International Journal of Pure and Applied Mathematics
Volume 87 No. 1 2013, 115-127
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v87i1.6

ijpam.eu

A NATURAL ALGORITHM FOR FINDING PARTICULAR SOLUTIONS FOR A CLASS OF LDE

Sever Achimescu[1], Angel Popescu[2], Marius Buliga[3 §]

[1,2]Department of Mathematics and Computer Science
Technical University of Civil Engineering Bucharest
B-ul Lacul Tei 124, 72302, Bucharest 38, ROMANIA

[3]Department of Mathematics
University of Pittsburgh-Bradford
Bradford, PA 16701, USA

Abstract: An algorithmic instructional elementary method is developed to find particular solutions for a class of LDE of order n. Some theoretical algebraic background are supplied.

AMS Subject Classification: 34-01, 34A30, 34A05
Key Words: linear differential operator, constant coefficients, particular solution

1. Introduction

Let $\varphi(x) = a_0x^n + a_1x^{n-1} + ... + a_n, a_0 \neq 0$, be a fixed polynomial of degree n with real coefficients, i.e. $\varphi \in \mathbb{R}[x]$, $\deg\varphi = n$. Let $D = \frac{d}{dx}, D^2 = \frac{d^2}{dx^2}, ..., D^n = \frac{d^n}{dx^n}$ be the usual differential operators and let

$$\varphi(D) = a_0D^n + a_1D^{n-1} + ... + a_nI$$

be the usual polynomial differential operator. Here I is the identity operator.

Received: May 9, 2013

§Correspondence author

For any function $y = y(x)$ of class C^n on a given interval $J \subset \mathbb{R}$ ($y \in C^n(J)$) one denotes $\varphi(D)(y)$ by $L[y]$.

For any $\mu \in \mathbb{C}$, the field of complex numbers, let

$$V_\mu = \{P(x)e^{\mu x} : P(x) \in \mathbb{C}[x]\}$$

be the infinite dimensional vector space (over $\mathbb{C}$) generated by $\left\{x^k e^{\mu x}\right\}$, $k = 0, 1, \ldots$.

We prove that if $\varphi(\mu) \neq 0$, then the restriction of L to V_μ is an isomorphism $L_\mu : V_\mu \to V_\mu$ and we supply an algorithm for finding $L_\mu^{-1}\left[Q(x)e^{\mu x}\right]$ for any $Q(x) \in \mathbb{C}[x]$. Moreover, if $Q(x) \in \mathbb{R}[x]$ and $\mu \in \mathbb{R}$, then $L_\mu^{-1}\left[Q(x)e^{\mu x}\right] = S(x)e^{\mu x}$, where $S(x) \in \mathbb{R}[x]$ (see theorem (1)).

If μ is a root of the equation $\varphi(x) = 0$ and if the algebraic multiplicity of μ is $m \geq 1$, then Ker $L_\mu \subset V_\mu$ is a vector subspace of dimension m, the restriction L_μ^* of L_μ to the subspace $V_\mu^* = \{x^m H(x)e^{\mu x} : H(x) \in \mathbb{C}[x]\}$, $L_\mu^* : V_\mu^* \to V_\mu$, is an isomorphism, Ker $L_\mu \oplus V_\mu^* = V_\mu$ and if $Q(x) \in \mathbb{C}[x]$, then $(L_\mu^*)^{-1}[Q(x)e^{\mu x}] = x^m H(x)e^{\mu x}$, where $H(x) \in \mathbb{C}[x]$ and $\deg H(x) = \deg Q(x)$ (see theorem (2)). We also supply easy and natural algorithms for finding the above polynomial $S(x)$ and this last polynomial $H(x)$ (see also theorem (2)).

The advantage of our algorithms is that they do not involve the knowledge of all the roots of the characteristic equation $\varphi(x) = 0$ and some approximate procedures can be developed if one does not know the exact value of that root which eventually produces a resonance.

These elementary and clear remarks together the corresponding algorithms can be very useful tools in an instructional process of teaching linear differential equations. I am sure that all these ideas are known, but the algebraic form in which we present them here maybe is new.

