if at least one \( R_i \) has characteristic \( \neq 2 \), then that \( R_i \) has at least one unit \( x \neq 1 \) (cf. Corollary 4 of [6]). Here \((1, 1, \ldots, 1)\) is not adjacent to \( e_i^{x-1} \). So diameter cannot be 1 in this case. In particular, for direct product of fields, \( diam(\Gamma_2(F_1 \times F_2 \cdots \times F_n)) \) has diameter 1 (i.e., a complete graph) if and only if \( F_i \cong \mathbb{Z}_2 \) for \( i = 1, 2, \ldots, n \).

We now give a result regarding the 2-connectedness of \( \Gamma_2(R) \).

**Theorem 2.11.** Let \( R \) be a commutative ring such that \( diam(\Gamma_2(R)) = 2 \). Then the graph \( \Gamma_2(R) \) is 2-connected.

**Proof.** Let diameter of \( \Gamma_2(R) \) be 2. We show that if we delete any one vertex from the graph, it still remains connected. Suppose \( u \) is the vertex deleted. If \( u \) is a unit, then the connected subgraph induced by the zero-divisors remain unchanged. So since every unit has an adjacent zero-divisor, the graph remains connected. So let \( u \) be a zero-divisor. Consider any two vertices \( x, y \). If there is a path between \( x, y \) without involving \( u \), then it is unaffected by the deletion. So suppose the only path between \( x, y \) is \( x - u - y \). (Note that since diameter is 2, there will be no other vertex in that path) So we have three possibilities as following:
Case I: Let \( xu = 0 \) and \( uy = 0 \). In this case \((x + y)u = 0\), so \( x + y \) is a zero-divisor (may be 0). So this case does not arise.

Case II: (Without loss of generality) Let \( xu = 0 \) and \( yu \neq 0 \) but \( y + u \) is a zero-divisor. If \( y + u = 0 \), then \( xy = -xu = 0 \), i.e., \( x \leftrightarrow y \). So \( y + u = z \) for some non-zero zero-divisor. Let \( zt = 0 \). If \( -z = u \), then \( 2u + y = 0 \). So we have the path \( x - 2u - y \). (Note that \( 2u \neq 0 \) as that would imply \( y = 0 \)). Now let \( -z \neq u \). If \( ut \neq u \), we have the path \( x - ut - (-z) - y \). If \( ut = u \), then \( uz = utz = 0 \), so \((x - z)u = 0\). Hence \( x \leftrightarrow -z \). So we have the path \( x - (-z) - y \).

Case III: Let \( xu \neq 0 \) and \( uy \neq 0 \), but \( x + u = z_1 \) and \( u + y = z_2 \) where \( z_1, z_2 \) are zero-divisors. Now both of \( z_1, z_2 \) cannot be 0 together. If, without loss of generality, \( z_1 = 0 \), then \( x - y = -z_2 \). So we have the path \( x - (-y) - y \). Note that \( -y \neq u \) since \( z_2 \neq 0 \). Now let both \( z_1, z_2 \) be non-zero. \( x - y = z_1 - z_2 = v \). If \( v \) is a zero-divisor, then we have the path \( x - (-y) - y \) as before. Now if \( v \) is a unit, then since \( x + u - v = z_2 \) and \( y + u + v = z_1 \), we have the path \( x - (u - v) - (u + v) - y \), since \( u \) is a zero-divisor.

So the above three cases show that \( \Gamma_2(R) \) is 2-connected.

**Corollary 2.12.** (i) \( \Gamma_2(\mathbb{Z}_n) \), when connected, is 2-connected.

(ii) \( \Gamma_2(R_1 \times R_2 \times \cdots \times R_n) \), where \( R_i \)'s are finite commutative rings \( n > 1, n \in \mathbb{N} \), is 2-connected.

**Remark 2.13.** (i) Let \( F \) be a finite field with \( |F| = p^n \) for some prime \( p \) and \( n \in \mathbb{N} \). Now \( F \) has no zero-divisors. So two vertices can be adjacent if and only if their corresponding elements are additive inverses of each other. As a result, \( \Gamma_2(F) \) is a disjoint union of \((p^n - 1)/2\) copies of \( K_2 \).

