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A FAMILY OF p-VALENT ANALYTIC FUNCTIONS DEFINED BY A FRACTIONAL CALCULUS OPERATOR

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Abstract: In this Paper a family $S(\alpha,\beta,\mu,p)$ of p-valent analytic functions involving fractional calculus operator $\Omega_z^{\mu,p}$ is studied and a sufficient coefficient condition for functions belonging to the family $S(\alpha,\beta,\mu,p)$ is proved and it is shown that this coefficient condition is necessary for its subfamily $TS(\alpha,\beta,\mu,p)$. Coefficient estimate, growth theorem and results on partial sums are obtained for the family $S(\alpha,\beta,\mu,p)$. Also an integral inequality is proved for functions belonging to the family $TS(\alpha,\beta,\mu,p)$.

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1. Prelimnaries

Let $A(p)$ denotes a family of functions of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \ (p \in N = \{1,2,3....\}) \tag{1}$$

which are analytic and p-valent in the open unit disk $U = \{z : z \in C \text{ and } |z| <$

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$1\}$ and $T(p)$ denotes a subfamily of $A(p)$ whose members are of the form:

$$f(z) = z^p - \sum_{k=1}^{\infty} |a_{p+k}| z^{p+k} \ (p \in N = \{1, 2, 3....\}). \tag{2}$$

Denote $A(1) \equiv S$.

The convolution or Hadamard product of $f(z)$ given by (1) and $g(z) \in A(p)$ given by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k} \tag{3}$$

is defined as:

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k} \tag{4}$$

which is analytic and p-valent in the unit disk U.

Let $S^*(p, \alpha)$ and $K(p, \alpha)$ denote respectively the family of starlike and convex functions $f(z) \in A(p)$ satisfying

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \text{ and } \text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha,\ 0 \leq \alpha < p \tag{5}$$

respectively. Again, let β-$UST(\alpha, p)$ and β-$UCV(\alpha, p)$ denote respectively the family of β-uniformly starlike and β-uniformly convex functions $f(z) \in A(p)$ satisfying

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta \left|\frac{zf'(z)}{f(z)} - p\right| + \alpha \tag{6}$$

and

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta \left|1 + \frac{zf''(z)}{f'(z)} - p\right| + \alpha \tag{7}$$

respectively for some $\beta \geq 0$, $0 \leq \alpha < p$ and $z \in U$. Clearly 0-$UST(\alpha, p) \equiv S^*(p, \alpha)$, 0-$UCV(\alpha, p) \equiv K(p, \alpha)$. Denote $S^*(1, \alpha) \equiv S^*(\alpha)$, $K(1, \alpha) \equiv K(\alpha)$ and β-$UST(\alpha, 1) \equiv \beta$-$UST(\alpha)$, β-$UCV(\alpha, 1) \equiv \beta$-$UCV(\alpha)$.

Saitoh [8] introduced a Carlson-Shaffer type operator $L_p(a, c)$ for $f(z) \in A(p)$ which is defined as:

$$L_p(a, c) f(z) := \phi_p(a, c, z) * f(z) \tag{8}$$

where $\phi_p(a, c, z)$ is defined as:

$$\phi_p(a,c,z) = z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{p+k} \tag{9}$$

for $a \in R_+$, $c \in R/Z_0^- = \{0,-1,-2,-3.........\}$ and $(a)_k$ is the Pochhammer symbol defined as:

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)(a+2)...(a+k-1),\ k \in N = 1,2,....$$

The Riemann-Liouville fractional derivative operator [9] of order $\mu\,(0 \leq \mu \leq 1)$ for analytic function $f(z)$ is defined as:

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\mu}} dt,\ \ 0 \leq \mu < 1 \tag{10}$$

$$D_z^1 f(z) = f'(z)$$

which is analytic in simply connected region of z-plane containing the origin, multiplicity of $(z-t)^{\mu}$ is removed by taking $\log(z-t)$ to be real when $(z-t) > 0$ and is well defined in the unit disk U.