2. Some Auxiliary Results

Let V be the complex vector space generated by all functions of one real variable $x, f(x) = P(x)e^{\lambda x}$, where $P(x) \in \mathbb{C}[x]$ is an arbitrary polynomial function with complex coefficients and $\lambda \in \mathbb{C}$ is an arbitrary complex number. This means that an element of V is of the following form:

$$v = \sum_{i=1}^{t} P_i(x)e^{\lambda_i x}, \tag{2.1}$$

where $P_i(x) \in \mathbb{C}[x]$ and $\lambda_i \in \mathbb{C}$ for any $i = 1, 2, ..., t$. We may assume that all λ_i in formula (2.1) are distinct.

For any fixed $\mu \in \mathbb{C}$ let V_μ be the vector subspace of V generated by the set of functions $\{P(x)e^{\mu x} : P(x) \in \mathbb{C}[x]\}$.

Proposition 1.

$$V = \bigoplus_{\mu \in \mathbb{C}} V_\mu,$$

i.e. any $v \in V$ can be uniquely written as in formula (2.1), where λ_i are distinct complex numbers.

Proof. Let t_0 be the least natural number such that there exists an equality of the following type:

$$\sum_{i=1}^{t_0} P_i(x)e^{\lambda_i x} = 0 \tag{2.2}$$

for any $x \in \mathbb{R}$, where $\lambda_1, ..., \lambda_{t_0} \in \mathbb{C}$ are distinct and the polynomials $P_1(x), ..., P_{t_0}(x)$ are not zero (as polynomials). If such an equality does not exist, we put $t_0 = 0$ and we have nothing more to prove. Obviously t_0 cannot be equal to 1. Let us assume that $t_0 > 1$. Since $P_i(x)$ is not identical to zero for any $i = 1, 2, ..., t_0$, let us divide (2.2) by $e^{\lambda_{t_0} x}$ say and find:

$$P_{t_0}(x) = \sum_{i=1}^{t_0-1} \left[-P_i(x)\right] e^{(\lambda_i - \lambda_{t_0})x} = 0 \tag{2.3}$$

for any $x \in \mathbb{R}$. Let $d_0 = \deg P_{t_0}(x)$ be the degree of $P_{t_0}(x)$. Let us differentiate $(d_0 + 1)$-times w.r.t. x the equality (2.3). We get:

$$0 = \sum_{i=1}^{t_0-1} Q_i(x)e^{(\lambda_i - \lambda_{t_0})x}, \tag{2.4}$$

where

$$Q_i(x) = -\sum_{j=0}^{d_0+1} \left[P_i(x)\right]^{(j)} \left(\lambda_i - \lambda_{t_0}\right)^{d_0+1-j} C_{d_0+1}^j. \tag{2.5}$$

Since $t_0 - 1 < t_0$, all $Q_i(x)$ must be zero as polynomials, i.e. for each $i = 1, 2, ..., t_0 - 1$, $P_i(x)$ is a solution of the following LDE of order $d_0 + 1$, with constant coefficients:

$$0 = (\lambda_i - \lambda_{t_0})^{d_0+1} C_{d_0+1}^0 y + ... + y^{(d_0)}. \tag{2.6}$$

Thus

$$(\lambda_i - \lambda_{t_0})^{d_0+1} P_i(x) = -(\lambda_i - \lambda_{t_0})^{d_0} (d_0 + 1)P_i'(x) - ... - P_i^{(d_0)}(x). \tag{2.7}$$

Since $\lambda_i \neq \lambda_{t_0}$ and since the degree of the polynomial on the left is greater than the degree of the polynomial on the right, one must have that $P_i(x) = 0$ for each $i = 1, 2, ..., t_0 - 1$. From (2.3) we get that $P_{t_0}(x) = 0$, a terrible contradiction (all $P_i(x)$ were not identical to zero). Hence the assumption $t_0 > 1$ was false and the proposition is now completely proved. $\square$

Since we want this presentation to be at an elementary level, let us recall some basic facts on complex differentiation and on series of complex functions (functions of a complex variable with complex values) which are to be used later in this note.

