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THE RELATIONSHIP BETWEEN QUASI-MEDIAL IDEMPOTENTS AND QUASI-ADEQUATE TRANSVERSALS

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Abstract: Some properties for an abundant semigroup with quasi-medial idempotents are obtained and then the relationship between quasi-medial idempotents and quasi-adequate transversal is characterized.

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1. Introduction and Preliminaries

Since Blyth and McFadden in [3] introduced the concept of *medial idempotent* and *inverse transversals* respectively for a regular semigroup, much attention has been paid to thoes classes of regular semigroups and many papers have been devoted to the theme of describing regular semigroups with medial idempotent (see [2], [7], [12] and its references) and to that of describing regular semigroup with an inverse transversal (see [1], [16] and its references).

Recall from [9] that an idempotent u of an abundant semigroup with a set E of idempotents is called *weak medial idempotent* if for any $e \in E$, $eue = e$. A weak medial idempotent u is called a *weak normal idempotent* if uSu is a

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adequate semigroup. In [15], a weak medial idempotent u is called a *quasi-medial idempotent* for a semigroup S if uSu is a quasi-adequate semigroup, which is the generalization of a weak normal idempotent. The construction of an abundant semigroup with a quasi-medial idempotent was given.

In this paper, some properties for quasi-medial idempotents are obtained. Then it is concerned with the relationship between quasi-medial idempotent and quasi-adequate transversal. We show that if S is a regular semigroup with quasi-medial idempotents u and w, then $uSw(uSu, wSw)$ is a *quasi-ideal orthodox transversal* (see [4]), which is a generalization of inverse transversal. As considering that the class of all abundant semigroups is an important class of generalized regular semigroups, we denote ourself to studying the relationship between quasi-medial idempotent and *quasi-ideal quasi-adequate transversal* (see [14]), which is an analogous concept of quasi-ideal orthodox transversal in the class of abundant semigroups. Interestingly, if S is an abundant semigroup with quasi-medial idempotents u and w then $uSw(uSu, wSw)$ is a quasi-ideal quasi-adequate transversal for S. At last, we consider that in some case an idempotent of an abundant semigroup with a quasi-ideal quasi-adequate transversal can be a quasi-medial idempotent.

The relations $\mathcal{L}^*$ and $\mathcal{R}^*$ on a semigroup S are generalization of the familiar Green's relations $\mathcal{L}$ and $\mathcal{R}$. Two elements a and b in S are said to be $\mathcal{L}^*$-related if and only if they are $\mathcal{L}$-related in some oersemigroup of S. The relation $\mathcal{R}^*$ is defined dually. A semigroup S is called *abundant* [8] if each $\mathcal{L}^*$-class and $\mathcal{R}^*$-class contains an idempotent. An abundant semigroup S is called *quasi-adequate* [6] if its idempotents form a subsemigroup. Quasi-adequate semigroups are analogues of orthodox semigroups in the range of abundant semigroups. An *adequate semigroup* is a quasi-adequate semigroup in which the idempotents commute. The class of all abundant semigroups is an important class of generalized regular semigroups. Many authors have investigated various types of abundant semigroups.

The reader is referred to [10], [6] and [8] for all the notation and terminology not defined in this paper.

We shall list some basic results which are used in the sequel. The following lemma is due to Fountain [8] which providing us an alternative description for $\mathcal{L}^*$ ($\mathcal{R}^*$).

Lemma 1.1. (see [8]) *Let S be a semigroup, $a \in S$ and e be an idempotent of S. Then the following conditions are equivalent:*

(1) $a \ \mathcal{L}^* \ e$ ($a \ \mathcal{R}^* \ e$);

(2) $a = ae$ ($ea = a$) and for all $x, y \in S^1$, $ax = ay$ ($xa = ya$) $\Rightarrow ex = ey$ ($xe = ye$).

Let S be an abundant semigroup and $a \in S$. Denote by a^* (a^+) a typical idempotent in $\mathcal{L}^*$-class ($\mathcal{R}^*$-class) of S containing a. The next lemma, is due to El-Qallali [5], will be used frequently in the sequel.

Lemma 1.2. (see [5]) *Let S be an abundant semigroup with the set of idempotents E and $x, y \in S$. If there exist $e, f \in E$ such that $x = eyf$ and $e \mathcal{L} y^+$, $f \mathcal{R} y^*$ for some $y^+, y^* \in E$, then $e \mathcal{R}^* x$ and $f \mathcal{L}^* x$.*

Lemma 1.3. (see [9]) *Let S be an abundant semigroup with a set E of idempotents and u be a weak medial idempotent of S. Then for any $x \in S$ and $e \in E$,*
(1) $ue, eu, ueu \in E$;
(2) $x^+u \mathcal{R}^* xu$, $ux^* \mathcal{L}^* ux$;
(3) $ux^+u \mathcal{R}^* uxu \mathcal{L}^* ux^*u$.

2. Quasi-Medial Idempotent

In this section, unless otherwise stated, we always suppose that S is an abundant semigroup with the set E of idempotents.