The image of power function z^k under the operator defined in (10) is given as:

$$D_z^{\mu}(z^k) = \frac{\Gamma(k+1)}{\Gamma(k-\mu+1)} z^{k-\mu},\ \ 0 \leq \mu \leq 1. \tag{11}$$

Thus, the normalized operator $\Omega_z^{\mu,p} : A(p) \to A(p)$ is defined as:

$$\Omega_z^{\mu,p} f(z) = z^{\mu} \frac{\Gamma(p+1-\mu)}{\Gamma(p+1)} D_z^{\mu} f(z),\ \ 0 \leq \mu \leq 1,\ p \in N \tag{12}$$

and its series expansion using (12) for $f(z) \in A(p)$ of the form (1) is given as:

$$\Omega_z^{\mu,p} f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} \phi_p^{\mu}(k) z^{p+k} \tag{13}$$

where

$$\phi_p^{\mu}(k) = \frac{(p+1)_k}{(p+1-\mu)_k}. \tag{14}$$

Involving operator $\Omega_z^{\mu,p}$ given in (13), a generalized family $S(\alpha,\beta,\mu,p)$ of functions $f(z) \in A(p)$ for some $\beta \geq 0$, $0 \leq \alpha < p$, $0 \leq \mu \leq 1$ is defined by

$$\mathrm{Re}\left\{ \frac{z\left(\Omega_z^{\mu,p} f(z)\right)'}{\Omega_z^{\mu,p} f(z)} \right\} > \beta \left| \frac{z\left(\Omega_z^{\mu,p} f(z)\right)'}{\Omega_z^{\mu,p} f(z)} - p \right| + \alpha. \tag{15}$$

It is examined that if $f(z) \in S(\alpha,\beta,\mu,p)$ with $0 \le \beta \le \frac{\alpha}{p}$, $0 \le \alpha < p$,then

$$\Omega_z^{\mu,p} f(z) \in S^*(p, \frac{\alpha-\beta p}{1-\beta}). \tag{16}$$

Note that $S(\alpha,\beta,0,p) \equiv \beta\text{-}UST(\alpha,p)$ and $S(\alpha,\beta,1,p) \equiv \beta\text{-}UCV(\alpha,p)$. Also, $TS(\alpha,\beta,\mu,p) \equiv S(\alpha,\beta,\mu,p) \cap T(p)$.

Further, it is observed that $S(\alpha,0,\mu,p) \equiv S_\mu(\alpha,p,1,-1)$,a family studied in chapter 2.

2. Coefficient Conditions

In this section, a sufficient coefficient condition for functions belonging to $S(\alpha,\beta,\mu,p)$ family is given and then it is proved that this condition is necessary for its subfamily $TS(\alpha,\beta,\mu,p)$.

Theorem 1. *Let $f(z) \in A(p)$ of the form (1), satisfies*

$$\sum_{k=1}^{\infty} \frac{\{k(1+\beta)+(p-\alpha)\}}{(p-\alpha)} \phi_p^\mu(k)|a_{p+k}| \le 1, \tag{17}$$

for $\beta \ge 0$, $0 \le \alpha < p$, $0 \le \mu \le 1$, $\phi_p^\mu(k) = \frac{(p+1)_k}{(p+1-\mu)_k}$, then $f(z) \in S(\alpha,\beta,\mu,p)$.

Proof. Let the inequality (17) holds true, then it is to prove that

$$\mathrm{Re}\left\{\frac{z\left(\Omega_z^{\mu,p} f(z)\right)'}{\Omega_z^{\mu,p} f(z)} - p\right\} > \beta\left|\frac{z\left(\Omega_z^{\mu,p} f(z)\right)'}{\Omega_z^{\mu,p} f(z)} - p\right| + \alpha - p$$

or,

$$\beta\left|\frac{z\left(\Omega_z^{\mu,p} f(z)\right)'}{\Omega_z^{\mu,p} f(z)} - p\right| - \mathrm{Re}\left\{\frac{z\left(\Omega_z^{\mu,p} f(z)\right)'}{\Omega_z^{\mu,p} f(z)} - p\right\} < p - \alpha.$$

Since, $\mathrm{Re}(w)\,\mathrm{or}\,\mathrm{Re}(-w) \le |w|$ for some complex number w, it follows that

$$\beta\left|\frac{z\left(\Omega_z^{\mu,p} f(z)\right)'}{\Omega_z^{\mu,p} f(z)} - p\right| - \mathrm{Re}\left\{\frac{z\left(\Omega_z^{\mu,p} f(z)\right)'}{\Omega_z^{\mu,p} f(z)} - p\right\}$$

$$\le \quad (1+\beta)\left|\frac{z\left(\Omega_z^{\mu,p} f(z)\right)'}{\Omega_z^{\mu,p} f(z)} - p\right|$$

$$\leq \quad (1+\beta)\frac{\sum_{k=1}^{\infty} k\phi_p^{\mu}(k)|a_{p+k}|}{1-\sum_{k=1}^{\infty}\phi_p^{\mu}(k)|a_{p+k}|}$$

whose right hand side expression is bounded above by $(p-\alpha)$ if, inequality (17) holds, which proves Theorem 1 □