Lemma 1. *Let Ω be an open subset of $\mathbb{C}$, let $\{f_n(z)\}$ be a sequence of continuous functions defined on Ω and let $S(z) = \sum_{n=0}^{\infty} f_n(z)$ be the corresponding series of these functions such that the set of convergence of the series contains Ω and the series of functions is locally uniformly convergent to $S(z)$. Let $z_0 \in \Omega$ be an arbitrary point of Ω. Then*

$$\lim_{z \to z_0, z \in \Omega} S(z) = \sum_{n=0}^{\infty} \lim_{z \to z_0, z \in \Omega} f_n(z)$$

or

$$\lim_{z \to z_0, z \in \Omega} S(z) = \sum_{n=0}^{\infty} f_n(z_0).$$

Proof. For a fixed $\varepsilon > 0$, let us take N large enough such that

$$\left| \sum_{n=0}^{\infty} f_n(z) - \sum_{n=0}^{N} f_n(z) \right| < \frac{\varepsilon}{3} \tag{2.8}$$

for any $z \in B(z_0, r)$, a disc centered at z_0 and of radius $r > 0$ (see the locally uniform convergence condition). Since $f_n(z)$ are continuous at z_0, there is a small $\delta_\varepsilon > 0$ and $\delta_\varepsilon < r$ such that if $|z - z_0| < \delta_\varepsilon$ then

$$\left| \sum_{n=0}^{N} f_n(z) - \sum_{n=0}^{N} f_n(z_0) \right| < \frac{\varepsilon}{3}. \tag{2.9}$$

Thus

$$\left| S(z) - \sum_{n=0}^{\infty} f_n(z_0) \right| \leq \left| S(z) - \sum_{n=0}^{N} f_n(z) \right| + \left| \sum_{n=0}^{N} f_n(z) - \sum_{n=0}^{N} f_n(z_0) \right| +$$

$$+\left|\sum_{n=0}^{N} f_n(z_0) - \sum_{n=0}^{\infty} f_n(z_0)\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

for $|z - z_0| < \delta_\varepsilon$ and the proof is completed. □

The exponential complex function e^z is defined by the following power series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \tag{2.10}$$

which is uniformly convergent on any bounded subset of $\mathbb{C}$ (see the Weierstrass test). Let z and w be two complex variables. Then, by definition

$$\frac{\partial}{\partial z}\left(e^{zw}\right)(z_0) = \lim_{z\to z_0} \frac{e^{zw} - e^{z_0 w}}{z - z_0}.$$

Applying now lemma (1) we get:

$$\frac{\partial}{\partial z}\left(e^{zw}\right)(z_0) = \lim_{z\to z_0} \sum_{n=0}^{\infty} \frac{\left(z^n - z_0^n\right)}{z - z_0}\frac{w^n}{n!} = \sum_{n=0}^{\infty} \lim_{z\to z_0} \frac{\left(z^n - z_0^n\right)}{z - z_0}\frac{w^n}{n!} =$$

$$= w\sum_{n=1}^{\infty} z_0^{n-1}\frac{w^{n-1}}{(n-1)!} = we^{z_0 w}.$$

Here

$$f_n(z) = \frac{\left(z^n - z_0^n\right)}{z - z_0}\frac{w^n}{n!} = \frac{w^n}{n!}\sum_{j=1}^{j=n} z^{n-j} z_0^{j-1}.$$

But

$$|f_n(z)| \le \frac{|w|^n}{n!} n M^{n-1} = \frac{|w|^n M^{n-1}}{(n-1)!},$$

for any z with $|z - z_0| < R$, R being a fixed positive number and $M = |z_0| + R$. Since

$$\sum_{n=1}^{\infty} \frac{|w|^n M^{n-1}}{(n-1)!} = |w|\, e^{|w|M}$$

is convergent, we apply Weierstrass test for uniform convergence of series of functions and find that $\sum_{n=0}^{\infty} \frac{\left(z^n - z_0^n\right)}{z - z_0}\frac{w^n}{n!}$ is locally uniformly convergent on $\mathbb{C}$. Finally we get that

$$\frac{\partial}{\partial z}\left(e^{zw}\right)(z_0) = we^{z_0 w}$$

for any z_0 in $\mathbb{C}$.