Lemma 2.1. *Let u be a weak medial idempotent for S and $x \in S$. Then*
(1) $x = x^+ux = xux^*$;
(2) $xu \mathcal{R} x \mathcal{L} ux$.

Proof. (1) $x = x^+x = x^+ux^+x = x^+ux$ and $x = xx^* = xx^*ux^* = xux^*$.
(2) It follows (1) immediately. □

Let U be an abundant subsemigroup of S. U is called a left (right) $*$-subsemigroup if for all $a \in U$, there exist $e \in U \cap E$ such that $a \mathcal{L}^*(S)\, e$ $(a \mathcal{R}^*(S)\, e)$. If U is both a left and a right $*$-subsemigroup, then we call it a $*$-subsemigroup.

Lemma 2.2. *Let u be a weak medial idempotent for S and $e \in E$. Then*
(1) $uS(Su, uSu)$ is a $*$-subsemigroup;
(2) $E(eS)e \subseteq E$ $(eE(Se) \subseteq E)$.

Proof. (1) Let $x \in uS$. Then by Lemma 1.3 and 2.1, $x^+u \mathcal{R}^* xu \mathcal{R} x$ and $ux^* \mathcal{L}^* ux \mathcal{L}^* x$. Hence $ux^+u \mathcal{R}^* ux = x$ and $ux^* \mathcal{L}^* x$ which together with $ux^+u, ux^* \in E \cap uS$ implies that uS is a $*$-subsemigroup.
(2) Let $x \in E(eS)$. Then $(xe)^2 = x^2e = xe$, whence $E(eS)e \subseteq E$. □

Proposition 2.3. *Let u and w be weak medial idempotents for S. Then $wSw \cong uSu$.*

Proof. Define a map $\phi : wSw \to uSu$ by $wxw \mapsto uwxwu$. Obviously, ϕ is fine. Suppose that $\phi(wxw) = \phi(wyw)$. Then $uwxwu = uwywu$ implies $wxw = wuwxwuw = wuwywuw = wyw$. It meas ϕ is injective. On the other hand, $\phi(wuxuw) = uwuxuwu = uxu$. Hence by $\phi(wxwyw) = uwxwywu = uwxwuwywu = \phi(wxw)\phi(wyw)$, ϕ is a isomorphism from wSw onto uSu. □

We have shown that if u is a weak medial idempotent for S then uS is a $*$-subsemigroup of S. In fact, if u is a quasi-medial idempotent then there are more exciting results.

Lemma 2.4. *Let u be a quasi-medial idempotent for S. Then $E(uS) = uE$ $(E(Su) = Eu, E(uSu) = uEu)$ is a band.*

Proof. (1) Let $e, f \in E(uS)$. Since $ueu, ufu \in uEu$,

$$\begin{aligned} (ef)^2 &= (ueuf)^2 = ueufueufuf \\ &= (ueufu)^2 f = ueufuf = ef \end{aligned}$$

It is trivial to check that $E(uS) \subseteq uE$. Hence $E(uS) = uE$ is a band. □

Lemma 2.5. *Let u be a weak medial idempotent for S. Then the following statements are equivalent:*

(1) u is a quasi-medial idempotent;
(2) For any $e \in E$, $E(eS)$ is a band;
(3) For any $e \in E$, $E(Se)$ is a band;
(4) For any $e \in E$, eSe is quasi-adequate $*$-subsemigroup;
(5) uS is a quasi-adequate $*$-subsemigroup;
(6) Su is a quasi-adequate $*$-subsemigroup;.

Proof. (1) $\Rightarrow$ (2) Let $x, y \in E(eS)$. Since $xe, ye \in E$,

$$\begin{aligned} (xy)^2 &= (exey)^2 = [(eue)x(eue)y]^2 \\ &= eu(xe)u(ye)u(xe)u(ye)y = eu(xe)u(ye)u(xe)u(ye)uey \\ &= eu[(uxeu)(uyeu)]^2 y = eu(uxeu)(uyeu)y \\ &= (eue)x(eue)y = exey = xy. \end{aligned}$$

Hence $E(eS)$ is a band.

(2) $\Rightarrow$ (3) Let $x, y \in E(Se)$. As $ex, ey \in E$,

$$(xy)^2 = xexeyexey = x(exey)^2 = xexey = xy, \text{i.e } xy \in E(Se).$$

Hence $E(Se)$ is a band.

(3) $\Rightarrow$ (4) Let $x, y \in E(eSe)$. By the similar argument before, $E(eSe)$ is a band. Let $x \in S$. Then by $ex = x$, $ex^+ = x^+$. It follows that $ex^+ex^+ = x^+$ and $x^+ex^+e = ex^+e$. Hence $ex^+e \in E(eSe)$ and $x \mathcal{R}^* x^+ \mathcal{R} ex^+e$. Dually, $x \mathcal{L}^* x^* \mathcal{L} ex^*e \in E(eSe)$. Therefore eSe is a quasi-adequate $*$-subsemigroup.