Theorem 2. *Let $f(z) \in T(p)$ be of the form (2), then $f(z) \in TS(\alpha,\beta,\mu,p)$ if and only if*

$$\sum_{k=1}^{\infty} \frac{\{k(1+\beta)+(p-\alpha)\}}{(p-\alpha)}\phi_p^{\mu}(k)|a_{p+k}| \leq 1, \tag{18}$$

for $\beta \geq 0$, $0 \leq \alpha < p$, $0 \leq \mu \leq 1$ and $\phi_p^{\mu}(k) = \frac{(p+1)_k}{(p+1-\mu)_k}$.

Proof. In view of Theorem 1, it needs only to prove necessary part. Let $f(z) \in TS(\alpha,\beta,\mu,p)$, then $f(z)$ of the form (2) satisfies

$$\mathrm{Re}\left\{\frac{z\left(\Omega_z^{\mu,p}f(z)\right)'}{\Omega_z^{\mu,p}f(z)}\right\} > \beta\left|\frac{z\left(\Omega_z^{\mu,p}f(z)\right)'}{\Omega_z^{\mu,p}f(z)} - p\right| + \alpha$$

or,

$$\beta\left|\frac{p-\sum_{k=1}^{\infty}(p+k)\phi_p^{\mu}(k)|a_{p+k}|z^k}{1-\sum_{k=1}^{\infty}\phi_p^{\mu}(k)|a_{p+k}|z^k} - p\right|$$
$$<\mathrm{Re}\left\{\frac{p-\sum_{k=1}^{\infty}(p+k)\phi_p^{\mu}(k)|a_{p+k}|z^k}{1-\sum_{k=1}^{\infty}\phi_p^{\mu}(k)|a_{p+k}|z^k} - \alpha\right\}.$$

Since, $\mathrm{Re}(w)$ or $\mathrm{Re}(-w) \leq |w|$,

$$\mathrm{Re}\beta\left\{p - \frac{p-\sum_{k=1}^{\infty}(p+k)\phi_p^{\mu}(k)|a_{p+k}|z^k}{1-\sum_{k=1}^{\infty}\phi_p^{\mu}(k)|a_{p+k}|z^k}\right\}$$
$$< \mathrm{Re}\left\{\frac{p-\sum_{k=1}^{\infty}(p+k)\phi_p^{\mu}(k)|a_{p+k}|z^k}{1-\sum_{k=1}^{\infty}\phi_p^{\mu}(k)|a_{p+k}|z^k} - \alpha\right\}.$$

Letting $z \to 1^-$ along the real axis, it gives the desired inequality

$$\sum_{k=1}^{\infty} \frac{\{k(1+\beta)+(p-\alpha)\}}{(p-\alpha)}\phi_p^{\mu}(k)|a_{p+k}| \leq 1.$$

This proves Theorem 3. □

3. Coefficient Estimate for the Family $S(\alpha,\beta,\mu,p)$

Theorem 3. *Let $f(z)\in A(p)$ of the form (1) be in the family $S(\alpha,\beta,\mu,p)$, $0\le\beta\le\frac{\alpha}{p}$, $0\le\alpha<p$, then*

$$|a_{p+k}|\le\frac{1}{k!\phi_p^{\mu}(k)}\prod_{j=1}^{k}\left(j-1+\frac{2(p-\alpha)}{(1-\beta)}\right),\ k\ge1 \tag{19}$$

where $\phi_p^{\mu}(k)=\frac{(p+1)_k}{(p+1-\mu)_k}$.

