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CONTRA p_s-CONTINUOUS FUNCTIONS

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Abstract: A new form of contra-continuity, called contra p_s-continuity, is introduced. It is shown that this class of functions is strictly between contra-complete continuity and contra precontinuity. Characterizations and properties of these functions are established. Relationships between these functions and other related classes of functions are also developed.

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1. Introduction

Contra-continuous functions were introduced by Dontchev (see [3]) in 1996. Since then many variations of contra-continuity have been investigated. In 2002 Jafari and Noiri (see [5]) introduced the concept contra-precontinuity and a weak form of contra-precontinuity, called almost precontinuity, was developed by Ekici (see [4]) in 2004. Recently the notions of a p_s-open set and a p_s-continuous function have been introduced by Khalaf and Assad (see [6]). In this note the concept of a p_s-open set is used to develop a new weak form of contra-precontinuity, which we call contra p_s-continuity. It is established that this class of functions is strictly between contra-complete continuity and contra-precontinuity. It is also shown that contra p_s-continuity implies a weak form of semi-continuity.

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2. Preliminaries

The symbols X, Y, and Z represent topological spaces with no separation properties assumed unless explicitly stated. All sets are considered to be subsets of topological spaces. The closure and interior of a set A are signified by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A set A is preopen (respectively, semi-open, regular open) if $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$, (respectively, $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$, $A = \mathrm{Int}(\mathrm{Cl}(A))$). A set A is preclosed (respectively, semi-closed, regular closed) provided its complement is preopen (respectively, semi-open, regular open). A set A is p_s-open (see [6]) if A is preopen and a union of semi-closed sets. A set is p_s-closed if its complement is p_s-open. The semi-interior (respectively, p_s-interior (see [6])) of a set A, denoted by $\mathrm{sInt}(A)$ (respectively, $p_s\mathrm{Int}(A)$) is the union of all semi-open (respectively, p_s-open) sets contained in A and the semi-closure (respectively, p_s-closure (see [7])) of A, denoted by $\mathrm{sCl}(A)$, (respectively, $p_s\mathrm{Cl}(A)$) is the intersection of all semi-closed (respectively, p_s-closed) sets containing A.

Definition 1. A function $f : X \to Y$ is said to be contra-completely continuous if $f^{-1}(F)$ is regular open for every closed subset F of Y.

Definition 2. A function $f : X \to Y$ is said to be contra-precontinuous (see [5]) if $f^{-1}(V)$ is preclosed for every open subset V of Y.

Definition 3. A function $f : X \to Y$ is said to be p_s-continuous (see [6]) (respectively almost p_s-continuous (see [7])) if, for every $x \in X$ and every open subset V of Y containing $f(x)$, there exists a p_s-open subset U of X containing x such that $f(U) \subseteq V$ (respectively, $f(U) \subseteq \mathrm{Int}(\mathrm{Cl}(V))$).

Note that a function is p_s-continuous if and only if the inverse image if every open set is p_s-open (see [7]).

3. Contra p_s-Continuous Functions

We define a function $f : X \to Y$ to be contra p_s-continuous provided that, for every open subset V of Y, $f^{-1}(V)$ is p_s-closed.

Definition 4. Let A be a subset of a space X. The kernel of A (see [9]), denoted by $\ker(A)$, is the intersection of all open subsets of X containing A.

Lemma 1. (see [5]) *The following statements hold for subsets A and B of a space X:*

(a) $x \in \ker(A)$ if and only if $A \cap F \neq \emptyset$ for every closed subset F of X containing x.

(b) $A \subseteq \ker(A)$ and $A = \ker(A)$ if A is open in X.
(c) If $A \subseteq B$, then $\ker(A) \subseteq \ker(B)$.

Theorem 5. *For a function $f : X \to Y$, the following statements are equivalent:*
(a) f is contra p_s-continuous.
(b) For every closed subset F of Y, $f^{-1}(F)$ is p_s-open.
(c) For every $x \in X$ and every closed subset F of Y containing $f(x)$, there exists a p_s-open subset U of X containing x such that $f(U) \subseteq F$.
(d) $f(p_s Cl(A)) \subseteq \ker(f(A))$ for every subset A of X.
(e) $p_s Cl(f^{-1}(B)) \subseteq f^{-1}(\ker(B))$ for every subset B of Y.

Proof. The implications (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) are clear.