(4) $\Rightarrow$ (5) As e is an arbitrary idempotent of S, uSu is quasi-adequate. Then uEu is a band. Let $g, h \in uE$. Then $(gh)^2 = (uguh)^2 = (uguhu)^2h = gh$. Hence $E(uS)$ is a band. By Lemma 2.2, uS is a quasi-adequate $*$-subsemigroup.

(5) $\Rightarrow$ (6) Let $g, h \in Eu$. Then $(gh)^2 = (guhu)^2 = g(uguh)^2 = guguh = gh$ since $E(uS) = uE$ is a band. It means $E(Su) = Eu$ is a band. Therefore Su is a quasi-adequate $*$-subsemigroup.

(6) $\Rightarrow$ (1) Let $g, h \in uEu$. Then $(gh)^2 = (guhu)^2 = guhu = gh$ since $E(Su) = Eu$ is a band. Hence u is a quasi-medial idempotent. $\square$

In what follows, we shall show that for any quasi-medial idempotents u and w for S, uw is also a quasi-medial idempotent for S and in this case the set of quasi-medial idempotents is a regular subband of S.

Lemma 2.6. *Let w and u be weak medial idempotents for S. Then*
(1) For any $e \in E$, $uew, euwe \in E$;
(2) For any $e \in E$, $ew \mathcal{L} uew \mathcal{R} ue$;
(3) For any $x \in S$, $x^*w \mathcal{L} ux^*w \mathcal{L}^* uxw \mathcal{R}^* ux^+w \mathcal{R} ux^+$;
(4) uSw is a $*$-subsemigroup.

Proof. (1) Since $ew, we \in E$, $(uew)^2 = u(ew)u(ew) = uew$ and $euweuwe = euwe$. Hence $uew, euwe \in E$.

(2) By Lemma 2.1, $ew \mathcal{R} e \mathcal{L} ue$. Since $\mathcal{R}$ is left congruence, $uew \mathcal{R} ue$. Dually, $ew \mathcal{L} uew$.

(3) Since $ux^* \mathcal{L}^* ux$ and $xw\mathcal{R}^* x^+w$, $ux^*w \mathcal{L}^* uxw \mathcal{R}^* ux^+w$. On the other hand, from $x^*w, ux^+ \in E$, follows that $x^*w \mathcal{L} ux^*w$ and $ux^+ \mathcal{R} ux^+w$.

(4) Follows from (3) immediately. $\square$

Lemma 2.7. *Let u and w be quasi-medial idempotents for S. Then*
(1) $uEw = uE \cap Ew$ is a band;
(2) $uwEuw$ is a band;
(3) uSw is a quasi-adequate $*$-subsemigroup with a band uEw.

Proof. (1) As $uEw \subseteq E$, $uEw \subseteq uE \cap Ew$. Obviously, $uE \cap Ew \subseteq uEw$. Hence By Lemma 2.4, $uEw = uE \cap Ew$ is a band.

(2) By (1) and its dual, $uwEuw \subseteq uEw \subseteq E$. For any $e, f \in E$,

$$\begin{aligned}(uweuwfuw)^2 &= uweuwfuweuwfuw \\ &= u(weuwfuw)^2 = uweuwfuw\end{aligned}$$

as $weuw, wfuw \in E$. Hence $uwEuw$ is a band.

(3) Follows from (1) and Lemma 2.6. □

Proposition 2.8. *Let u and w be quasi-medial idempotens for S. Then uw is also a quasi-medial idempotent.*

Proof. Let $e \in E$. Then by Lemma 2.6, $e \mathcal{R}\ eu \mathcal{L}\ weu \mathcal{R}\ we$. It follows that $e \mathcal{R}\ eu \mathcal{R}\ euwe \mathcal{L}\ we \mathcal{L}\ e$, whence $e \mathcal{R}\ euwe \mathcal{L}\ e$. Hence $e = euwe$, which together with Lemma 2.7(2) implies that uw is a quasi-medial idempotent. □

Theorem 2.9. *Let S be an abundant semigroup with a quasi-medial idempotent u. Then the set U of quasi-medial idempotents is a regular subband of S.*

Proof. Obviously, U is not empty. Let $v, w \in U$. Then $uvw = uw(uvw)uw = uw$ as uvw is a quasi-medial idempotent for S. Hence U is a regular subband of S. □

Let S be an abundant semigroup with the set of idempotents E. Denote by $\bar{E}$ the subsemgroup of S generated by E. From [11] follows that an idempotent u of S is called medial idempotent if for any $x \in \bar{E}$, $xex = x$.

