SYMmetric BI-DERIVATIONS ON TM–ALGEBRAS

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Abstract: Recently an algebra based on propositional calculi was introduced by Tamilarasi and Mekalai in the year 2010 known as TM–algebras, see [6]. In our paper [1] we introduced the notion of derivation on TM-algebras. In this paper, we introduce the notion of symmetric bi-derivation on TM-algebras and study some of its properties.

AMS Subject Classification: 03G25, 06F35
Key Words: BCK/BCI algebras, TM-algebras, derivations, symmetric bi-derivations

1. Introduction

It is well known that BCK and BCI-algebras are two classes of algebras of logic. They were introduced by Imai and Iseki [3] and have been extensively investigated by many researchers. Recently another algebra based on propositional calculi was introduced by Tamilarasi and Mekalai [6] in the year 2010 known as TM–algebras.

Motivated by the notion of derivations on rings and near-rings Jun and Xin [4] studied the notion of derivation on BCI-algebras. In our paper [1],

Received: March 21, 2013

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we introduced the notion of derivation on $T M$–algebras. In [5], the authors have discussed the notion of symmetric bi-derivation on BCI-algebras. This motivated us to introduce the notion of symmetric bi-derivation on TM-algebras in this paper. We study the properties of symmetric bi-derivations on TM-algebras and prove that the set of all symmetric bi-derivations on a TM-algebra forms a semigroup under a suitably defined binary composition.

2. Preliminaries

In this section, we recall some basic definitions and results that are needed for our work.

**Definition 2.1.** A $T M$–algebra $(X, *, 0)$ is a non-empty set $X$ with a constant 0 and a binary operation $*$ satisfying the following axioms:

1. $x * 0 = x$
2. $(x * y) * (x * z) = z * y \forall x, y, z \in X.$

**Definition 2.2.** A $T M$–algebra $X$ is said to be associative if $(x * y) * z = x * (y * z)$ for all $x, y, z \in X$.

**Definition 2.3.** For any $T M$–algebra $(X, *, 0)$. We define the set $G(X) = \{x \in X \mid 0 * x = x\}$.

**Remark 2.4.** In a $T M$–algebra $X$, by definition, $x \wedge y = y * (y * x)$. However, in a TM-algebra, $x = y * (y * x)$. Hence, in a TM-algebra, we have $x \wedge y = x \forall x, y \in X$.

**Definition 2.5.** [2] Let $X$ be a $T M$–algebra. If we define an operation $+$, called addition, as $x + y = x * (0 * y)$ for all $x, y \in X$, then $(X, +)$ is an abelian group with identity 0 and the additive inverse $-x = 0 * x \forall x \in X$.

**Remark 2.6.** If we have a $T M$–algebra $(X, *, 0)$ it follows from the above definition that $(X, +)$ is an abelian group with $-y = 0 * y \forall y \in X$. Then we have $x - y = x * y \forall x, y \in X$. On the other hand if we choose an abelian group $(X, +)$ with an identity 0 and define $x * y = x - y$, we get a $T M$–algebra $(X, *, 0)$ where $x + y = x * (0 * y) \forall x, y \in X$.

**Definition 2.7.** Let $(X, *, 0)$ be a $T M$–algebra. A self map $d : X \to X$ is said to be a $(l, r)$–derivation on $X$, if $d(x * y) = (d(x) * y) \wedge (x * d(y))$. $d$ is said to be a $(r, l)$–derivation on $X$, if $d(x * y) = (x * d(y)) \wedge (d(x) * y)$. It is said to be a derivation on $X$ if $d$ is both a $(l, r)$–derivation and a $(r, l)$–derivation on $X$. 
3. Symmetric BI-Derivations

We start this section with the definition of Cartesian product of $TM$–algebras.

**Definition 3.1.** Let $X, Y$ be $TM$–algebras. An operation $*$ on the Cartesian product $X \times Y$ of $X, Y$ is defined as follows.