Thus we just obtained the following formulas:

$$\frac{\partial}{\partial z}\left(e^{zw}\right) = we^{zw}, \frac{\partial}{\partial w}\left(e^{zw}\right) = ze^{zw} \tag{2.11}$$

This last elementary formula lead us to the following relation:

$$\frac{\partial}{\partial w}\left(\frac{\partial}{\partial z}\left(e^{zw}\right)\right) = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial w}\left(e^{zw}\right)\right). \tag{2.12}$$

(this is a very known result for those people who had a course in complex functions of many variables; it is the complex variant for Schwarz theorem from real analysis). By using elementary extensions of differentiability rules for real variables functions, one can easily find

$$\frac{\partial}{\partial w}\left(\frac{\partial}{\partial z}\left(z^k e^{zw}\right)\right) = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial w}\left(z^k e^{zw}\right)\right) \tag{2.13}$$

for any z, w in $\mathbb{C}$ and any $k \in \mathbb{N}$.

3. The Main Results

Let $a_0 \neq 0, a_1, ..., a_n$ be $n+1$ real numbers and let

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + ... + a_n \tag{3.1}$$

be the usual linear differential operator of order n w.r.t. x. We considered it to be restricted to the above vector space V.

Proposition 2. *For any $P(x) \in \mathbb{C}[x]$, $P(x) = b_0 + b_1 x + ... + b_h x^h$, $b_h \neq 0$ and $\lambda \in \mathbb{C}$ one has:*

$$L\left[P(x)e^{\lambda x}\right] = \left[\sum_{j=0}^{h} b_j \sum_{i=0}^{j} C_j^i \varphi^{(i)}(\lambda) x^{j-i}\right] e^{\lambda x}, \tag{3.2}$$

where

$$\varphi(\lambda) = a_0 \lambda^n + a_1 \lambda^{n-1} + ... + a_n,$$

$\varphi^{(i)}(\lambda)$ *is its i-th derivative and* $C_j^i = \frac{j!}{i!(j-i)!}$.

Proof. Let us take $x^r e^{\lambda x}$, where r is a natural number, x is a real variable and λ is a complex variable. Since formula (2.13) implies

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial \lambda}\left(x^r e^{\lambda x}\right)\right) = \frac{\partial}{\partial \lambda}\left(\frac{\partial}{\partial x}\left(x^r e^{\lambda x}\right)\right),$$

one can write:

$$L[x^r e^{\lambda x}] = L\left[\frac{\partial^r}{\partial \lambda^r}(e^{\lambda x})\right] = \frac{\partial^r}{\partial \lambda^r} L\left(e^{\lambda x}\right) = \frac{\partial^r}{\partial \lambda^r}\left(\varphi(\lambda) e^{\lambda x}\right). \tag{3.3}$$

We use now Leibniz formula and find:

$$L[x^r e^{\lambda x}] = \left[\sum_{i=0}^{r} C_r^i \varphi^{(i)}(\lambda) x^{r-i}\right] e^{\lambda x}. \tag{3.4}$$

The linearity of L and formula (3.4) will immediately lead us to formula (3.2). □

Corollary 1. *Let $\lambda = \lambda_1$ be a root of $\varphi(\lambda)$ of algebraic multiplicity $m \in \mathbb{N}$ $m \leq n$, and let $P(x) \in \mathbb{C}[x]$ be a polynomial of degree $h < m$. Then*

$$L\left[P(x) e^{\lambda_1 x}\right] = 0. \tag{3.5}$$

Proof. Since $\varphi(\lambda_1) = 0, \varphi'(\lambda_1) = 0, ..., \varphi^{(m-1)}(\lambda_1) = 0$, we see from formula (3.2) that $L\left[P(x) e^{\lambda_1 x}\right] = 0$. □

Corollary 2. *For any $\lambda \in \mathbb{C}$, V_λ is an invariant vector subspace of V.*

Proof. Since any element of V_λ is of the form $P(x) e^{\lambda x}$, from formula (3.2) we see that

$$L\left[P(x) e^{\lambda x}\right] \in V_\lambda.$$ □

This means that in order to study the operator $L : V \to V$, it is enough to study its restriction L_λ to V_λ for any $\lambda \in \mathbb{C}$.