Now we suppose that u and w in Proposition 2.8 and Theorem 2.9 are medial idempotents. Then

Corollary 2.10. *Let u and w be medial idempotens for S. Then uw is also a medial idempotent.*

Proof. Let $x \in \bar{E}$. Since $xux = x$, $xu \in E$ and so $xuw \in E$. By Lemma 2.1, $x \mathcal{R}^*\ xu \mathcal{R}\ xuw$. It meas $xuwx = x$. Hence uw is a medial idempotent. □

Corollary 2.11. *Let S be an abundant semigroup with a medial idempotent u. Then the set U of medial idempotents is a regular subband of S.*

3. Relationship between Quasi-Medial Idempotents and Quasi-Adequate Transversals

In this section, it shall be concerned with the relationship between quasi-medial idempotents and quasi-ideal transversals. In fact, let S be a semigroup with quasi-medial idempotents u and w. Firstly, we will show that if S is regular then $uSw(uSu, wSw)$ is a quasi-ideal orthodox transversal for S and that if S is an abundant semigroup then $uSw(uSu, wSw)$ is a quasi-ideal quasi-adequate transversal for S. Next, let S be an abundant semigroup with a quasi-ideal quasi-adequate transversal S°. We shall consider in which case an idempotent of S is a quasi-medial idempotent.

For any subsemigroup T of a semigroup S, let $V_T(x) = V(x) \cap T$ where $x \in S$. Moreover, T is a quasi-ideal of S if $TST \subseteq T$.

Lemma 3.1. *Let S be an abundant semigroup with weak medial idempotents u and w. Then*

(1) For any $x' \in V(S)$, $xux' = xx'$ and $x'ux = x'x$;

(2) $(\forall x \in S)\ V(x) \neq \emptyset \Rightarrow uV(x)w = V_{uSw}(x) = V_{uSw}(uxw)$

(3) $(\forall x \in S)\ V(x) \neq \emptyset \Rightarrow V_{uSw}(x) = V_{uSw}(wx) = V_{uSw}(xu) = V_{uSw}(wxu)$;

(4) $(\forall x \in S)\ V(x) \neq \emptyset \Rightarrow V_{uSw}(x) = V_{uSu}(x)xV_{wSw}(x)$.

Proof. (1) Obviously, $x'x, xx' \in E$. Hence $xux' = xx'xux'xx' = xx'x = x$ and $x'ux' = x'xx'ux'xx' = x'xx' = x'$.

(2) Let $x' \in V(x)$. Then $ux'wxux'w = ux'xx'w = ux'w$ and $wxux'wxu = uxx'xw = wxu$. It means $uV(x)w \subseteq V_{uSw}(x)$, i.e $V_{uSw}(x) \neq \emptyset$. Obviously, $V_{uSw}(x) \subseteq uV(x)w$. Therefore $uV(x)w = V_{uSw}(x)$.

Next we shall prove that $V_{uSw}(x) = V_{uSw}(uxw)$. Let $y \in V_{uSu}(x)$. Take $x^+ = xy$ and $x^* = yx$. Then by Lemma 2.1,

$$\begin{aligned} xyx &= (x^+uxwx^*)y(x^+uxwx^*) \\ &= x^+uxw(yx)y(xy)uxwx^* \quad (\text{since } x^+ = xy, x^* = yx) \\ &= x^+uxwyuxwx^* \quad (\text{since } y \in V_{uSw}(x)) \\ &= x^+uxyxwx^* \\ &= x^+uxwx^* = x \end{aligned}$$

and $y = yxy = yx^+uxwx^*y = yxyuxwyxy = yuxwy$. Hence $V_{uSu}(x) \subseteq V_{uSu}(uxu)$. Conversely, let $y \in V_{uSw}(uxw)$. Notice that

$$x^+w \ \mathcal{L} \ ux^+w \ \mathcal{R}^* \ uxw \ \mathcal{R}^* \ uxwy \ \mathcal{L}^* \ y$$

and that

$$ux^* \ \mathcal{R} \ ux^*w \ \mathcal{L}^* \ uxw \ \mathcal{L}^* \ yuxw \ \mathcal{R}^* \ y.$$

Then $x^+w(uxwy)\ \mathcal{L}^*\ y\ \mathcal{R}^*\ (yuxw)ux^*$. By commuting,

$$\begin{aligned} xyx &= [x^+w(uxw)ux^*]y[x^+w(uxw)ux^*] \\ &= [x^+w(uxw)y(uxw)ux^*]y[x^+w(uxw)y(uxw)ux^*] \\ &= x^+w(uxw)[(yuxw)ux^*yx^+w(uxwy)](uxw)ux^* \\ &= x^+w(uxw)y(uxw)ux^* \\ &= x^+wuxwux^* = x \end{aligned}$$

and

$$\begin{aligned} yxy &= y[x^+w(uxw)ux^*]y \\ &= yx^+w[(uxw)y(uxw)y(uxw)]ux^*y \\ &= y(x^+wuxwy)(uxw)(yuxwux^*)y \\ &= y(uxw)y = y. \end{aligned}$$

Hence $V_{uSw}(uxw) \subseteq V_{uSw}(x)$.