Or, equivalently,

$$|a_{p+k}|\le\frac{\left(\frac{2(p-\alpha)}{1-\beta}\right)_k(p+1-\mu)_k}{(1)_k(p+1)_k},\ k\ge1.$$

Proof. Since $f(z)\in S(\alpha,\beta,\mu,p)$ for $0\le\beta\le\frac{\alpha}{p}$, $0\le\alpha<p$, it gives

$$\operatorname{Re}\left\{\frac{z\left(\Omega_z^{\mu,p}f(z)\right)'}{\Omega_z^{\mu,p}f(z)}\right\}>\frac{(\alpha-\beta p)}{(1-\beta)}.$$

Let $q(z)=1+q_1z+q_2z^2+...$, be defined as

$$q(z)=\frac{(1-\beta)\left\{\frac{z\left(\Omega_z^{\mu,p}f(z)\right)'}{\Omega_z^{\mu,p}f(z)}\right\}-(\alpha-\beta p)}{(p-\alpha)} \tag{20}$$

which is analytic in U with $q(0)=1$ and $\operatorname{Re}\{q(z)\}>0$ for $z\in U$, then

$$z\left(\Omega_z^{\mu,p}f(z)\right)'-\frac{(\alpha-\beta p)}{(1-\beta)}\Omega_z^{\mu,p}f(z)=\frac{(p-\alpha)}{(1-\beta)}q(z)\Omega_z^{\mu,p}f(z).$$

On writing their respective series expansions, it gives

$$z\left(pz^{p-1}+\sum_{k=1}^{\infty}(p+k)\phi_p^{\mu}(k)a_{p+k}z^{p+k-1}\right)-\frac{(\alpha-\beta p)}{(1-\beta)}\left(z^p+\sum_{k=1}^{\infty}\phi_p^{\mu}(k)a_{p+k}z^{p+k}\right)$$
$$=\frac{(p-\alpha)}{(1-\beta)}(1+q_1z+q_2z^2+...)\left(z^p+\sum_{k=1}^{\infty}\phi_p^{\mu}(k)a_{p+k}z^{p+k}\right).$$

On comparing the coefficients of z^{p+k} on both sides in the above equation,

$$k\phi_p^{\mu}(k)a_{p+k}\ =\ \frac{(p-\alpha)}{(1-\beta)}\{q_k+\phi_p^{\mu}(1)a_{p+1}q_{k-1}+\phi_p^{\mu}(2)a_{p+2}q_{k-2}+$$

$$... + \phi_p^{\mu}(k-1)a_{p+k-1}q_1\right\}.$$

Hence, on using the coefficient estimate $|q_k| \leq 2$, $k \geq 1$ for Caratheodory functions $q(z)$ [2], for $k \geq 1$,

$$|a_{p+k}| \leq \frac{2(p-\alpha)}{k\phi_p^{\mu}(k)(1-\beta)}\left\{1 + \phi_p^{\mu}(1)|a_{p+1}| + \phi_p^{\mu}(2)|a_{p+2}| + \right. \tag{21}$$
$$\left. ... + \phi_p^{\mu}(k-1)|a_{p+k-1}|\right\}.$$

At $k = 1$, (21) gives

$$|a_{p+1}| \leq \frac{2(p-\alpha)}{1!\phi_p^{\mu}(1)(1-\beta)} \tag{22}$$

which proves (19) for $k = 1$, also at $k = 2$, (21) gives

$$|a_{p+2}| \leq \frac{2(p-\alpha)}{2!\phi_p^{\mu}(2)(1-\beta)}\left\{1 + \frac{2(p-\alpha)}{(1-\beta)}\right\}. \tag{23}$$

Thus, (22) and (23) prove that (19) is true for $k = 1, 2$. Let (19) is true for $k = n$

i.e.

$$|a_{p+n}| \leq \frac{1}{n!\phi_p^{\mu}(n)}\prod_{j=1}^{n}\left(j - 1 + \frac{2(p-\alpha)}{(1-\beta)}\right), \quad n \geq 1. \tag{24}$$

Now writing inequality (21) for $k = n + 1$,

$$|a_{p+n+1}| \leq \frac{2(p-\alpha)}{(n+1)\phi_p^{\mu}(n+1)(1-\beta)}\left\{1 + \phi_p^{\mu}(1)|a_{p+1}| + \phi_p^{\mu}(2)|a_{p+2}| + \right.$$
$$\left. ... + \phi_p^{\mu}(n)|a_{p+n}|\right\}.$$