(ii) \( \Gamma_2 \)-graph taken over direct products of finite fields is connected from Theorem 2.8. In particular, the subgraph of \( \Gamma_2(F_1 \times F_2) \) induced by the zero divisors is complete, because zero-divisors there are either of the form \((a, 0)\) or \((0, b)\) and hence any two of them are adjacent. Again, if any one of the fields \( F_i \) is \( \mathbb{Z}_2 \), then the subgraph induced by the units is complete. Also, subgraph of \( \Gamma_2(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \cdots \times \mathbb{Z}_3) \) induced by the units is a complete graph.

(iii) Let \( n > 1 \) be a positive integer. Then \( K_{2^n - 1} \cong \Gamma_2(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2(\text{n copies})) \cong \Gamma_2(R) \) where \( R \) is any boolean ring with \( 2^n \) elements.

**Proposition 2.14.** \( \Gamma_2(R) \cong \Gamma_2(S) \) does not imply \( R \cong S \). Here we give two examples in favour of this proposition:

(i) \( \Gamma_2(\mathbb{Z}_2[x, y]/<x^2, xy, y^2>) \cong \Gamma_2(\mathbb{Z}_8) \cong K_3 \cup K_4 \)

(ii) \( \Gamma_2(\mathbb{Z}_p[x]/<x^n>) \cong \Gamma_2(\mathbb{Z}_p^n) \cong K_{p^{n-1}-1} \cup (p - 1)/2 \) copies of \( K_{p^{n-1}}, p^{n-1} \).
3. Degrees and Other Properties of $\Gamma_2(Z_n)$

Now we look at $\Gamma_2(Z_n)$ in more details. We start with the degree patterns of $\Gamma_2(Z_n)$. For this graph maximum possible degree of any vertex is $(n-2)$, since there are $(n-1)$ vertices in the graph, and a vertex is not adjacent to itself. Also note that $\Gamma_2(Z_2)$ is a single vertex graph and $\Gamma_2(Z_3)$ has an isolated vertex. In our next result, we show that $\Gamma_2(Z_n)$ has no isolated vertex for any other value of $n$ and also give the necessary and sufficient condition for attaining the maximum possible degree.

**Theorem 3.1.** For $n > 4$, $\Gamma_2(Z_n)$ has no vertex of degree 0. A vertex can have maximum degree i.e. $(n-2)$, if and only if the prime-power factorization of $n$ has more than one prime and there is an idempotent $x$ such that $2x = 0$.

**Proof.** $\Leftarrow$ (Necessity of the condition):

Case I: Let $n = p^m$ for some prime $p$. Here subgraph of $\Gamma_2(Z_n)$ induced by zero-divisors is complete and the subgraph induced by units is union of complete bipartite graphs. So a vertex cannot have degree 0. Again since the subgraph of $\Gamma_2(Z_n)$ induced by zero-divisors is disjoint from the subgraph induced by units here, a vertex cannot be adjacent to all other vertices and hence we do not have a vertex of degree $(n-2)$.

Case II: Let $n = p_1^{q_1}p_2^{q_2}...p_k^{q_k}$ where $k \geq 2$. Here $\Gamma_2(Z_n)$ is connected and hence no vertex can have degree 0. If possible, let $x$ be a vertex of degree $n-2$. Now $x \neq 1$, since 1 is not adjacent to $-2$. In fact, $x$ cannot be a unit, because then $x$ is not adjacent to $-2x$ (Note that $-2x \neq 0, x$). Hence $x$ must be a zero-divisor. So $x \leftrightarrow 1$. Since 0 is not taken as a vertex, $x \neq 0$, so $x.1 \neq 0$. So we must have $x+1 = z_1$ for some zero divisor $z_1$. We show that $z_1 \neq 0$. Let $z_1 = 0$. Then $x = -1$. So $-1$ is adjacent to all other vertices. Note that the number of units in $Z_n$ is $\phi(n) = \phi(p_1^{q_1}p_2^{q_2}...p_k^{q_k}) \geq 4$. So there exists a unit other than 1, $-1, 2$. Let that unit be $y$. Since $-1 \leftrightarrow y$, we have that $y - 1$ is a zero-divisor. So $1 - y$ is a zero-divisor. Again $1 - y \neq 0, -1$ and so $-1 \leftrightarrow (1 - y)$. Hence $-1 + 1 - y = -y$ is a zero-divisor i.e. $y$ is a zero-divisor. This is a contradiction. So $z_1 \neq 0$. Now $x - z_1 = -1$ which is not a zero-divisor. So since $x \leftrightarrow (-z_1)$, we have $xz_1 = 0$. Note that $x \neq -z_1$ as that would have implied that $2x = -1$, thus contradicting the fact that $x$ is a zero-divisor.

