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ON THE OSTROWSKI-GRÜSS-LIKE TYPE INEQUALITY AND AN ANALOGUE OF OSTROWSKI-LIKE TYPE INEQUALITY

Jaekeun Park
Department of Mathematics
Hanseo University
Seosan, Chungnam, 356-706, KOREA

Abstract: In this article, we establish some new generalizations of the Ostrowski-Grüss-like type inequality for twice continuously differentiable functions and new generalized analogues of the Ostrowski-like type inequality in three different cases.

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Key Words: Ostrowski's inequality, Grüss' inequality

1. Introduction

In [10], Ostrowski proved the following theorem:

Theorem 1.1. *Let $f : I \to R$ be a differentiable mapping in the interior I^0 of an interval I and $a, b \in I^0$ with $a < b$. If $\left|f'(t)\right| \le M$ for any $t \in [a, b]$, then we have*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \le \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2}\right](b-a)M \tag{1}$$

for $x \in [a, b]$.

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Recently there has been considerable interest in the study of Ostrowski type inequalities. Some new types of inequalities are established, for example inequalities of Ostrowski-Grüss type and inequalities of Ostrowski-Chebyshev type. In [8], Milovanović and Pečarić gave a generalization of Ostrowski's inequality and some related applications.

An Ostrowski-Grüss type inequality was given for the first time by Dragomir and Wang in [3]. In [5], Matić *et al.* generalized and improved this inequality. For generalizations, improvements and recent results, see the papers [3, 4, 11, 12, 13, 14]. Recently, in [13], Ujević proved the following theorem:

Theorem 1.2. *Let $f : I \to R$ be a differentiable function in the interior I^0 of an interval I in R and $a, b \in I^0$ with $a < b$. If there exist constants $\gamma, \Gamma \in R$ such that $\gamma \le f'(t) \le \Gamma$ for any $t \in [a,b]$ and $f' \in L^1[a,b]$, then we have*

$$\Big|f(x) - \big(x - \frac{a+b}{2}\big)\frac{f(b)-f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(t)dt\Big| \le \frac{(b-a)}{2}(S-\gamma) \tag{2}$$

and

$$\Big|f(x) - \big(x - \frac{a+b}{2}\big)\frac{f(b)-f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(t)dt\Big| \le \frac{(b-a)}{2}(\Gamma - S), \tag{3}$$

where $S = \frac{f(b)-f(a)}{b-a}$.

Theorem 1.3. *Let $f : I \to R$ be a twice continuously differentiable function in the interior I^0 of an interval I in R with $f'' \in L^2[a,b]$ and $a, b \in I^0$ with $a < b$. Then we have*

$$\Big|f(x) - \big(x - \frac{a+b}{2}\big)\frac{f(b)-f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(t)dt\Big| \le \frac{(b-a)^{\frac{3}{2}}}{2\sqrt{3}\pi} \parallel f'' \parallel_2, \tag{4}$$

for $x \in [a,b]$.

For two absolutely continuous functions $f, g : [a,b] \to R$ such that $f, g, fg \in L^1[a,b]$, the Chebyshev functional $T(\cdot,\cdot)$ is defined by

$$T_a^b(f,g) = \frac{1}{(b-a)}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx \int_a^b g(x)dx. \tag{5}$$

A well-known inequality in the literature, which is related to the Chebyshev functional, is the Ostrowski inequality [8].

The Ostrowski type inequality also plays an important role in numerical quadrature rules [7].

For various generalizations, extensions and related Ostrowski type inequalities for functions of one or several variables, see the monograph [2] and the references therein. For the related result, see the papers [1, 2, 5, 8, 7].