(3) By (2), $V_{uSw}(x) \neq \emptyset$. Here we just prove that $V_{uSw}(x) = V_{uSw}(wx)$. The remainder part can be proved by the similar arguments. In fact, we suppose that $x' \in V_{uSw}(x)$. Then $x'(wx)x' = x'xx' = x'$ and $wxx'wx = wxx'x = wx$, whence $V_{uSw}(x) \subseteq V_{uSw}(wx)$. Conversely, let $y \in V_{uSw}(wx)$. Then $yxy = (yw)xy = y(wx)y = y$ and $xyx = x^+wxywxwx^* = x^+wxwx^* = x$. It means $V_{uSw}(wx) \subseteq V_{uSw}(x)$.

(4) Let $y \in V_{uSu}(x)$ and $z \in V_{wSw}(x)$. Then by computing, $(yxz)x(yxz) = y(xzx)yxz = yxyxz = yxz$ and $x(yxz)x = (xyx)zx = xzx = x$. It follows that $V_{uSu}(x)xV_{uSw}(x) \subseteq V_{uSw}(x)$. Conversely, let $a \in V_{uSw}(x)$. Then for any $x' \in V(x)$, $a = axa = ax(ux'u)x(wx'w)a = (axx'u)x(wx'xa)$. Notice that $(axx'u)x(axx'u) = axaxx'u = axx'u$ and $x(axx'u)x = xax = x$. Hence by $ua = u$, $axx'u \in V_{uSu}(x)$. Similarly, we have $wx'xa \in V_{wSw}(x)$. Therefore $V_{uSw}(x) \subseteq V_{uSu}(x)xV_{uSw}(x)$. □

As we all know, a regular semigroup S is orthodox if and only if

$$(\forall\ a, b \in S)\ V(a) \cap V(b) \neq \emptyset\ \Rightarrow\ V(a) = V(b).$$

Here we shall show that a regular semigroup with two quasi-medial idempotents have the similar result.

Lemma 3.2. *Let S be a regular semigroup with quasi-medial idempotents u and w. Then*

(1) $(\forall x \in S)\ V_{uSw}(x) \neq \emptyset$;

(2) $(\forall x, y \in S)\ V_{uSw}(x) \cap V_{uSw}(y) \neq \emptyset \Rightarrow V_{uSw}(x) = V_{uSw}(y)$;

(3) $(\forall x, y \in S)\ \{\ x, y\ \} \cap uSw \neq \emptyset \Rightarrow V_{uSw}(x)V_{uSw}(y) \subseteq V_{uSw}(yx)$.

Proof. (1) Let $x' \in V(x)$. In view of the proof of Lemma 3.1 (2), $ux'w \in V_{uSw}(x)$.

(2) Since $V_{uSw}(x) = V_{uSw}(uxw)$ and $V_{uSw}(y) = V_{uSw}(uyw)$, $V_{uSw}(uxw) \cap V_{uSw}(uyw) \neq \emptyset$. Notice that uSw is orthodox. We have $V_{uSw}(uxw) = V_{uSw}(uyw)$, whence $V_{uSw}(x) = V_{uSw}(y)$.

(3) Suppose that $y \in uSw$. As uSw is an orthodox semigroup, we have

$$\begin{aligned} V_{uSw}(x)V_{uSw}(y) &= V_{uSw}(wx)V_{uSw}(y) \\ &= V_{uSw}(uwxw)V_{uSu}(y) \\ &\subseteq V_{uSw}(yuwxw) \\ &= V_{uSw}(uywuwxw) \\ &= V_{uSw}(uyxw) = V_{uSu}(yx). \end{aligned}$$

□

Recall from [4] that a subsemigroup S° of a regular semigroup S is called an orthodox transversal for S if

$(O1)$ $(\forall x \in S) \quad V_{S^\circ}(x) \neq \emptyset$,

$(O2)$ $(\forall x, y \in S)$ $\{ x, y \} \cap S^\circ \neq \emptyset \Rightarrow V_{S^\circ}(x)V_{S^\circ}(y) \subseteq V_{S^\circ}(yx)$.

Then by Lemma 2.7 and 3.2, we have

Theorem 3.3. *Let S be a regular semigroup with quasi-medial idempotents u and w. Then uSw is a quasi-ideal orthodox transversal for S. In particular, uSu and wSw are also quasi-ideal orthodox transversals for S.*

For convenience, we list some notaion about quasi-adequate transversals as follows [13]. The reader is referred to [13] and [14] for all the notation, terminology and some results not mentioned in this section.

Let S be an abundant semigroup and S° an abundant subsemigroup with the set of idempotents E°. S° is called an *abundant transversal* for S if for any $x \in S$, there exist $x^\circ \in S^\circ$, $e, f \in E$ such that $x = ex^\circ f$, where $e \ \mathcal{L}^* \ x^{\circ +}$, $f \ \mathcal{R}^* \ x^{\circ *}$ for some $x^{\circ +}, x^{\circ *} \in E^\circ$. In this case, let

$$C_{S^\circ}(x) = \{x^\circ \in S^\circ | x = ex^\circ f, e\mathcal{L}x^{\circ +}, f\mathcal{R}x^{\circ *}, \ x^{\circ +}, x^{\circ *} \in E^\circ\},$$