Similarly $x \leftrightarrow -1 \Rightarrow (x - 1) = z_2$ for some zero-divisor $z_2$. Proceeding as before, $xz_2 = 0$.

Now $xz_1 = 0 \Rightarrow x(x + 1) = 0 \Rightarrow x^2 + x = 0$ ....(i)

and $xz_2 = 0 \Rightarrow x(x - 1) = 0 \Rightarrow x^2 = x$ ....(ii)
From (i) and (ii), we have \( 2x = 0 \) and \( x^2 = x \). So the necessary condition to have a \( (n-2) \) degree vertex is to have more than one prime factor in the prime-power factorization of \( n \) and to have an idempotent element \( x \) such that \( 2x = 0 \).

\[ \Rightarrow \text{(Sufficiency of the condition): Let } x \text{ be an element such that } x^2 = x \text{ and } 2x = 0. \text{ Now } 2x = 0 \Rightarrow 2\mid n \text{ and } (n/2)\mid x. \text{ So } x = (n/2). \text{ Again } x^2 = x \Rightarrow n\mid x(x-1). \text{ Now } x(x-1) = (n^2)/4 - n/2. \text{ So } n\mid x(x-1) \Rightarrow (n^2 - 2n)/4 = kn \text{ for some integer } k. \text{ This gives that } (n-2)/4 \text{ is an integer, say } t. \text{ So } n = 2 + 4t = 2(2t + 1) = 2r \text{ where } r \text{ is odd. Now we have } 2 \leftrightarrow x. \text{ Let } z \text{ be a zero-divisor other than } 2 \text{ and } x. \text{ So } x \text{ and } z \text{ have } \text{a common factor } p_i, \text{ because } x = n/2 = r \text{ and } z \text{ being not } 2, \text{ has a common factor with } n/2 = r. (Note that power of } 2 \text{ in the prime power factorization of } n \text{ is } 1, \text{ since } n = 2r \text{ and } r \text{ is odd). So } x + z \text{ is a zero-divisor and hence } x \leftrightarrow z. \text{ Again, let } u \text{ be a unit. So } u \text{ is odd. Now } x \text{ being odd, } x + u \text{ is even, and hence a zero divisor. So } x \leftrightarrow u. \text{ So } x \text{ is adjacent to all other vertices and hence } \deg(x) = n - 2. \]

Next we consider the degree pattern of a unit in \( \Gamma_2(\mathbb{Z}_n) \).

**Theorem 3.2.** Let \( u \) be a unit in \( \mathbb{Z}_n \). Then in \( \Gamma_2(\mathbb{Z}_n) \), \( \deg(u) = (n - \phi(n) - 1) \) or \( (n - \phi(n)) \) according as \( n \) is even or odd.

**Proof.** Consider a unit \( u \). Clearly it is adjacent to precisely the vertices of the form \( (z - u) \), where \( z \) is any zero divisor. Now \(|\{z - u : z \in \mathbb{Z}(\mathbb{Z}_n)\}| = |\{z : z \in \mathbb{Z}(\mathbb{Z}_n)\}| = n - \phi(n)\). Now \( u \in \{z - u : z \in \mathbb{Z}(\mathbb{Z}_n)\} \) according as \( n \) is even or odd. Hence \( \deg(u) = n - \phi(n) \) if \( n \) is odd.

**Corollary 3.3.** \( \Gamma_2(\mathbb{Z}_n) \) is not Eulerian for any positive integer \( n \).

**Proof.** Let \( n \) be even. So from Theorem 3.2, we have that \( \deg(1) = n - \phi(n) - 1 = \) an odd number. Again, if \( n \) is odd, \( \deg(1) = n - \phi(n) = \) an odd number in this case. So \( \deg(1) \) is always odd. So \( \Gamma_2(\mathbb{Z}_n) \) cannot be eulerian for any \( n \).

Next, we have some results about the zero-divisors in \( \Gamma_2(\mathbb{Z}_n) \).

**Theorem 3.4.** (i) Let \( z \) be a non-zero zero-divisor in \( \mathbb{Z}_n \), such that \( z^2 = 0 \). Then \( \deg(z) = n - \phi(n) - 2 \).