(c) $\Rightarrow$ (d) Let $A \subseteq X$ and let $y \in f(p_s\mathrm{Cl}(A))$. Suppose $y \notin \ker(f(A))$. Then there exists an open subset V of Y such that $f(A) \subseteq V$ and $y \notin V$. Let $x \in p_s\mathrm{Cl}(A)$ such that $y = f(x)$. Then $f(x) \in Y - V$, which is closed in Y. By (c) there exists a p_s-open subset U of X for which $x \in U$ and $f(U) \subseteq Y - V$. Since $f(A) \subseteq V$, $A \cap U = \emptyset$. Since U is p_s-open, it follows that $x \notin p_s\mathrm{Cl}(A)$. This contradiction proves that $y \in \ker(f(A))$.

(d) $\Rightarrow$ (e) Let $B \subseteq Y$. It follows from (d) that

$$f(p_s\mathrm{Cl}(f^{-1}(B))) \subseteq \ker(f(f^{-1}(B))) \subseteq \ker(B)$$

and thus $p_s\mathrm{Cl}(f^{-1}(B)) \subseteq f^{-1}(\ker(B))$.

(e) $\Rightarrow$ (a) Let V be an open subset of Y. Using (e) we obtain $p_s\mathrm{Cl}(f^{-1}(V)) \subseteq f^{-1}(\ker(V)) = f^{-1}(V)$ and, since $p_s\mathrm{Cl}(f^{-1}(A))$ is p_s-closed, it follows that $f^{-1}(V)$ is p_s-closed. □

Definition 6. A function $f : X \to Y$ is said to be usc-continuous if, for every closed subset F of Y, $f^{-1}(F)$ is a union of semi-closed sets.

The implications below follow from the definition of a p_s-open set and the fact that regular open sets are both preopen and semi-closed.

contra-complete cont. $\Rightarrow$ contra p_s-cont. $\Rightarrow$ contra-precont.
$\Downarrow$
usc-continuous

The following examples show that none of the above implications are reversible.

Example 7. Let X be the real numbers and let σ be the usual topology on X. Let $A = (0,1) \cup (1,2)$ and let $\tau = \{X, \emptyset, X - A\}$. The identity mapping $f : (X,\sigma) \to (X,\tau)$ is contra p_s-continuous but not contra-completely continuous.

Example 8. Let X be the real numbers with the usual topology. The identity mapping on X is obviously semi-continuous and hence usc-continuous. However, it is not contra p_s-continuous.

Example 9. Assume $X = \{a, b, c\}$ has the topology $\sigma = \{X, \emptyset, \{a\}\}$ and let $\tau = \{X, \emptyset, \{b, c\}\}$. The identity mapping $f : (X, \sigma) \to (X, \tau)$ is contra-precontinuous but not contra p_s-continuous.

Since the identity on the real numbers with the usual topology is p_s-continuous but not contra p_s-continuous (Example 8), p_s-continuity does not imply contra p_s-continuity. Also, since the function in Example 7 is contra p_s-continuous, but not p_s-continuous, p_s-continuity and contra p_s-continuity are independent

Definition 10. A space X is said to be a C-space (see [2]) if every open subset of X is a union of closed sets.

Theorem 11. *If $f : X \to Y$ is contra p_s-continuous and Y is a C-space, then f is p_s-continuous.*

Proof. Let $x \in X$ and let V be an open subset of Y containing $f(x)$. Since Y is a C-space, there exists a closed subset F of Y such that $x \in F \subseteq V$. Then, since f is contra p_s-continuous, Theorem 5 implies that there exists a p_s-open subset U of X containing x such that $f(U) \subseteq F$. Hence $f(U) \subseteq V$, which proves that f is p_s-continuous. □

Corollary 12. *If $f : X \to Y$ is contra p_s-continuous and Y is either regular or T_1, then f is p_s-continuous.*

Definition 13. A function $f : X \to Y$ is said to be p_s-preopen (respectively, M-preopen (see [8]), if for every p_s-open (respectively, preopen) subset U of X, $f(U)$ is preopen.

Theorem 14. *If $f : X \to Y$ is p_s-preopen and contra p_s-continuous, then f is almost p_s-continuous.*

Proof. Let $x \in X$ and let V be an open subset of Y containing $f(x)$. Then by Theorem 5 there exists a p_s-open subset U of X for which $x \in U$ and $f(U) \subseteq \mathrm{Cl}(V)$. Since f is p_s-preopen, $f(U)$ is preopen and therefore $f(U) \subseteq \mathrm{Int}(\mathrm{Cl}(V))$, which proves that f is almost p_s-continuous. □

Corollary 15. *If $f : X \to Y$ is M-preopen and contra p_s-continuous, then f is almost p_s-continuous.*

Corollary 16. *If $f : X \to Y$ has the property that images of semi-closed sets are preopen and f is contra p_s-continuous, then f is almost p_s-continuous.*

Definition 17. A function $f : X \to Y$ is said to be almost weakly p_s-continuous if for every open subset V of Y, $f^{-1}(V) \subseteq p_s\mathrm{Int}(\mathrm{Cl}(f^{-1}(\mathrm{Cl}(V))))$.