On applying (22), (23) and (24),

$$|a_{p+n+1}| \leq \frac{2(p-\alpha)}{(n+1)\phi_p^{\mu}(n+1)(1-\beta)}\left\{1 + \frac{2(p-\alpha)}{1!(1-\beta)} + \right.$$
$$\left. + \frac{2(p-\alpha)}{2!(1-\beta)}\left(1 + \frac{2(p-\alpha)}{(1-\beta)}\right) + ... + \frac{1}{n!}\prod_{j=1}^{n}\left(j - 1 + \frac{2(p-\alpha)}{(1-\beta)}\right)\right\}$$

or,

$$|a_{p+n+1}| \leq \frac{2(p-\alpha)}{(n+1)\phi_p^{\mu}(n+1)(1-\beta)}\left\{\frac{1}{(n-1)!}\left(1 + \frac{2(p-\alpha)}{(1-\beta)}\right)...\right.$$

$$\left(n-1+\frac{2(p-\alpha)}{(1-\beta)}\right)+\frac{1}{n!}\prod_{j=1}^{n}\left(j-1+\frac{2(p-\alpha)}{(1-\beta)}\right)\Bigg\}$$

or,

$$\begin{aligned}|a_{p+n+1}| \leq \; & \frac{1}{(n+1)!\phi_p^{\mu}(n+1)}\left\{\frac{2(p-\alpha)}{(1-\beta)}\left(1+\frac{2(p-\alpha)}{(1-\beta)}\right)\cdots\right. \\ & \left.\left(n-1+\frac{2(p-\alpha)}{(1-\beta)}\right)\left(n+\frac{2(p-\alpha)}{(1-\beta)}\right)\right\}\end{aligned}$$

or,

$$|a_{p+n+1}| \leq \frac{1}{\phi_p^{\mu}(n+1)(n+1)!}\prod_{j=1}^{n+1}\left(j-1+\frac{2(p-\alpha)}{(1-\beta)}\right), \; n\geq 1,$$

which shows that, the result is true for $k=n+1$. Thus by mathematical induction, (19) holds true for any $n\geq 1$. This proves the result.

Taking $\mu=0$, $p=1$ and $\mu=1$, $p=1$ respectively in Theorem 3, following results of Owa et al. [5] are obtained. □

Corollary 4. *[5] Let $f(z)\in S$ of the form (1) be in the family β-$UST(\alpha)$, $\beta\geq 0$, $0\leq\alpha<1$, then*

$$|a_{1+k}| \leq \frac{1}{k!}\prod_{j=1}^{k}\left(j-1+\frac{2(1-\alpha)}{(1-\beta)}\right), \; k\geq 1.$$

Corollary 5. *[5] Let $f(z)\in S$ of the form (1) be in the family β-$UCV(\alpha)$, $\beta\geq 0$, $0\leq\alpha<1$, then*

$$|a_{1+k}| \leq \frac{1}{(k+1)!}\prod_{j=1}^{k}\left(j-1+\frac{2(1-\alpha)}{(1-\beta)}\right), \; k\geq 1.$$

Again, taking $\beta=0$ in Corollary 5, following results of Robertson [6] are obtained.

Corollary 6. *[6] Let $f(z)\in S$ of the form (1) be in the family $S^*(\alpha)$, $0\leq\alpha<1$, then*

$$|a_{1+k}| \leq \frac{1}{k!}\prod_{j=1}^{k}(j+1-2\alpha), \; k\geq 1.$$

Corollary 7. *[6] Let $f(z) \in S$ of the form (1) be in the family $K(\alpha)$, $0 \leq \alpha < 1$, then*

$$|a_{1+k}| \leq \frac{1}{(k+1)!} \prod_{j=1}^{k} (j+1-2\alpha), \; k \geq 1.$$

Further, taking $\mu = 0, \beta = 0$ and $\mu = 1$, $\beta = 0$ respectively in Theorem 3, following results of Goyal and Bhagtani [3] are obtained.

Corollary 8. *[3] Let $f(z) \in A(p)$ of the form (1) be in the family $S^*(p, \alpha)$, $0 \leq \alpha < p$, then*

$$|a_{p+k}| \leq \frac{1}{k!} \prod_{j=1}^{k} (j-1+2p-2\alpha), \; k \geq 1.$$

Corollary 9. *[3] Let $f(z) \in A(p)$ of the form (1) be in the family $K(p, \alpha)$, $0 \leq \alpha < p$, then*

$$|a_{p+k}| \leq \frac{1}{(p+k)k!} \prod_{j=1}^{k} (j-1+2p-2\alpha), \; k \geq 1.$$

4. Growth Theorem

In this section, growth result of $f(z) \in S(\alpha, \beta, \mu, p)$ using coefficient estimate obtained in Theorem 3 and Carlson Shaffer type operator defined in (8) is proved.