1. $(x_1, y_1) * (x_2, y_2) = (x_1 * x_2, y_1 * y_2)$.
2. $(0, 0) = 0$,

**Lemma 3.2.** Cartesian product of two $TM$–algebras is again a $TM$–algebra.

**Proof.** Let $X$ and $Y$ be two $TM$–algebras. Consider the cartesian product $X \times Y$.

$$(x, y) * (0, 0) = (x * 0, y * 0) = (x, y),$$

$$(x_1, y_1) * (x_2, y_2) * ((x_1, y_1) * (x_3, y_3)) = (x_1 * x_2, y_1 * y_2) * ((x_1 * x_3, y_1 * y_3))$$

$$= ((x_1 * x_2) * (x_1 * x_3), (y_1 * y_2) * (y_1 * y_3))$$

$$= ((x_3 * x_2), (y_3 * y_2))$$

$$= (x_3, y_3) * (x_2, y_2)$$

Therefore $(X \times Y, *, 0)$ is a $TM$–algebra.

**Definition 3.3.** Let $X$ be a $TM$–algebra. A mapping $D: X \times X \rightarrow X$ is a symmetric map if $D(x, y) = D(y, x)$ holds for all pairs of elements $x, y \in X$.

**Example 3.4.** Let $(X, *, 0)$ be a $TM$–algebra with the Cayley table.

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The map $D: X \times X \rightarrow X$ defined by $D(x, y) = x * (0 * y)$ is a symmetric map.

**Definition 3.5.** Let $X$ be a $TM$–algebra and $D: X \times X \rightarrow X$ be a symmetric mapping. A mapping $d: X \rightarrow X$ defined by $d(x) = D(x, x)$ is called trace of $D$. 
Example 3.6. Let \((X, *, 0)\) be a TM–algebra with the Cayley table.

\[
\begin{array}{c|cccc}
* & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 2 & 1 & 3 \\
1 & 1 & 0 & 3 & 2 \\
2 & 2 & 3 & 0 & 1 \\
3 & 3 & 1 & 2 & 0
\end{array}
\]

The map \(D : X \times X \to X\) defined by \(D(x, y) = x*(0*y) = x+y\) is a symmetric map.

Since \(x = 0\), \(D(0, 0) = 0 + 0 = 0\). \(x = 1\), \(D(1, 1) = 1 + 1 = 3\). \(x = 2\), \(D(2, 2) = 2 + 2 = 3\). \(x = 3\), \(D(3, 3) = 3 + 3 = 0\).

Thus the mapping \(d : X \to X\) given by \(d(x) = D(x, x) = \begin{cases} 0 & \text{if } x = 0, 3 \\ 3 & \text{if } x = 1, 2 \end{cases}\)
is the trace of the symmetric mapping \(D\).

Definition 3.7. Let \(X\) be a TM–algebra and \(D : X \times X \to X\) be a symmetric mapping. If \(D\) satisfies the identity, \(D(x*y, z) = (D(x, z)*y) \land (x*D(y, z))\) for all \(x, y, z \in X\), then \(D\) is called left-right symmetric bi-derivation. ((\(l, r\) symmetric bi-derivation)

If \(D\) satisfied the identity, \(D(x*y, z) = (x*D(y, z)) \land (D(x, z)*y)\) for all \(x, y, z \in X\), then \(D\) is called right-left symmetric bi-derivation. ((\(r, l\)–symmetric bi-derivation)

If \(D\) is both an \((l, r)\)symmetric bi-derivation and an \((r, l)\) symmetric bi-derivation then \(D\) is called a symmetric bi-derivation.

Example 3.8. Consider in example 3.6. Define a mapping \(D : X \times X \to X\) by \(D(x, y) = x*(0*y)\) for all \(x, y \in X\). Then \(D\) is a \((l, r)\)–symmetric Bi-derivation.

Example 3.9. Consider in example 3.4. Define \(D(x, y) = x*(0*y)\) for all \(x, y \in X\) is a symmetric map. Then \(D\) is a symmetric Bi-derivation.