Theorem 1. *Let $\alpha \in \mathbb{C}$ be a complex number such that $\varphi(\alpha) \neq 0$. Then $L_\alpha : V_\alpha \to V_\alpha$ is an isomorphism. If*

$$Q(x) = c_0 + c_1 x + ... + c_q x^q \in \mathbb{C}[x], c_q \neq 0,$$

is a polynomial of degree q, then there exist a unique polynomial

$$P(x) = b_0 + b_1 x + ... + b_h x^h \in \mathbb{C}[x], b_h \neq 0,$$

such that

$$L_\alpha\left[P(x)e^{\alpha x}\right]=Q(x)e^{\alpha x}.$$

Moreover $h=q$, *i.e.* $\deg P=\deg Q$ *and* $b_0,b_1,...,b_q$ *is the unique solution of the following triangular system:*

$$\begin{cases} \sum_{j=0}^{q} b_j C_j^j \varphi^{(j)}(\alpha)=c_0 \\ \sum_{j=1}^{q} b_j C_j^{j-1} \varphi^{(j-1)}(\alpha)=c_1 \\ \sum_{j=2}^{q} b_j C_j^{j-2} \varphi^{(j-2)}(\alpha)=c_2 \\ \vdots \\ \sum_{j=q}^{q} b_j C_j^{j-q} \varphi^{(j-q)}(\alpha)=c_q \end{cases}. \tag{3.6}$$

Proof. Since $L_\alpha : V_\alpha \to V_\alpha$ is a linear operator, it will be enough to prove that L_α is also an injective and a surjective mapping.

Let us search for a polynomial $P(x)=b_0+b_1x+...+b_hx^h\in\mathbb{C}[x]$, such that $L_\alpha\left[P(x)e^{\alpha x}\right]=Q(x)e^{\alpha x}$. From formula (3.2), by rearranging it after the power of x, we get:

$$\begin{cases} \sum_{j=0}^{h} b_j C_j^j \varphi^{(j)}(\alpha)=c_0 \\ \sum_{j=1}^{h} b_j C_j^{j-1} \varphi^{(j-1)}(\alpha)=c_1 \\ \vdots \\ \sum_{j=q}^{h} b_j C_j^{j-q} \varphi^{(j-q)}(\alpha)=c_q \\ \sum_{j=q+1}^{h} b_j C_j^{j-q-1} \varphi^{(j-q-1)}(\alpha)=0 \\ \vdots \\ \sum_{j=h}^{h} b_j C_j^{j-h} \varphi^{(j-h)}(\alpha)=0 \end{cases}. \tag{3.7}$$

This is a triangular system of $h+1$ equations and $h+1$ unknowns $b_0, b_1,...,b_h$. Since $\varphi(\alpha)\neq 0$, from the last $h-q$ equations we get that $b_h=0, b_{h-1}=0,...,b_{q+1}=0$. Substituting these values in the first $q+1$ equations we get exactly the linear system (3.6), which has a unique solution because on the diagonal we have $\varphi(\alpha)\neq 0$. Thus $h=q$ and the polynomial $P(x)=b_0+b_1x+...+b_qx^q$ is uniquely determined and so L_α is an isomorphism. □

In (3.6) first of all we get $b_q=\frac{c_q}{\varphi(\alpha)}$, then from the equation

$$\sum_{j=q-1}^{q} b_j C_j^{j-q+1}\varphi^{(j-q+1)}(\alpha)=c_{q-1}$$

or

$$b_{q-1}\varphi(\alpha)+b_q\cdot q\cdot\varphi'(\alpha)=c_{q-1},$$

we get b_{q-1}, etc. This is the well known "down to up" method for solving an upper triangular system.

Theorem 2. *Let $\lambda = \lambda_1$ be a root of multiplicity $m \leq n$ of the polynomial $\varphi(\lambda)$. Then $L_{\lambda_1} : V_{\lambda_1} \to V_{\lambda_1}$, the restriction of L to V_{λ_1}, is a surjective linear operator with $KerL = \left\{T(x)e^{\lambda_1 x} : T(x) \in \mathbb{C}[x], \deg T \leq m-1\right\}$ a vector subspace of V_{λ_1} of dimension m. If $Q(x) = c_0 + c_1 x + ... + c_q x^q, c_q \neq 0$, then there exists a unique polynomial $H(x)$ of degree q such that the inverse image of $Q(x)e^{\lambda_1 x}$ through L_{λ_1} is of the form*