(ii) Let \( z \) be a non-zero zero-divisor in \( \mathbb{Z}_n \), where \( n \) is squarefree. Then \( \deg(z) = n - \phi(n) - 2 - \phi(p_1p_2 \cdots p_i) \), where \( p_1, p_2, \cdots, p_i \) are distinct prime factors of \( z \).

**Proof.** These can be proved using Theorem 3.3 and 3.4 of [8].
Now we consider the girth and planarity in $\Gamma_2(\mathbb{Z}_n)$.

**Theorem 3.5.**

$$girth(\Gamma_2(\mathbb{Z}_n)) = \begin{cases} 4 & \text{if } n = 9 \\ \infty & \text{if } n = 2, 3, 4 \text{ or } n \text{ is a prime} \\ 3 & \text{otherwise.} \end{cases}$$

**Proof.** $\Gamma_2(\mathbb{Z}_2)$ is a single vertex graph. It is clear that $girth(\Gamma_2(\mathbb{Z}_n))$ is $\infty$ for $n = 2, 3, 4$ or when $n$ is a prime(from Remark 2.13). Now we first consider the cases when $n = p^m (> 4)$, where $p$ is a prime and $m > 1$. If $p = 2$, then from Theorem 2.7, the subgraph of $\Gamma_2(\mathbb{Z}_n)$ induced by the zero-divisors is a $K_{2^m - 1}$ graph. Now $2^m > 4 \implies 2^m - 1 \geq 3$. So a 3-cycle exists. Next, let $p \geq 3$. In this case $p^m \geq 9$. Now from Figure 2, $\Gamma_2(\mathbb{Z}_9) = K_2 \cup K_{3,3}$, so although it has no 3-cycle, it has a 4-cycle and hence girth is 4. For $p^m > 9$, the subgraph of $\Gamma_2(\mathbb{Z}_{p^m})$ induced by zero-divisors is a $K_{p^m - 1}$ graph from Theorem 2.6. Now $p^m > 9$ and $p \geq 3 \implies p^m - 1 > 3$. So a 3-cycle exists and the girth is 3.

Now we come to the cases where $n$ has more than one prime factor. Let one of the prime factors of $n$, say $p_i$ is greater than 3. Then if $p_1$ is the least prime factor of $n$, then $n \geq p_1 p_i \geq 5 p_1$ and hence we have a 3-cycle $p_1 - 2p_1 - 3p_1 - p_1$. Note that even if $p_1 > 3$, still this gives the same 3-cycle. Again if 2 and 3 are the only prime factors then for $n > 6$ we have the 3-cycle $2 - 4 - 6 - 2$ and for $n = 6$ we have the 3-cycle $2 - 3 - 4 - 2$. Hence the result. \qed
Remark 3.6. From the proof of Theorem 3.5, it is clear that the subgraph of $\Gamma_2(\mathbb{Z}_n)$ induced by the zero-divisors has girth 3 for all $n$ except when $n = 4, 9$ or when $n$ is a prime (in those cases the girth is $\infty$). Again, from part (5) of Theorem 3.14 in [1], girth of the subgraph induced by the units is $\leq 4$ if the subgraph contains a cycle.

Theorem 3.7. $\Gamma_2(\mathbb{Z}_n)$ is planar if and only if $n = 4, 6, 8$ or $n$ is a prime.