$$I_x = \{e \in E | (\exists x^\circ \in C_{S^\circ}(x)) x = ex^\circ f, e\mathcal{L}x^{\circ +}, f\mathcal{R}x^{\circ *}, \ x^{\circ +}, x^{\circ *} \in E^\circ\},$$

$$\Lambda_x = \{f \in E | (\exists x^\circ \in C_{S^\circ}(x)) x = ex^\circ f, e\mathcal{L}x^{\circ +}, f\mathcal{R}x^{\circ *}, \ x^{\circ +}, x^{\circ *} \in E^\circ\},$$

$$I = \bigcup_{x \in S} I_x, \quad \Lambda = \bigcup_{x \in S} \Lambda_x.$$

S° is called a quasi-adequate transversal for S if S° is quasi-adequate and

$(QA1)$ $(\forall x \in S)$ $C_{S^\circ}(x) \neq \emptyset$,

$(QA2)$ $(\forall e \in E)$ $(\forall g \in E^\circ)$

$$C_{S^\circ}(e)C_{S^\circ}(g) \subseteq C_{S^\circ}(ge) \text{ and } C_{S^\circ}(g)C_{S^\circ}(e) \subseteq C_{S^\circ}(eg).$$

A quasi-adequate transversal S° is multiplicative if
(M) $(\forall x, y \in S)$ $\Lambda_x I_y \subseteq E^\circ$.

A quasi-adequate transversal S° is a quasi-ideal quasi-adequate transversal for S if S° is a quasi-ideal of S (i.e, $S^\circ S S^\circ \subseteq S^\circ$).

In what follows, unless otherwise stated, we always suppose that S is an abundant semigroup with the set E of idempotents.

Lemma 3.4. *Let u and w be quasi-medial idempotents for S. For any $x \in S$,*
(1) $uxw \in C_{uSw}(x)$;
(2) $x^+w \in I_x$ and $ux^* \in \Lambda_x$.

Proof. (1), (2) Obviously, $x = x^+wuxwux^*$ since wu is a quasi-medial idempotent. Notice that $x^+w, ux^* \in E$. Then by Lemma 1.3 and 2.1,

$$x^+w \;\mathcal{L}\; ux^+w \;\mathcal{R}^*\; uxw \text{ and } ux^* \;\mathcal{R}\; ux^*w \;\mathcal{L}^*\; uxw.$$

Hence $uxw \in C_{uSw}(x)$ and $x^+w \in I_x$, $ux^* \in \Lambda_x$. □

Lemma 3.5. *Let u and w be quasi-medial idempotents for S. Then*
(1) $I = Ew$ and $\Lambda = uE$;
(2) $I \cap \Lambda = Ew \cap uE = uEw$;
(3) $\Lambda I \subseteq uSw$.

Proof. (1) By Lemma 3.4, uSw is a abundant transversal for S. Then for any $g \in I$, there exists $x \in S$ and $x^\circ \in C_{uSw}(x)$ such that $g \;\mathcal{L}\; x^{\circ+}$ for some $x^{\circ+} \in uEw$. It follows that $g = gx^{\circ+} = gx^{\circ+}w = gw \in Ew$. On the other hand, let $h \in Ew$. Then $h = hw \;\mathcal{L}\; uhw \in uEw$. It means $h \in I_h \subseteq I$. Hence $I = Ew$. The remained proof is a dual.

(2), (3) It is trivial. □

Let S be a quasi-adequate semigroup with a band B. For $e \in B$, denote by $E(e)$ the $\mathcal{J}$-class of B containing e. Define a relation δ on S by: for $a, b \in S$,

$$a \;\delta\; b \;\Leftrightarrow\; E(a^+)aE(a^*) = E(b^+)bE(b^*) \text{ for some } a^+, a^*, b^+, b^*.$$

It follows from [6] that δ is an equivalence relation which contained in any adequate congruence on S. In particular, if S is an orthodox semigroup, then δ is the least inverse congruence on S.

Lemma 3.6. *Let u and w be quasi-medial idempotents for S. Then for any $e \in E$ and $g \in uEw$,*

(1) $C_{uSw}(e) = \delta_{uew} \subseteq uEw$;

(2) $C_{uSw}(e)C_{uSw}(g) \subseteq C_{uSw}(ge)$ and $C_{uSw}(g)C_{uSw}(e) \subseteq C_{uSw}(eg)$.