$$\left[\mu_0 + \mu_1 x + ... + \mu_{m-1} x^{m-1} + x^m H(x)\right] e^{\lambda_1 x},$$

where $\mu_0, \mu_1, ..., \mu_{m-1}$ are free elements in $\mathbb{C}$. This means that the canonical isomorphism

$$\overline{L} : \frac{V_{\lambda_1}}{KerL} \overset{\simeq}{\to} V_{\lambda_1}$$

is such that $(\overline{L})^{-1}[Q(x)e^{\lambda_1 x}] = x^m H(x) e^{\lambda_1 x}$. The coefficients $b_m^*, b_{m+1}^*, ..., b_{m+q}^*$ of the unique polynomial $H(x) = b_m^* + b_{m+1}^* x + ... + b_{m+q}^* x^q$ can be uniquely obtained by solving the following triangular linear system in the unknowns $b_m, b_{m+1}, ..., b_{m+q}$:

$$\left\{\begin{array}{c} \sum_{j=m}^{m+q} b_j C_j^j \varphi^{(j)}(\lambda_1) = c_0 \\ \sum_{j=m+1}^{m+q} b_j C_j^{j-1} \varphi^{(j-1)}(\lambda_1) = c_1 \\ \sum_{j=m+2}^{m+q} b_j C_j^{j-2} \varphi^{(j-2)}(\lambda_1) = c_2 \\ \vdots \\ \sum_{j=m+q}^{m+q} b_j C_j^{j-q} \varphi^{(j-q)}(\lambda_1) = c_q \end{array}\right. . \tag{3.8}$$

Proof. Let us search for a polynomial $P(x) = b_0 + b_1 x + ... + b_h x^h$ with $h \geq m+q$ such that $L\left[P(x)e^{\lambda_1 x}\right] = Q(x)e^{\lambda_1 x}$.

We get again the triangular linear system (3.7). Since $\varphi(\lambda_1) = 0, \varphi'(\lambda_1) = 0, ..., \varphi^{(m-1)}(\lambda_1) = 0$ and $\varphi^{(m)}(\lambda_1) \neq 0$, this last system becomes:

$$\left\{\begin{array}{c} \sum_{j=m}^{h} b_j C_j^j \varphi^{(j)}(\lambda_1) = c_0 \\ \sum_{j=m+1}^{h} b_j C_j^{j-1} \varphi^{(j-1)}(\lambda_1) = c_1 \\ \vdots \\ \sum_{j=m+q}^{h} b_j C_j^{j-q} \varphi^{(j-q)}(\lambda_1) = c_q \\ \sum_{j=m+q+1}^{h} b_j C_j^{j-q-1} \varphi^{(j-q-1)}(\lambda_1) = 0 \\ \vdots \\ \sum_{j=h}^{h} b_j C_j^{j-h} \varphi^{(j-h)}(\lambda_1) = 0 \end{array}\right. . \tag{3.9}$$

From the last $h-m-q$ equations (if $h=m+q$ we say nothing) we get $b_h=0, b_{h-1}=0, ..., b_{m+q+1}=0$. Substituting these values in the first $m+q$ equations, we get exactly the linear system (3.8). Since $b_0, b_1, ..., b_{m-1}$ does not appear in (3.9), they remain free. But the system (3.8) has a unique solution $b_m^*, b_{m+1}^*, ..., b_{m+q}^*$, because $\varphi^{(m)}(\lambda_1) \neq 0$. Finally we get that

$$P(x) = b_0 + b_1 x + ... + b_{m-1}x^{m-1} + x^m \left(b_m^* + b_{m+1}^* x + ... + b_{m+q}^* x^q\right).$$

So $H(x) = b_m^* + b_{m+1}^* x + ... + b_{m+q}^* x^q$ is uniquely determined. Moreover, $Q(x) = 0$ if and only if $H(x) = 0$, i.e.

$$KerL_{\lambda_1} = \left\{ \left(b_0 + b_1 x + ... + b_{m-1}x^{m-1}\right) e^{\lambda_1 x} : b_0, b_1, ..., b_{m-1} \in \mathbb{C} \right\}. \qquad \square$$

The formulas 3.6 and 3.8 can also be find in the book [1].