Example 3.10. Consider the TM–algebra with the Cayley-Table as in exampmle 3.4. Define the symmetric map \(D : X \times X \to X\) such that
\[
D(x, x) = 3, \text{ if } x = 0, 1, 2, 3. \\
D(0, 3) = D(3, 0) = D(1, 2) = D(2, 1) = 0. \\
D(0, 2) = D(2, 0) = D(1, 3) = D(3, 1) = 1. \\
D(0, 1) = D(1, 0) = D(2, 3) = D(3, 2) = 2. \\
\]
Then \(D\) is a symmetric Bi-derivation.

Proposition 3.11. Let \(X\) be a TM–algebra. Define a symmetric map \(D : X \times X \to X\) by \(D(x, y) = x+y\) for all \(x, y \in X\). Then \(D\) is a \((l, r)\)–symmetric
Bi-derivation.

Proof. 

\[ D(x \ast y, z) = (x \ast y) + z \quad \text{for all } x, y, z \in X \]

\[ = (x \ast y) \ast (0 \ast z) \]

\[ = (x \ast (0 \ast z)) \ast y \quad (\because (x \ast y) \ast z = (x \ast z) \ast y) \]

\[ = (x + z) \ast y \]

\[ = ((x \ast (y + z)) \ast ((x \ast (y + z)) \ast ((x + z) \ast y)) \quad (\because y \ast (y \ast x) = x) \]

\[ = (x + z) \ast y \land (x \ast (y + z)) \]

\[ = (D(x, z) \ast y) \land (x \ast D(y, z)) \]

This proves that \( D \) is a \((l, r)\)-symmetric Bi-derivation.

**Theorem 3.12.** Let \( X \) be an associative \( TM \)-algebra. Then the symmetric map \( D : X \times X \to X \) defined by \( D(x, y) = x + y \) for all \( x, y \in X \) is a symmetric bi-derivation.

Proof. By the above proposition, \( D \) is a \((l, r)\)-symmetric bi-derivation.

\[ D(x \ast y, z) = (x \ast y) + z \]

\[ = (x \ast y) \ast (0 \ast z) \]

\[ = (x \ast (0 \ast z)) \ast y \quad (\because X \text{ is associative}) \]

\[ = (x \ast 0) \ast z \ast y \quad (\because X \text{ is associative}) \]

\[ = (x \ast z) \ast y = (x \ast y) \ast z \quad \cdots \cdots (1) \]

\[ (x \ast D(y, z)) \land (D(x, z) \ast y) = x \ast D(y, z) \quad (\because x \land y = y \ast (y \ast x) = x) \]

\[ = x \ast (y + z) \]

\[ = x \ast (y \ast (0 \ast z)) \]

\[ = x \ast ((y \ast 0) \ast z) \quad (\because X \text{ is associative}) \]

\[ = x \ast (y \ast z) \]

\[ = (x \ast y) \ast z \quad \cdots \cdots (2) \quad (\because X \text{ is associative}) \]

From (1) and (2), \( D(x \ast y, z) = (x \ast D(y, z)) \land (D(x, z) \ast y) \) for all \( x, y, z \in X \).

This proves that \( D \) is \((r, l)\)-symmetric bi-derivation and hence a symmetric bi-derivation.

**Proposition 3.13.** Let \( X \) be a \( TM \)-algebra and \( D : X \times X \to X \) be a symmetric map. Then
1. If $D$ is a $(l, r)$–symmetric bi-derivation then $D(x, y) = D(x, y) \land x$ for all $x, y \in X$.

2. If $D$ is a $(r, l)$–symmetric bi-derivation then $D(x, y) = x \land D(x, y)$ for all $x, y \in X$ if and only if $D(0, y) = 0$ for all $y \in X$.

Proof.