Theorem 10. *Let $f(z) \in A(p)$ of the form (1) be in the family $S(\alpha, \beta, \mu, p)$, then for $|z| = r < 1$,*

$$|f(z)| \leq L_p(p+1-\mu, \, p+1) \left(\frac{r^p}{(1-r)^{\frac{2(p-\alpha)}{(1-\beta)}}} \right).$$

Proof. Since $f \in A(p)$ of the from (1) be in the family $S(\alpha, \beta, \mu, p)$, Theorem 3 gives

$$|a_{p+k}| \leq \frac{\left(\frac{2(p-\alpha)}{(1-\beta)}\right)_k (p+1-\mu)_k}{(1)_k (p+1)_k}. \tag{25}$$

Thus,

$$\begin{aligned}
|f(z)| &\leq |z|^p + \sum_{k=1}^{\infty} |a_{p+k}||z|^{p+k} \\
&\leq |z|^p + \sum_{k=1}^{\infty} \frac{\left(\frac{2(p-\alpha)}{(1-\beta)}\right)_k (p+1-\mu)_k}{(1)_k (p+1)_k} |z|^{p+k} \\
&= \sum_{k=0}^{\infty} \frac{\left(\frac{2(p-\alpha)}{(1-\beta)}\right)_k (p+1-\mu)_k}{(1)_k (p+1)_k} |z|^{p+k} \\
&= \left\{ \sum_{k=0}^{\infty} \frac{(p+1-\mu)_k}{(p+1)_k} |z|^{p+k} \right\} * \left\{ \sum_{k=0}^{\infty} \frac{\left(\frac{2(p-\alpha)}{(1-\beta)}\right)_k}{(1)_k} |z|^{p+k} \right\} \\
&= \phi_p(p+1-\mu, p+1, |z|) * \phi_p\left(\frac{2(p-\alpha)}{(1-\beta)}, 1, |z|\right) \\
&= \phi_p(p+1-\mu, p+1, |z|) * \frac{|z|^p}{(1-|z|)^{\frac{2(p-\alpha)}{(1-\beta)}}} \\
&= L_p(p+1-\mu, p+1)\left(\frac{r^p}{(1-r)^{\frac{2(p-\alpha)}{(1-\beta)}}}\right).
\end{aligned}$$

This proves Theorem 10. □

For $\mu = 0$, $\beta = 0$, $p = 1$ and $\mu = 1$, $\beta = 0$, $p = 1$ respectively in Theorem 10, following results of Banerji and Shenan [1] are obtained.

Corollary 11. *[1] Let $f(z) \in S$ be in the family $S^*(\alpha)$, then*

$$|f(z)| \leq \frac{r}{(1-r)^{2(1-\alpha)}}, \; |z| = r < 1.$$

Corollary 12. *[1] Let $f(z) \in S$ be in the family $K(\alpha)$, then*

$$|f(z)| \leq \phi(2(1-\alpha), 2; r), \; |z| = r < 1.$$

5. Partial Sums

In this section, inequalities involving partial sums of $f(z) \in A(p)$ are obtained. Let non-zero partial sums of $f(z) \in A(p)$ of the form (1) be defined as follows:

$$f_0(z) = z^p \text{ and } f_n(z) = z^p + \sum_{k=1}^{n} a_{p+k} z^{p+k}, \; k \geq 1. \tag{26}$$

Theorem 13. *Let $f(z) \in A(p)$ of the form (1) satisfies*

$$\sum_{k=1}^{\infty} c_{p+k} |a_{p+k}| \phi_p^{\mu}(k) \leq 1 \tag{27}$$

where $c_{p+k} := \frac{\{k(1+\beta)+(p-\alpha)\}}{(p-\alpha)} \frac{(p+1)_k}{(p+1-\mu)_k}$, $\beta \geq 0$, $0 \leq \alpha < p$, $0 \leq \mu \leq 1$, *then* $f \in S(\alpha, \beta, \mu, p)$ *and*

$$Re\left\{\frac{f(z)}{f_n(z)}\right\} > 1 - \frac{1}{c_{p+n+1}}, \tag{28}$$

$$Re\left\{\frac{f_n(z)}{f(z)}\right\} > \frac{c_{p+n+1}}{1 + c_{p+n+1}}, \tag{29}$$

$$Re\left\{\frac{f'(z)}{f_n'(z)}\right\} > 1 - \frac{p+n+1}{c_{p+n+1}}. \tag{30}$$