In [6], Masjed-Jamei and Dragomir introduced a new analogue of the Ostrowski type inequality in three different cases and apply them for some quadrature rules and proved the following theorem:

Theorem 1.4. *Let $f : I \to R$ be a differentiable function in the interior I^0 of an interval I in R and $a, b \in I^0$ with $a < b$. If $\alpha(x) \leq f'(x) \leq \beta(x)$ for any $\alpha, \beta \in C[a, b]$ and $x \in [a, b]$, then the following inequality holds:*

$$\begin{aligned} &\frac{1}{b-a}\Big\{\int_a^x (t-a)\alpha(t)dt + \int_x^b (t-b)\beta(t)dt\Big\} \\ &\leq f(x) - \frac{1}{b-a}\int_a^b f(t)dt \\ &\leq \frac{1}{b-a}\Big\{\int_a^x (t-a)\beta(t)dt + \int_x^b (t-b)\alpha(t)dt\Big\}. \end{aligned} \tag{6}$$

Theorem 1.5. *Let $f : I \to R$ be a differentiable function in the interior I^0 of an interval I in R and $a, b \in I^0$ with $a < b$. If $\alpha(x) \leq f'(x)$ for any $\alpha \in C[a, b]$ and $x \in [a, b]$, then the following inequalities hold:*

$$\begin{aligned} &\frac{1}{b-a}\Big[\int_a^x (t-a)\alpha(t)dt + \int_x^b (t-b)\alpha(t)dt \\ &\quad - \max\{x-a, b-x\}\Big\{f(b) - f(a) - \int_a^b \alpha(t)dt\Big\} \\ &\leq f(x) - \frac{1}{b-a}\int_a^b f(t)dt \\ &\leq \frac{1}{b-a}\Big[\int_a^x (t-a)\alpha(t)dt + \int_x^b (t-b)\alpha(t)dt \\ &\quad + \max\{x-a, b-x\}\Big\{f(b) - f(a) - \int_a^b \alpha(t)dt\Big\}. \end{aligned} \tag{7}$$

In this article, in Section 2 we will prove the Ostrowski-Grüss-like type inequalities similar to above Theorem 1.2 and Theorem 1.3 but now involving twice differentiable functions, and in Section 3 we will introduce a new generalized analogue of the Ostrowski-like type inequality in three different cases by using the more generalized kernel function.

2. Ostrowski-Grüss Type Inequality for Twice Differentiable Functions

In this section we will give the generalization of the Ostrowski-Grüss-like type inequality for twice continuously differentiable functions. To do this, let us consider the following kernel function $K_1(\cdot,\cdot) : [a+\lambda\frac{b-a}{2}, b-\lambda\frac{b-a}{2}]^2 \to R$ defined by

$$K_1(x,t) = \begin{cases} \frac{t}{n}\{t-n(a+\lambda\frac{b-a}{2})\} & t\in[a+\lambda\frac{b-a}{2}, x] \\ \frac{t}{n}\{t-n(b-\lambda\frac{b-a}{2})\} & t\in[x, b-\lambda\frac{b-a}{2}] \end{cases}$$

for any $\lambda \in [0,1]$ and integer $n \geq 2$.

To begin with, let us start with the following theorem.

Theorem 2.1. *Let $f : I \to R$ be a twice continuously differentiable function in the interior I^0 of an interval I in R and $a, b \in I^0$ with $a < b$. If there exist constants $\gamma, \Gamma \in R$ such that $\gamma \leq f''(t) \leq \Gamma$ for any $t \in [a+\lambda\frac{b-a}{2}, b-\lambda\frac{b-a}{2}]$ for $\lambda \in [0,1]$ and $f'' \in L^2[a+\lambda\frac{b-a}{2}, b-\lambda\frac{b-a}{2}]$, then, for any integer $n \geq 2$ we have the following inequalities:*

$$\begin{aligned}\Big|R_n(x)\Big| \leq & \frac{(1-\lambda)(b-a)}{12n}\Big[\big\{3n(\lambda-2)+2(\lambda+2)\big\}a \\ & + \big\{8-(3n+2)\lambda\big\}b\Big](S(\lambda)-\gamma)\end{aligned} \tag{8}$$

and

$$\begin{aligned}\Big|R_n(x)\Big| \leq & \frac{(1-\lambda)(b-a)}{12n}\Big[\big\{3n(\lambda-2)+2(\lambda+2)\big\}a \\ & + \big\{8-(3n+2)\lambda\big\}b\Big](\Gamma-S(\lambda)),\end{aligned} \tag{9}$$