1. Let $D$ be a $(l, r)$–symmetric bi-derivation.

$$D(x, y) = D(x * 0, y) \quad \text{for all } x, y \text{ in } X$$

$$= (D(x, y) * 0) \land (x * D(0, y))$$

$$= D(x, y) \land (x * D(0, y))$$

$$= (x * D(0, y)) * ((x * D(0, y)) * D(x, y))$$

$$= (x * D(0, y)) * ((x * D(x, y)) * D(0, y))$$

$$= x * (x * D(x, y)) \quad (\because (x * y) * z = (x * z) * y)$$

$$= D(x, y) \land x$$

2. Let $D$ be a $(r, l)$–symmetric bi-derivation and $D(0, y) = 0$ for all $y \in X$.

$$D(x, y) = D(x * 0, y)$$

$$= (x * D(0, y)) \land (D(x, y) * 0)$$

$$= (x * 0) \land D(x, y)$$

$$= x \land D(x, y).$$

Conversely, if $D(x, y) = x \land D(x, y)$ for all $x, y \in X$. Then

$$D(0, y) = 0 \land D(0, y) = D(0, y) * (D(0, y) * 0) = D(0, y) * D(0, y) = 0.$$

**Proposition 3.14.** Let $X$ be a $TM$–algebra and $D : X \times X \to X$ be a $(l, r)$–symmetric bi-derivation. Then

1. $D(a, y) = D(0, y) * (0 * a) = D(0, y) + a$ for all $a, y \in X$.

2. $D(a + b, y) = D(a, y) + D(b, y) - D(0, y)$ for all $a, b, y \in X$.

3. $D(a, y) = a$ for all $a, y \in X$ if and only if $D(0, y) = 0$.

Proof.
1. Let $a = 0 \ast (0 \ast a)$.

\[
D(a, y) = D(0 \ast (0 \ast a), y)
= (D(0, y) \ast (0 \ast a)) \land (0 \ast D(0 \ast a, y))
= D(0, y) \ast (0 \ast a) \quad (\because x \land y = x)
= D(0, y) + a
\]

2. By (1)

\[
D(a + b, y) = D(0, y) + a + b
= D(0, y) + a + D(0, y) + b - D(0, y)
= D(a, y) + D(b, y) - D(0, y)
\]

3. $D(a, y) = a$ for all $a, y \in X$.

Put $a = 0$, $D(0, y) = 0 \quad \forall y \in X$.

Conversely if $D(0, y) = 0$, then $D(a, y) = D(0, y) + a = 0 + a = a$.

**Proposition 3.15.** Let $X$ be a $TM$–algebra and $D : X \times X \rightarrow X$ be a $(r, l)$–symmetric bi-derivation. Then

1. $D(a, y) \in G(X)$ for all $a \in G(X)$.
2. $D(a, y) = a \ast D(0, y) = a + D(0, y)$ for all $a, y \in X$.
3. $D(a + b, y) = D(a, y) + D(b, y) - D(0, y)$ for all $a, b, y \in X$.
4. $D(a, y) = a$ for all $a, y \in X$ if and only if $D(0, y) = 0$.

**Proof.**

1. $0 \ast a = a \quad (\because a \in G(X))$.

\[
D(a, y) = D(0 \ast a, y) \quad \text{for all } a, y \in X
= (0 \ast D(a, y)) \land (D(0, y) \ast a)
= (D(0, y) \ast a) \ast ((D(0, y) \ast a) \ast (0 \ast D(a, y)))
= 0 \ast D(a, y) \quad (\because y \ast (y \ast x) = x)
\]

This shows that $D(a, y) \in G(X)$.
2. 
\[ D(a, y) = D(a * 0, y) \text{ for all } a, y \in X \]
\[ = (a * D(0, y)) \land (D(a, y) * 0) \]
\[ = (a * D(0, y)) \land D(a, y) \]
\[ = D(a, y) * (D(a, y) * (a * D(0, y))) \]
\[ = a * D(0, y) \]

Again \( D(a, y) = a * D(0, y) \)
\[ = a * D(0 * 0, y) \]
\[ = a * ((0 * D(0, y)) \land (D(0, y) * 0)) \]
\[ = a * (0 * D(0, y)) \]
\[ = a + D(0, y) \]