4. Conclusions to Theorems 1 and 2

A. Let

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + ... + a_n, a_0 \neq 0, a_0, a_1, ..., a_n \in \mathbb{R}$$

be the linear differential operator of formula (3.1) and let

$$\varphi(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + ... + a_n = 0$$

be the characteristic algebraic equation associated to L.

Let $\lambda_1 \in \mathbb{C}$ be a root of $\varphi(\lambda) = 0$ of algebraic multiplicity $m \in \mathbb{N}$. Let $Q(x)$ be a polynomial of degree q, with complex coefficients. Then the equation

$$L[P(x)e^{\lambda_1 x}] = Q(x)e^{\lambda_1 x} \tag{4.1}$$

has:

A$_1$. A unique solution $P^*(x)e^{\lambda_1 x}$ in V if $m = 0$. Here $P^*(x) \in \mathbb{C}[x]$, $\deg P^*(x) = \deg Q(x)$ and

A$_2$. An infinite number of solutions of the type $[J(x) + x^m H^*(x)]e^{\lambda_1 x}$, where $J(x), H^*(x) \in \mathbb{C}[x]$, $H^*(x)$ is a fixed by $Q(x)$ polynomial of degree q and $J(x)$ is an arbitrary polynomial of degree $m-1$.

This means that the set of all solutions of (4.1) in this last case is an affine variety in V_{λ_1}, which passes through the "fixed point" $x^m H^*(x)$. this variety reduces to a point if $m = 0$.

Both unique polynomials $P^*(x)$ and $H^*(x)$ can be computed by solving a linear system of the type (3.6) or (3.8).

B. Let $\lambda_1 = \gamma + i\delta$, $\gamma, \delta \in \mathbb{R}, \delta \neq 0$ be the above root of multiplicity m of the equation $\varphi(\lambda) = 0$. Let $P(x) = P_1(x) + iP_2(x)$, where $P_1, P_2 \in \mathbb{R}[x]$. If $m \geq 1$ we change $P(x)$ with $H(x)$ and consider the equation:

$$L[x^m H(x) e^{\lambda_1 x}] = Q(x) e^{\lambda_1 x}, \tag{4.2}$$

where this time $Q(x) \in \mathbb{R}[x]$ is a polynomial with real coefficients. When $m = 0$, this $H(x)$ is the above $P(x)$. Let us denote

$$H(x) = H_1(x) + iH_2(x),$$

where $H_1(x), H_2(x) \in \mathbb{R}[x]$, the unique solution of (4.2). Since

$$\begin{aligned} H(x)e^{\lambda_1 x} &= e^{\gamma x} \left[H_1(x) \cos \delta x - H_2(x) \sin \delta x \right] + \\ &+ ie^{\gamma x} \left[H_1(x) \sin \delta x + H_2(x) \cos \delta x \right] \end{aligned}$$

and

$$Q(x)e^{\lambda_1 x} = Q(x)e^{\gamma x} \cos \delta x + iQ(x)e^{\gamma x} \sin \delta x,$$

the equation (4.2) is equivalent with the following two "real" equations:

$$L\left[x^m e^{\gamma x} \left\{H_1(x) \cos \delta x - H_2(x) \sin \delta x\right\}\right] = Q(x)e^{\gamma x} \cos \delta x. \tag{4.3}$$

and

$$L\left[x^m e^{\gamma x} \left\{H_1(x) \sin \delta x + H_2(x) \cos \delta x\right\}\right] = Q(x)e^{\gamma x} \sin \delta x. \tag{4.4}$$

Both these equations have unique solutions:

$$H_1^*(x) = \operatorname{Re} H^*(x), H_2^*(x) = \operatorname{Im} H^*(x),$$

where $H^*(x) = H_1^*(x) + iH_2^*(x)$ is the unique solution of (4.2).