where

$R_n(x)$

$$
\begin{aligned}
&= xf'(x) - f(x) + \Big(\frac{2}{n(1-\lambda)(b-a)}\Big) \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} f(t)dt \\
&\quad + \Big(\frac{n-1}{n}\Big)\Big\{\frac{(a+\lambda\frac{b-a}{2})^2 f'(a+\lambda\frac{b-a}{2}) - (b-\lambda\frac{b-a}{2})^2 f'(b-\lambda\frac{b-a}{2})}{(1-\lambda)(b-a)}\Big\} \\
&\quad - \Big(\frac{n-2}{n}\Big)\Big\{\frac{(a+\lambda\frac{b-a}{2}) f(a+\lambda\frac{b-a}{2}) - (b-\lambda\frac{b-a}{2}) f(b-\lambda\frac{b-a}{2})}{(1-\lambda)(b-a)}\Big\} \\
&\quad - \Big\{\frac{1}{2}x^2 - \Big(\frac{3n-2}{24n}\Big)\big\{(1-\lambda)^2(b-a)^2 + 3(a+b)^2\big\}\Big\} \\
&\quad\quad \times \Big\{\frac{f'(b-\lambda\frac{b-a}{2}) - f'(a+\lambda\frac{b-a}{2})}{(1-\lambda)(b-a)}\Big\},
\end{aligned}
$$

and

$$
S(\lambda) = \frac{f'(b-\lambda\frac{b-a}{2}) - f'(a+\lambda\frac{b-a}{2})}{(1-\lambda)(b-a)}.
$$

Proof. By the definition of $K_1(x,t)$, we have

$$
\begin{aligned}
&\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} K_1(x,t) f''(t)dt \\
&= \frac{2}{n}\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} f(t)dt \\
&\quad + (1-\lambda)(b-a)xf'(x) - (1-\lambda)(b-a)f(x) \\
&\quad + \Big(\frac{n-1}{n}\Big)\Big\{(a+\lambda\frac{b-a}{2})^2 f'(a+\lambda\frac{b-a}{2}) \\
&\quad - (b-\lambda\frac{b-a}{2})^2 f'(a+\lambda\frac{b-a}{2})\Big\} \\
&\quad - \Big(\frac{n-2}{n}\Big)\Big\{(a+\lambda\frac{b-a}{2}) f(a+\lambda\frac{b-a}{2}) \\
&\quad - (b-\lambda\frac{b-a}{2}) f(b-\lambda\frac{b-a}{2})\Big\}.
\end{aligned}
\tag{10}
$$

By the simple calculation, we have

$$
\frac{1}{(1-\lambda)(b-a)}\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} K_1(x,t)dt
$$

$$= \Big\{\frac{1}{2}x^2 - \Big(\frac{3n-2}{24n}\Big)\Big((1-\lambda)^2(b-a)^2 + 3(a+b)^2\Big)\Big\} \tag{11}$$

and

$$\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} f''(t)dt = f'(b-\lambda\frac{b-a}{2}) - f'(a+\lambda\frac{b-a}{2}). \tag{12}$$

By using (10),(11) and (12), we get

$$\begin{aligned}
&T_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}(K_1(x,\cdot), f'') \\
&= (\frac{2}{n(1-\lambda)(b-a)}) \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} f(t)dt + xf'(x) - f(x) \\
&\quad + (\frac{n-1}{n})\Big\{\frac{(a+\lambda\frac{b-a}{2})^2 f'(a+\lambda\frac{b-a}{2}) - (b-\lambda\frac{b-a}{2})^2 f'(a+\lambda\frac{b-a}{2})}{(1-\lambda)(b-a)}\Big\} \\
&\quad - (\frac{n-2}{n})\Big\{\frac{(a+\lambda\frac{b-a}{2}) f(a+\lambda\frac{b-a}{2}) - (b-\lambda\frac{b-a}{2}) f(b-\lambda\frac{b-a}{2})}{(1-\lambda)(b-a)}\Big\} \\
&\quad - \Big\{\frac{1}{2}x^2 - (\frac{3n-2}{24n})\Big((1-\lambda)^2(b-a)^2 + 3(a+b)^2\Big)\Big\} \\
&\quad \times \Big\{\frac{f'(b-\lambda\frac{b-a}{2}) - f'(a+\lambda\frac{b-a}{2})}{(1-\lambda)(b-a)}\Big\}.
\end{aligned}$$