3. \( D(a + b, y) = a + b + D(0, y) = a + D(0, y) + b + D(0, y) - D(0, y) \)
\[ = D(a, y) + D(b, y) - D(0, y). \]

4. If \( D(0, y) = 0 \), then \( D(a, y) = D(a * 0, y) = a * D(0, y) = a * 0 = a. \) (By (2)) Conversely if \( D(a, y) = a \ \forall \ a \in X, \ D(0, y) = 0. \)

4. Semigroup of Symmetric Bi-Derivations

**Definition 4.1.** Let \( \mathcal{B}_L \) denote the set of all \((l, r)\)–symmetric bi-derivation on \( X \). Define the binary operation \( \land \) on \( \mathcal{B}_L \) as follows: For \( D_1, D_2 \in \mathcal{B}_L \) define \( (D_1 \land D_2)(x, y) = D_1(x, y) \land D_2(x, y) \) for all \( x, y \in X \).

**Proposition 4.2.** Let \( D_1 \) and \( D_2 \) are \((l, r)\)–symmetric bi-derivation on \( X \), then \( (D_1 \land D_2) \) is also a \((l, r)\)–symmetric bi-derivation.

**Proof.** We will prove the following implication
\[ (D_1 \land D_2)(x * y, z) = ((D_1 \land D_2)(x, z) * y) \land (x * ((D_1 \land D_2)(y, z))). \]

\[ (D_1 \land D_2)(x * y, z) = D_1(x * y, z) \land D_2(x * y, z) \]
\[ = D_2(x * y, z) * (D_2(x * y, z) * D_1(x * y, z)) \]
\[ = D_1(x * y, z) \]
Combining (1) and (2), we get, \( (D_1 \wedge D_2) \) is a \((l,r)\)-symmetric bi-derivation.

**Proposition 4.3.** The binary composition \( \wedge \) defined on \( \mathcal{A}_L \) is associative.

**Proof.** Let \( X \) be a \( TM \)-algebra. Let \( D_1, D_2, D_3 \) are \((l,r)\)-symmetric bi-derivation.

\[
(D_1 \wedge D_2)(x, y, z) = (D_1(x, y, z)) \wedge (D_2(x, y, z))
\]

\[
= (D_1(x, y) \wedge D_2(x, y)) (x, y) \wedge (D_3(x, y, z)) \wedge D_3(x, y, z)
\]

\[
= D_3(x, y, z) \wedge (D_3(x, y, z)) \wedge D_1(x, y, z)
\]

\[
= D_1(x, y, z)
\]

Combining (1) and (2) we get, \( (D_1 \wedge D_2) \wedge D_3 = D_1(D_2 \wedge D_3) \).

This proves that, \( \wedge \) is associative.

Combining the above two propositions, we get the following theorem.

**Theorem 4.4.** \( \mathcal{A}_L \) is a semigroup under the binary composition \( \wedge \) defined by \( (D_1 \wedge D_2)(x, y) = D_1(x, y) \wedge D_2(x, y) \) for all \( x, y \in X \) and \( D_1, D_2 \in \mathcal{A}_L \).
Analogously we can prove that,

**Theorem 4.5.** $\mathcal{D}_R$ is a semigroup under the binary operation $\wedge$ defined by $(D_1 \wedge D_2)(x, y) = D_1(x, y) \wedge D_2(x, y)$ for all $x, y \in X$ and $D_1, D_2 \in \mathcal{D}_R$ where $\mathcal{D}_R$ is the set of all $(r, l)$--symmetric bi-derivation.

Combining the above two theorem we get the following theorem.

**Theorem 4.6.** If $\mathcal{D}$ denotes the set of all symmetric bi-derivation on $X$, it is a semi-group under the binary operation $\wedge$ defined by $(D_1 \wedge D_2)(x, y) = D_1(x, y) \wedge D_2(x, y)$ for all $x, y \in X$ and $D_1, D_2 \in \mathcal{D}$.

**References**