Proof. Since $f(z) \in A(p)$ of the form (1) satisfy (27), from Theorem 1, $f(z) \in S(\alpha, \beta, \mu, p)$. Further, from (27), as it is easily seen that

$$c_{p+n+1} > c_{p+n} > 1, \tag{31}$$

$$\sum_{k=1}^{n} |a_{p+k}| + c_{p+n+1} \sum_{k=n+1}^{\infty} |a_{p+k}| \leq \sum_{k=1}^{\infty} c_{p+k} |a_{p+k}| \leq 1. \tag{32}$$

Set

$$g_1(z) = c_{p+n+1} \left\{ \frac{f(z)}{f_n(z)} - \left(1 - \frac{1}{c_{p+n+1}}\right) \right\} \tag{33}$$

which is analytic in U and $g_1(0) = 1$.

$$\begin{aligned} \left| \frac{g_1(z) - 1}{g_1(z) + 1} \right| &= \left| \frac{c_{p+n+1} \sum_{k=n+1}^{\infty} a_{p+k} z^k}{2 + 2\sum_{k=1}^{n} a_{p+k} z^k + c_{p+n+1} \sum_{k=n+1}^{\infty} a_{p+k} z^k} \right| \\ &\leq \frac{c_{p+n+1} \sum_{k=n+1}^{\infty} |a_{p+k}|}{2 - 2\sum_{k=1}^{n} |a_{p+k}| - c_{p+n+1} \sum_{k=n+1}^{\infty} |a_{p+k}|} \\ &\leq 1 \end{aligned}$$

if (32) holds, which readily yields that $\mathrm{Re}\{g_1(z)\} > 0$, this proves assertion (28) of Theorem 10.

Similarly, set

$$g_2(z) = (1+c_{p+n+1})\left\{\frac{f_n(z)}{f(z)} - \frac{c_{p+n+1}}{1+c_{p+n+1}}\right\} \tag{34}$$
$$= \left\{1 - \frac{(1+c_{p+n+1})\sum_{k=n+1}^{\infty} a_{p+k}z^k}{1+\sum_{k=1}^{\infty} a_{p+k}z^k}\right\}$$

and making use of (32),

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| = \left|\frac{(1+c_{p+n+1})\sum_{k=n+1}^{\infty} a_{p+k}z^k}{2+2\sum_{k=1}^{\infty} a_{p+k}z^k - (1+c_{p+n+1})\sum_{k=n+1}^{\infty} a_{p+k}z^k}\right|$$
$$\leq \frac{(1+c_{p+n+1})\sum_{k=n+1}^{\infty} |a_{p+k}|}{2-2\sum_{k=1|}^{n} |a_{p+k}| - (c_{p+n+1}-1)\sum_{k=n+1}^{\infty} |a_{p+k}|}$$
$$\leq 1$$

which proves the assertion (29).

Further, set

$$g_3(z) = \frac{c_{p+n+1}}{p+n+1}\left\{\frac{f'(z)}{f'_n(z)} - \left(1 - \frac{p+n+1}{c_{p+n+1}}\right)\right\} \tag{35}$$
$$= 1 + \frac{\frac{c_{p+n+1}}{p+n+1}\sum_{k=n+1}^{\infty} \frac{(p+k)}{p} a_{p+k}z^k}{1+\sum_{k=1}^{n} \frac{(p+k)}{k} a_{p+k}z^k}.$$

$$\left|\frac{g_3(z)-1}{g_3(z)+1}\right| = \left|\frac{\frac{c_{p+n+1}}{p+n+1}\sum_{k=n+1}^{\infty} \frac{(p+k)}{p} a_{p+k}z^k}{2+2\sum_{k=1}^{n} \frac{(p+k)}{p} a_{p+k}z^k + \frac{c_{p+n+1}}{p+n+1}\sum_{k=n+1}^{\infty} \frac{(p+k)}{p} a_{p+k}z^k}\right| \tag{36}$$
$$\leq \frac{\frac{c_{p+n+1}}{p+n+1}\sum_{k=n+1}^{\infty} \frac{(p+k)}{p} |a_{p+k}|}{2-2\sum_{k=1}^{n} \frac{(p+k)}{p} |a_{p+k}| - \frac{c_{p+n+1}}{p+n+1}\sum_{k=n+1}^{\infty} \frac{(p+k)}{p} |a_{p+k}|} \leq 1$$