Proof. When $n$ is a prime, then $\Gamma_2(\mathbb{Z}_n)$ is a disjoint union of copies of $K_2$ (from Theorem 2.13). So the graph is planar when $n$ is a prime. From Figure 1, $\Gamma_2(\mathbb{Z}_6)$ is planar and so is $\Gamma_2(\mathbb{Z}_4)(\cong K_1 \cup K_2)$. Again from Theorem 2.7, $\Gamma_2(\mathbb{Z}_8)$ is $\cong K_3 \cup K_4$ and hence is planar. Now let $n = 2^m$ where $2^m > 8$. From Theorem 2.7, the subgraph of $\Gamma_2(\mathbb{Z}_n)$ induced by the units is a $K_{2m-1}$. Clearly $2^{m-1} \geq 8$ and hence the graph contains a $K_5$, so it cannot be a planar. For $p \geq 3$, $\Gamma_2(\mathbb{Z}_{p^m})$, where $m > 1$, contains a $K_{p^{m-1}+p-1}$ from Theorem 2.6. Since $p^{m-1} \geq 3$, the graph contains a $K_{3,3}$ and hence cannot be planar. Now we come to the cases where $n$ has more than one prime factor. First let $n$ be even. If $n = 10$, then the subgraph induced by $\{2, 4, 6, 8, 5\}$ forms a $K_5$ and for $n \geq 12$, the subgraph induced by $\{2, 4, 6, 8, 10\}$ forms a $K_5$. So the graph is not planar in these cases. Now let $n$ be odd. Then if the prime factors of $n$, arranged in increasing order are $p_1, p_2, \ldots, p_k$, then $p_2 \geq 5$. So $n \geq 5p_1$. Here the subgraph induced by vertices $\{p_1, 2p_1, 3p_1, 4p_1, n/p_1\}$ forms a $K_5$ and hence the graph is not planar. Hence the result. \qed

4. $\Gamma_2(R)$ over Non-Commutative Rings and Infinite Rings

Let us consider the $\Gamma_2(R)$ graphs for non-commutative rings. Redmond[7] defined an undirected zero-divisor graph for a non-commutative ring [7], and that graph is always connected (unless $Z(R) \neq \{0\}$)[5]. Definition of $\Gamma_2(R)$ graphs clearly makes room for non-commutative rings. A particular case of interest is the ring of matrices with entries from any commutative ring.

Theorem 4.1. $\Gamma_2(M_n(R))$, where $R$ is a commutative ring and $n$ is a natural number $> 1$, is connected.

Proof. Note that a matrix $A$ is either a left or a right zero-divisor in $M_n(R)$ for a commutative ring $R$ if and only if $\det(A) \in Z(R)$ (cf. Theorem 9.1 of [4]). So in $M_n(R)$, the singular matrices, i.e., the matrices with determinant 0, are zero-divisors. Now from Theorem 2.2 of [5], it is clear that between any two zero-divisors, there is a path. Hence, we try to find an adjacent zero-divisor
for an arbitrary unit. If it can be found, then that implies that the graph is connected.

Let us consider a unit in $M_n(R)$, say $A = (a_{ij})$. Since it is a unit, $detA \neq 0$. Now consider the matrix $B = (b_{ij})$ defined by $b_{1i} = -a_{1i}$ for all $i = 1, 2, .., n$ and all other elements are 0. So $detB = 0$, i.e., $B$ is a zero-divisor. Note that since $detA \neq 0$, $a_{1j} \neq 0$ for at least one $j \in \{1, 2, .., n\}$. So for that $j$, $b_{1j} \neq 0$. Hence $B$ is not the zero matrix. Now all the elements of the first row of $A+B$ is zero, so $det(A+B) = 0$, i.e., $A+B$ is a zero-divisor and hence $B$ is adjacent to $A$. So every unit has an adjacent zero-divisor. So $\Gamma_2(M_n(R))$ is connected.

Finally, we have a look at the graph $\Gamma_2(R)$ where $R$ is an infinite ring. The first ring that naturally comes in our mind is the ring $\mathbb{Z}$. Now $\Gamma_2(\mathbb{Z})$ is disjoint union of infinite number of copies of $K_2$ since two vertices $u, v$ are adjacent if and only if $u+v = 0$. However, when we move to the direct product of $\mathbb{Z}$ with any ring $R$, then the graph is connected.

**Theorem 4.2.** Let $R$ be a ring $\neq \{0\}$. Then $\Gamma_2(\mathbb{Z} \times R)$ is connected with diameter $\leq 2$ and girth 3. Moreover it contains a cycle of length $k$ for any natural number $k > 2$.

**Proof.** First part of the theorem statement follows from Theorem 2.8. Now it is easy to see that for any positive integer $k(>2)$, there is a cycle of length $k$ viz. $(1,0) - (2,0) \ldots - (k,0) - (1,0)$ as a subgraph of this graph. In particular, its girth is 3.

**Remark 4.3.** If $R$ in the above theorem contains at least one element $u$ which is not a zero-divisor (which happens for rings with unity), then diameter is necessarily 2, because $(1,0)$ is not adjacent to $(1,u)$ in that case. e.g.- $\Gamma_2(\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z})$ is connected with diameter 2 and girth 3.

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