Proof. (1) Let $e^{\circ} \in C_{uSu}(e)$. Then there exists $i_e \in I_e$ and $\lambda_e \in \Lambda_e$ such that $e = i_e e^{\circ} \lambda_e$ and $i_e \ \mathcal{L} \ e^{\circ+}, \lambda_e \ \mathcal{R} \ e^{\circ*}$ for some $e^{\circ+}, e^{\circ*} \in uEw$. It is easy to check that $e^{\circ} = e^{\circ+}ee^{\circ*} = (e^{\circ+}ew)uew(uee^{\circ*})$ as uw and wu are quasi-medial idempotents. By Lemma 1.2, $i_e \ \mathcal{R} \ e \ \mathcal{L} \ \lambda_e$. Hence $ew \ \mathcal{R} \ i_e \ \mathcal{L} \ e^{\circ+}$ and $ue \ \mathcal{L} \ \lambda_e \ \mathcal{R} \ e^{\circ*}$. Then $uew \ \mathcal{L} \ ew \ \mathcal{L}^* \ e^{\circ+}ew$ and $uew \ \mathcal{R} \ ue \ \mathcal{R}^* \ uee^{\circ+}$. Obviously, $e^{\circ+}ew, uee^{\circ+} \in uEw$. Therefore $e^{\circ+}ew, uee^{\circ+} \in E(uew)$ and so $e^{\circ} \in \delta_{uew}$.

Conversely, let $\bar{e} \in \delta_{uew}$. Obviously, $\delta_{uew} \subseteq uEw$. Then $\bar{e} \in uEw$ and $uew\bar{e} \ \mathcal{L} \ \bar{e} \ \mathcal{R} \ \bar{e}uew$. Notice that $\bar{e}, ew \in I$. We have $\bar{e} \ \delta(I) \ ew$. It follows that $ew \ \mathcal{R} \ ew\bar{e} \ \mathcal{L} \ \bar{e}$. Dually, $ue \ \mathcal{L} \ \bar{e}ue \ \mathcal{R} \ \bar{e}$. Hence $ew\bar{e} \ \mathcal{L} \ uew\bar{e} \ \mathcal{L} \ \bar{e}$ and $\bar{e}uew \ \mathcal{R} \ \bar{e}ue \ \mathcal{R} \ \bar{e}$. On the other hand, we have $e = ewue = ew(uew)\bar{e}(uew)ue = ew\bar{e}ue$. Therefore $\bar{e} \in C_{uSu}(e)$.

(2)As $g = ugw$ and $euwe = e$, $ugew = gwew = gw(uwew)(uew)$ and $gw(uew)(uwew) = gwuew = guew$. For any $x, y \in uEw$, $x \ \delta(uEw) \ y$ if and only $x \ \delta \ y$. Hence

$$uewg \ \delta \ guew = gw(uew)(uwew) \ \delta \ gw(uwew)(uew) = ugew.$$

It follows that $\delta_{uew}\delta_g \subseteq \delta_{uewg} = \delta_{ugew}$, whence $C_{uSu}(e)C_{uSu}(g) \subseteq C_{uSu}(ge)$. Similarly, $C_{uSu}(g)C_{uSu}(e) \subseteq C_{uSu}(eg)$. □

Theorem 3.7. *Let S be an abundant semigroup with quasi-medial idempotents u and w. Then uSw is a quasi-ideal quasi-adequate transversal for S. In particular, uSu and wSw are also quasi-ideal quasi-adequate transversals for S.*

Proof. By Lemma 3.4 and 3.6. □

Corollary 3.8. *Let S be a abundant semigroup with medial idempotents u and w. Then uSw is a multiplicative quasi-adequate transversal for S. In particular, uSu and wSw are also multiplicative quasi-adequate transversals for S.*

Proof. Let $e, f \in E$. Then $ew \in Ew, uf \in uE$. By Corollary 2.10, $(ufew)^2 = u(fewufe)w = ufew$. It means $\Lambda I \subseteq uEw$. Hence by Theorem 3.7, uSw, uSu and wSw are multiplicative quasi-adequate transversals. □

Lemma 3.9. *Suppose that S° is a quasi-ideal quasi-adequate transversal for S. Let $x \in S$ and $i_x \in I_x$, $x^\circ \in C_{S^\circ}(x)$, $\lambda_x \in \Lambda_x$ such that $x = i_x x^\circ \lambda_x$ and $i_x \mathcal{L} x^{\circ+}, \lambda_x \mathcal{R} x^{\circ*}$. If $x^\circ \in E^\circ$ and $u \in E^\circ$ is an identity of E°, then $\lambda_x u i_x = x^{\circ*}x^{\circ+}$.*

Proof. In fact, since u is an identity of E°,

$$\lambda_x u \lambda_x = \lambda_x u(x^{\circ*}\lambda_x) = \lambda_x(ux^{\circ*})\lambda_x = \lambda_x(x^{\circ*}\lambda_x) = \lambda_x.$$

It follows that $\lambda_x u \in E$. It is trivial to check that $\lambda_x u \mathcal{R} \lambda_x$. Similarly, $ui_x \mathcal{L} i_x$. Notice that $x^\circ \in E^\circ$. We have $x^{\circ*} \mathcal{R} x^{\circ*}x^{\circ+} \mathcal{L} x^{\circ+}$. Then $ui_x \mathcal{L} x^{\circ*}x^{\circ+} \mathcal{R} \lambda_x u$. From $\lambda_x u, ui_x \in E^\circ$ follows that $\lambda_x u \; \delta(E^\circ) \; ui_x$. It means $\lambda_x u \mathcal{R} \lambda_x u i_x \mathcal{L} ui_x$. Hence $\lambda_e u i_e \; \mathcal{H}^* \; e^{\circ*}e^{\circ+}$, which together with $\lambda_x u i_x, x^{\circ*}x^{\circ+} \in E^\circ$ implies that $\lambda_x u i_x = x^{\circ*}x^{\circ+}$. □