Here, let

$$R_n(x) = T_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}(K_1(x,\cdot), f'').$$

By the definition of Chebyshev functional, we have

$$\begin{aligned}
R_n(x) &= \frac{1}{(1-\lambda)(b-a)} \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} K_1(x,t) f''(t)dt \\
&\quad - \frac{1}{(1-\lambda)^2(b-a)^2} \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} f''(t)dt \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} K_1(x,t)dt \\
&= \frac{1}{(1-\lambda)(b-a)} \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} (f''(t) - C) \\
&\quad \times \Big[K_1(x,t) - \frac{1}{(1-\lambda)(b-a)} \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} K_1(x,s)ds\Big] dt.
\end{aligned} \tag{13}$$

Also, we know that

$$\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} \Big[K_1(x,t) - \frac{1}{(1-\lambda)(b-a)} \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} K_1(x,s)ds\Big]dt = 0.$$

So, if we choose $C = \gamma$, then we get

$$\begin{aligned} R_n(x) =& \frac{1}{(1-\lambda)(b-a)} \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} (f''(t)-\gamma) \\ & \times \Big[K_1(x,t) - \frac{1}{(1-\lambda)(b-a)} \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} K_1(x,s)ds\Big]dt. \end{aligned}$$

Note that

$$\begin{aligned} &\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} \Big|(f''(t)-\gamma)\Big|dt \\ &= f'(b-\lambda\frac{b-a}{2}) - f'(a+\lambda\frac{b-a}{2}) - \gamma(1-\lambda)(b-a) \\ &= (1-\lambda)(b-a)(S(\lambda)-\gamma), \end{aligned} \tag{14}$$

where

$$S(\lambda) = \frac{f'(b-\lambda\frac{b-a}{2}) - f'(a+\lambda\frac{b-a}{2})}{(1-\lambda)(b-a)}.$$

Also, we have

$$\begin{aligned} \Big|R_n(x)\Big| \le & \frac{1}{(1-\lambda)(b-a)} \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} \Big|(f''(t)-\gamma)\Big|dt \\ & \times \max\Big|K_1(x,t) - \frac{1}{(1-\lambda)(b-a)} \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} K_1(x,s)ds\Big|. \end{aligned} \tag{15}$$

Note that

$$\begin{aligned} &\max\Big|K_1(x,t) - \frac{1}{(1-\lambda)(b-a)} \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} K_1(x,s)ds\Big| \\ &= \max\Big|K_1(x,t) - \Big\{\frac{1}{2}x^2 - (\frac{3n-2}{24n})\big((1-\lambda)^2(b-a)^2 + 3(a+b)^2\big)\Big\}\Big| \\ &= \frac{(1-\lambda)(b-a)}{12n}\Big[\{3n(\lambda-2)+2(\lambda+2)\}a + \{8-(2+3n)\lambda\}b\Big]. \end{aligned} \tag{16}$$

By (15) and (16), we get the desired result (8):

$$\begin{aligned}
&\left|R_n(x)\right| \\
&\le (1-\lambda)(b-a)(S(\lambda)-\gamma) \\
&\quad \times \left(\frac{1}{12n}\right)\Big[\big\{3n(\lambda-2)+2(\lambda+2)\big\}a+\{8-(2+3n)\lambda\big\}b\Big] \\
&= \frac{(1-\lambda)(b-a)}{12n}\Big[\big\{3n(\lambda-2)+2(\lambda+2)\big\}a \\
&\qquad\qquad + \{8-(2+3n)\lambda\big\}b\Big]\big(S(\lambda)-\gamma\big),
\end{aligned}$$