if

$$\sum_{k=1}^{n} \frac{(p+k)}{p}|a_{p+k}| + \frac{c_{p+n+1}}{p+n+1}\sum_{k=n+1}^{\infty} \frac{(p+k)}{p}|a_{p+k}| \leq 1 \tag{37}$$

which holds if the left hand side of (37) is bounded above by $\sum_{k=1}^{\infty} c_{p+k}|a_{p+k}|$ i.e. if

$$\sum_{k=1}^{n} \left(c_{p+k} - \frac{(p+k)}{p}\right) |a_{p+k}| \tag{38}$$

$$+\sum_{k=n+1}^{\infty}\left(c_{p+k}-\frac{c_{p+n+1}}{p+n+1}\frac{(p+k)}{p}\right)|a_{p+k}|\geq 0.$$

As $\frac{c_{p+k}}{p+k}$ is an increasing function of k, (38) is true. Thus $\mathrm{Re}\{g_3(z)\}>0$ which readily yields the assertion (30). □

Remark 14. Note that taking $\beta=1$, $p=1$ and $\mu=0$, $\mu=1$ respectively, above results of Theorem 13 coincide with results obtained by Rosy [7].

6. Integral Mean Inequality

The following subordination result due to Littlewood [4] is used in next Theorem.

Lemma 15. *[4] Let $f(z)$ and $g(z)$ be analytic in U with $f(z)\prec g(z)$, then*

$$\int_0^{2\pi}\left|f(re^{i\theta})\right|^q d\theta\leq\int_0^{2\pi}\left|g(re^{i\theta})\right|^q d\theta \tag{39}$$

where $q>0$, $z=re^{i\theta}$ and $0<r<1$.

Theorem 16. *Let $f(z)\in T(p)$ of the form (2) be in the family $TS(\alpha,\beta,\mu,p)$, then for $z=re^{i\theta}$, $0<r<1$ and $q>0$*

$$\int_0^{2\pi}\left|f(re^{i\theta})\right|^q d\theta\leq\int_0^{2\pi}\left|f_1(re^{i\theta})\right|^q d\theta \tag{40}$$

where

$$f_1(z)=z^p-\frac{(p-\alpha)(p+1-\mu)}{\{(1+\beta)+(p-\alpha)\}(p+1)}z^{p+1}. \tag{41}$$

Proof. For

$$f(z)=z^p-\sum_{k=1}^{\infty}|a_{p+k}|z^{p+k}$$

and $f_1(z)$ given by (41), to prove (40), it is equivalent to show

$$\int_0^{2\pi}\left|1-\sum_{k=1}^{\infty}|a_{p+k}|z^k\right|^q d\theta\leq\int_0^{2\pi}\left|1-\frac{(p-\alpha)(p+1-\mu)}{\{(1+\beta)+(p-\alpha)\}(p+1)}z\right|^q d\theta.$$

Hence, by Lemma 15, it suffices to show

$$1-\sum_{k=1}^{\infty}|a_{p+k}|z^k \prec 1-\frac{(p-\alpha)(p+1-\mu)}{\{(1+\beta)+(p-\alpha)\}\,(p+1)}z$$

which is true if there exists a Schwartz function $w\,(z)$ such that

$$1-\sum_{k=1}^{\infty}|a_{p+k}|z^k = 1-\frac{(p-\alpha)(p+1-\mu)}{\{(1+\beta)+(p-\alpha)\}\,(p+1)}w(z). \tag{42}$$

Using (18) of Theorem 2,

$$\begin{aligned} |w(z)| &= \left|\frac{\{(1+\beta)+(p-\alpha)\}\,\phi_p^{\mu}(1)}{(p-\alpha)}\sum_{k=1}^{\infty}|a_{p+k}|z^k\right| \\ &\leq |z| \\ &< 1. \end{aligned}$$

This completes the proof of Theorem 16. □

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