Theorem 18. *If $f : X \to Y$ is contra p_s-continuous, then f is almost weakly p_s-continuous.*

Proof. Assume V is an open subset of Y. Then, since $f^{-1}(\mathrm{Cl}(V))$ is p_s-open in X, $f^{-1}(V) \subseteq f^{-1}(\mathrm{Cl}(V)) \subseteq p_s\mathrm{Int}(\mathrm{Cl}(f^{-1}(\mathrm{Cl}(V))))$, which proves that f is almost weakly p_s-continuous. □

Definition 19. The p_s-frontier of a subset A of a space X is the set given by $p_s\mathrm{Fr}(A) = p_s\mathrm{Cl}(A) \cap p_s\mathrm{Cl}(X - A)$.

Theorem 20. *Let $f : X \to Y$ be a function and let $x \in X$. Then f is not contra p_s-continuous at x if and only if x is a member of the p_s-frontier of the inverse image of a closed subset of Y containing $f(x)$.*

Proof. ($\Rightarrow$) Assume f is not contra p_s-continuous at x. Then by Theorem 5 there exists a closed subset F of Y such that $f(x) \in F$ and, for every p_s-open subset U of X containing x, $f(U) \nsubseteq F$. Hence $U \cap f^{-1}(Y - F) \neq \emptyset$ for every p_s-open subset U of X containing x, which implies that $x \in p_s\mathrm{Cl}(f^{-1}(Y - F))$ and hence $x \in p_s\mathrm{Cl}(f^{-1}(F)) \cap p_s\mathrm{Cl}(f^{-1}(Y - F)) = p_s\mathrm{Fr}(f^{-1}(F))$.

($\Leftarrow$) Let $x \in X$ and assume that $x \in p_s\mathrm{Fr}(f^{-1}(F))$ for some closed subset F of Y containing $f(x)$. Suppose f is contra p_s-continuous at x. Then there exists a p_s-open subset U of X containing x such that $f(U) \subseteq F$. Then we see that $x \in U \subseteq f^{-1}(F)$ and hence that $x \notin p_s\mathrm{Cl}(Y - f^{-1}(F))$ and thus $x \notin p_s\mathrm{Fr}(f^{-1}(F))$. Therefore f is not contra p_s-continuous at x. □

4. Properties

Recall that the graph of a function $f : X \to Y$ is the subset $G(f) = \{(x, y) : y = f(x)\}$ of the product space $X \times Y$.

Definition 21. A function $f : X \to Y$ is said to have a p_s-closed graph if for every $(x, y) \in X \times Y - G(f)$ there exists a p_s-open set U of X such that $x \in U \subseteq X$ and an open set V such that $y \in V \subseteq Y$ for which $(x, y) \in U \times V \subseteq X \times Y - G(f)$.

Theorem 22. *If $f : X \to Y$ is contra p_s-continuous and Y is T_2, then $G(f)$ is p_s-closed.*

Proof. Assume $(x, y) \in X \times Y - G(f)$. then, since $y \neq f(x)$, there exist disjoint open subsets V and W of X and Y, respectively, such that $f(x) \in V$ and $y \in W$. Since f is contra p_s-continuous, there exists a p_s-open subset U of X such that $x \in U$ and $f(U) \subseteq \text{Cl}(V)$. Because $\text{Cl}(V) \cap W = \emptyset$, $f(U) \cap W = \emptyset$ and thus $(x, y) \in U \times W \subseteq X \times Y - G(f)$. Therefore $G(f)$ is p_s-closed. □

Definition 23. A function $f : X \to Y$ is said to have a contra p_s-closed graph if, for every $(x, y) \in X \times Y - G(f)$, there exists a p_s-open set U of X such that $x \in U \subseteq X$ and a closed set F such that $y \in F \subseteq Y$ for which $(x, y) \in U \times F \subseteq X \times Y - G(f)$.