Proposition 3.10. *Let S° be a quasi-ideal quasi-adequate transversal for S. If for any $e \in E$, $C_{S^\circ}(e) \subseteq E^\circ$ and u is an identity of E°, then u is a quasi-medial idempotent for S.*

Proof. Let $i_e \in I_e$, $e^\circ \in C_{S^\circ}(e)$ and $\lambda_e \in \Lambda_e$ be such that $e = i_e e^\circ \lambda_e$ and $i_e \mathcal{L} e^{\circ+}, \lambda_e \mathcal{R} e^{\circ*}$. Then $eue = i_e e^\circ \lambda_e u i_e e^\circ \lambda_e = i_e e^\circ e^{\circ*} e^{\circ+} e^\circ \lambda_e = i_e e^\circ \lambda_e = e$. On the other hand, $uEu \subseteq S^\circ \cap E = E^\circ$ as S° is a quasi-ideal. It follows that $(uEu)(uEu) \subseteq E^\circ \cap uSu \subseteq uEu$. Hence uEu is a band and so u is a quasi-medial idempotent for S. □

Proposition 3.11. *Let S° be a multiplicative quasi-adequate transversal for S. If u is an identity of E°, then u is a medial idempotent for S.*

Proof. Let $e, f \in E$. Then there exists $i_e \in I_e, i_f \in I_f$, $e^\circ \in C_{S^\circ}(ef), f^\circ \in C_{S^\circ}(f)$ and $\lambda_e \in \Lambda_e, \lambda_f \in \Lambda_f$ such that $e = i_e e^\circ \lambda_e$, $i_e \mathcal{L} e^{\circ+}, \lambda_e \mathcal{R} e^{\circ*}$ for some $e^{\circ+}, e^{\circ*} \in E^\circ$ and $f = i_f f^\circ \lambda_f$, $i_f \mathcal{L} f^{\circ+}$, $\lambda_f \mathcal{R} f^{\circ*}$ for some $f^{\circ+}, f^{\circ*} \in E^\circ$. Since S° is a multiplicative quasi-adequate transversal for S, $e^\circ, f^\circ \in E^\circ$ by Lemma 2.3 of [14] and $e^\circ \lambda_e i_f f^\circ \in C_{S^\circ}(ef)$ by Theorem 2.9 of [14]. Notice that $\lambda_e i_f \in E^\circ$. We have $e^\circ \lambda_e i_f f^\circ \in E^\circ$. On the other hand, since $e^{\circ+}(e^\circ \lambda_e i_f f^\circ) = e^\circ \lambda_e i_f f^\circ$, $e^{\circ+}(e^\circ \lambda_e i_f f^\circ)^+ = (e^\circ \lambda_e i_f f^\circ)^+$ for some $(e^\circ \lambda_e i_f f^\circ)^+ \in E^\circ$. Then by $i_e \mathcal{L} e^{\circ+}$, $i_e(e^\circ \lambda_e i_f f^\circ)^+ \mathcal{L} (e^\circ \lambda_e i_f f^\circ)^+$. Dually, $(e^\circ \lambda_e i_f f^\circ)^* \lambda_f \mathcal{R} (e^\circ \lambda_e i_f f^\circ)^*$ for some $(e^\circ \lambda_e i_f f^\circ)^* \in E^\circ$. Obviously,

$$ef = i_e e^\circ \lambda_e i_f f^\circ \lambda_f = i_e(e^\circ \lambda_e i_f f^\circ)^+(e^\circ \lambda_e i_f f^\circ)(e^\circ \lambda_e i_f f^\circ)^* \lambda_f.$$

Hence in view of the proof of Lemma 3.9,

$$(e^\circ \lambda_e i_f f^\circ)^* \lambda_f u i_e (e^\circ \lambda_e i_f f^\circ)^+ = (e^\circ \lambda_e i_f f^\circ)^* (e^\circ \lambda_e i_f f^\circ)^+.$$

It follows that $efuef = i_e(e^\circ\lambda_e i_f f^\circ)\lambda_f = ef$. Therefore by Mathematic Induction, for any $x \in \bar{E}$, $xux = x$. □

Remark. The converse of Theorem 3.11 is false.

Example. Let S be a four-element semigroup given by the following Cayley table:

	e	f	g	a
e	e	f	e	a
f	f	f	f	a
g	g	f	g	a
a	a	a	a	f

Then S is a quasi-adequate semigroup with a band $E(S) = \{e, f, g\}$ and $f\ \mathcal{H}^*\ a$, $g\ \mathcal{L}\ e$. Obviously, S is a multiplicative quasi-adequate transversal for itself. It is easy to check that e is a medial idempotent for S for all but an identity of $E(S)$.

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