In particular, if one search for a particular solution in V of the LDE of order n :

$$L[y] = Q(x)e^{\gamma x} \cos \delta x + R(x)e^{\gamma x} \sin \delta x, \tag{4.5}$$

where $Q(x), R(x) \in \mathbb{R}[x]$, one can construct the unique solution $S^*(x) = S_1^*(x) + iS_2^*(x)$, where $S_1^*(x), S_2^*(x) \in \mathbb{R}[x]$ of the equation

$$L[x^m S(x) e^{\lambda_1 x}] = R(x) e^{\lambda_1 x}, \tag{4.6}$$

and then we have to put together solutions of (4.3) and of

$$L\left[x^m e^{\gamma x}\left\{S_1(x)\sin\delta x + S_2(x)\cos\delta x\right\}\right] = R(x)e^{\gamma x}\sin\delta x. \tag{4.7}$$

We finally obtain a particular solution of the equation (4.5):

$$y_P^*(x) = x^m e^{\gamma x}\left[\left\{H_1^*(x) + S_2^*(x)\right\}\cos\delta x + \left\{S_1^*(x) - H_2^*(x)\right\}\sin\delta x\right]. \tag{4.8}$$

Since

$$\max\left\{\deg H_1^*(x), \deg H_2^*(x)\right\} = \deg H^* = \deg Q$$

and

$$\max\left\{\deg S_1^*(x), \deg S_2^*(x)\right\} = \deg S^* = \deg R,$$

usually one can search for a particular solution in V of the following type:

$$y_P(x) = x^m e^{\gamma x}\left[U_1(x)\cos\delta x + U_2(x)\sin\delta x\right],$$

where $U_1(x)$, $U_2(x)$ are real polynomials of degrees equal to

$$\max\{\deg Q(x), \deg R(x)\}.$$

Such a solution always exists and it is unique because of the formula (4.8).

5. An Example

Let us use the above theory in order to find a particular solution of the following LDE:

$$y^{(IV)} + 2y''' - 2y' - y = x^3 e^{-x}$$

The associated characteristic algebraic equation is:

$$\varphi(\lambda) = \lambda^4 + 2\lambda^3 - 2\lambda - 1.$$

Its roots are $\lambda_1 = -1$ with the algebraic multiplicity $m = 3$ and $\lambda_2 = 1$ with multiplicity 1.

In this case $m = q = 3$, so

$$y_P(x) = x^3 H(x) e^{-x},$$

where $H(x) = b_3 + b_4 x + b_5 x^2 + b_6 x^3$.

But

$$\varphi'(\lambda) = 4\lambda^3 + 6\lambda^2 - 2,$$

$$\begin{aligned} \varphi''(\lambda) &= 12\lambda^2 + 12\lambda, \\ \varphi'''(\lambda) &= 24\lambda + 12, \\ \varphi^{(IV)}(\lambda) &= 24 \end{aligned}$$

and

$$\begin{aligned} \varphi(-1) &= 0 \\ \varphi'(-1) &= 0 \\ \varphi''(-1) &= 0 \\ \varphi'''(-1) &= -12 \\ \varphi^{(IV)}(-1) &= 24. \end{aligned}$$

The coefficients b_3, b_4, b_5 and b_6 are the unique solution of the linear system (3.8), which in this particular case becomes:

$$\left\{ \begin{array}{lllll} b_3\varphi'''(-1)+ & b_4\varphi^{(IV)}(-1) & & & = 0 \\ & b_4C_4^3\varphi'''(-1)+ & b_5C_5^4\varphi^{(IV)}(-1) & & = 0 \\ & & b_5C_5^3\varphi'''(-1)+ & b_6C_6^4\varphi^{(IV)}(-1) & = 0 \\ & & & b_6C_6^3\varphi'''(-1) & = 1 \end{array} \right. .$$

The solution of this triangular system is

$$b_3 = -\frac{1}{16}, b_4 = -\frac{1}{32}, b_5 = -\frac{1}{80}, b_6 = -\frac{1}{240},$$

i.e.

$$H(x) = -\frac{1}{16} - \frac{1}{32}x - \frac{1}{80}x^2 - \frac{1}{240}x^3.$$

Thus the searched particular solution for is:

$$y_P(x) = x^3 \left[-\frac{1}{16} - \frac{1}{32}x - \frac{1}{80}x^2 - \frac{1}{240}x^3 \right] e^{-x}.$$

References

[1] E.A. Coddington, *An Introduction to Ordinary Differential Equations*, Dover Publications Inc., New York (1989).