Theorem 24. *If $f : X \to Y$ is contra p_s-continuous and Y is Urysohn, then $G(f)$ is contra p_s-closed.*

Proof. Assume $(x, y) \in X \times Y - G(f)$. Then, since $y \neq f(x)$, there exist open subsets V and W of X and Y, respectively, such that $f(x) \in V$ and $y \in W$ and $\text{Cl}(V) \cap \text{Cl}(W) = \emptyset$. Since f is contra p_s-continuous, there exists a p_s-open subset U of X containing x such that $f(U) \subseteq \text{Cl}(V)$. Then we have $(x, y) \in U \times \text{Cl}(W) \subseteq X \times Y - G(f)$, which proves that $G(f)$ is contra p_s-closed. □

Theorem 25. *Assume Y is an Urysohn space and that $f_i : X_i \to Y$ for $i = 1, 2$ is contra p_s-continuous for each i. The set $A = \{(x_1, x_2) : f_1(x_1) = f_2(x_2)\}$ is p_s-closed in the product space $X_1 \times X_2$.*

Proof. Suppose $(x_1, x_2) \in (X_1 \times X_2) - A$. Then $f_1(x_1) \neq f_2(x_2)$ and, since Y is Urysohn, there exist open sets V_1 and V_2 containing $f_1(x_1)$ and $f_2(x_2)$, respectively, such that $\text{Cl}(V_1) \cap \text{Cl}(V_2) = \emptyset$. Since products of p_s-open sets are p_s-open, $f_1^{-1}(\text{Cl}(V_1)) \times f_2^{-1}(\text{Cl}(V_2))$ is p_s-open in $X_1 \times X_2$. Then we see that $(x_1, x_2) \in f_1^{-1}(\text{Cl}(V_1)) \times f_2^{-1}(\text{Cl}(V_2)) \subseteq (X_1 \times X_2) - A$. Since unions of p_s-open sets are p_s-open (see [6]), it follows that A is closed. □

Recall that a set A is called regular semi-open if $A = \text{sInt}(\text{sCl}(A))$.

Lemma 2. (see [6]) *If X is a space, A is a regular semi-open subset of X and B is a p_s-open subset of X, then $A \cap B$ is p_s-open in A.*

Theorem 26. *If $f : X \to Y$ is contra p_s-continuous and A is a regular semi-open subset of X, then $f|_A : A \to Y$ is contra p_s-continuous.*

Proof. Let F be a closed subset of Y. Since $f^{-1}(F)$ is p_s-open in X, using Lemma 2 we have $f|_A^{-1}(F) = f^{-1}(F) \cap A$ is p_s-open in A. Thus $f|_A : A \to Y$ is contra p_s-continuous. □

The proof of the following theorem is straightforward.

Theorem 27. *If $f : X \to Y$ is contra p_s-continuous and $g : Y \to Z$ is continuous, then $g \circ f : X \to Z$ is contra p_s-continuous.*

Theorem 28. *Let $f_\alpha : X \to Y_\alpha$ be a function for every $\alpha \in \mathcal{A}$ and let $f : X \to \Pi_{\alpha \in \mathcal{A}} Y_\alpha$ be the product function given by $f(x) = (f_\alpha(x))_{\alpha \in \mathcal{A}}$ for every $x \in X$. If f is contra p_s-continuous, then f_α is contra p_s-continuous for every $\alpha \in \mathcal{A}$.*

Proof. Let $\beta \in \mathcal{A}$ and let $p_\beta : \Pi_{\alpha \in \mathcal{A}} Y_\alpha \to Y_\beta$ be the βth projection function. Since $f_\beta = p_\beta \circ f$, it follows from Theorem 27 that f_β is contra p_s-continuous. □

Theorem 29. *If $f : X \to Y$ is a function, $g : Y \to Z$ is injective and closed, and $g \circ f : X \to Z$ is contra p_s-continuous, then f is contra p_s-continuous.*

Proof. Let F be a closed subset of Y. Since g is closed, $g(F)$ is closed in Z. Then, since $g \circ f$ is contra p_s-continuous and g is injective, we see that $f^{-1}(F) = f^{-1}(g^{-1}(g(F)))$ is p_s-open in X, which proves that f is contra p_s-continuous. □

Definition 30. A function $f : X \to Y$ is said to be M-p_s-open if for every p_s-open subset U of X, $f(U)$ is p_s-open in Y.

The proof of the following result is analogous to the above proof and is omitted.

Theorem 31. *If $f : X \to Y$ is surjective and M-p_s-open, $g : Y \to Z$ is a function, and $g \circ f : X \to Z$ is contra p_s-continuous, then g is contra p_s-continuous.*

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