Secondly, if we choose $C=\Gamma$ in (13), then by a similar argument we get

$$\begin{aligned}
\left|R_n(x)\right| &= \frac{1}{(1-\lambda)(b-a)}\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}\left|(f''(t)-\Gamma)\right|dt \\
&\qquad \times \max\left|K_1(x,t)-\frac{1}{(1-\lambda)(b-a)}\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}K_1(x,s)ds\right| \qquad (17)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}\left|(f''(t)-\Gamma)\right|dt \\
&= -\Big\{f'(b-\lambda\frac{b-a}{2})-f'(a+\lambda\frac{b-a}{2})\Big\}+\Gamma(1-\lambda)(b-a) \\
&= (1-\lambda)(b-a)(\Gamma-S(\lambda)). \qquad (18)
\end{aligned}$$

By (16),(17) and (18), we get the desired result (9).

Theorem 2.2. *Let $f: I\to R$ be a twice continuously differentiable function in the interior I^0 of an interval I in R and $a,b\in I^0$ with $a<b$. If $f''\in L_2[a+\lambda\frac{b-a}{2}, b-\lambda\frac{b-a}{2}]$ for $\lambda\in[0,1]$, then, for any integer $n\ge 2$ we have the following inequalities:*

$$\begin{aligned}
\left|R_n(x)\right| \le & \frac{(1-\lambda)(b-a)}{12n}\Big[\big\{3n(\lambda-2)+2(\lambda+2)\big\}a \\
&\qquad + \big\{8-(3n+2)\lambda\big\}b\Big](S(\lambda)-f''(\frac{a+b}{2})) \qquad (19)
\end{aligned}$$

where

$$S(\lambda)=\frac{f'(b-\lambda\frac{b-a}{2})-f'(a+\lambda\frac{b-a}{2})}{(1-\lambda)(b-a)}.$$

Proof. Let $R_n(x)$ be defined as in the equality (13) with $C \in R$ an arbitrary constant. If we choose $C = f''(\frac{a+b}{2})$, we get

$$\begin{aligned}
&\Big|R_n(x)\Big| \\
&\leq \frac{1}{(1-\lambda)(b-a)} \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} \Big|f''(t) - f''(\frac{a+b}{2})\Big|dt \\
&\quad \times \max\Big|K_1(x,t) - \Big\{\frac{1}{2}x^2 - (\frac{3n-2}{24n})\{(1-\lambda)^2(b-a)^2 + 3(a+b)^2\}\Big\}\Big| \\
&= \frac{(1-\lambda)(b-a)}{12n}\Big[\{3n(\lambda-2)+2(\lambda+2)\}a + \{8-(2+3n)\lambda\}b\Big] \\
&\quad \times \big(S(\lambda) - f''(\frac{a+b}{2})\big),
\end{aligned}$$

which completes the proof.

3. Ostrowski-Type Inequality for Differentiable Functions

In this section we will prove new analogues of the Ostrowski-like type inequality for differentiable functions in three different cases, which are generalizations of Theorem 3.1 and Theorem 3.2.

To do this, let us consider the following kernel function $K_2(\cdot,\cdot) : [a + \lambda\frac{b-a}{2}, b - \lambda\frac{b-a}{2}]^2 \to R$ defined by

$$K_2(x,t) = \begin{cases} t - (a + \lambda\frac{b-a}{2}) & t \in [a + \lambda\frac{b-a}{2}, x] \\ t - (b - \lambda\frac{b-a}{2}) & t \in [x, b - \lambda\frac{b-a}{2}] \end{cases}$$

for any $\lambda \in [0,1]$.

Theorem 3.1. *Let $f : I \to R$ be a differentiable function in the interior I^0 of an interval I in R and $a, b \in I^0$ with $a < b$. If $\alpha(x) \leq f'(x) \leq \beta(x)$ for any $\alpha, \beta \in C[a + \lambda\frac{b-a}{2}, b - \lambda\frac{b-a}{2}]$ and $x \in [a + \lambda\frac{b-a}{2}, b - \lambda\frac{b-a}{2}]$ for $\lambda \in [0,1]$, then the following inequality holds:*

$$\int_{a+\lambda\frac{b-a}{2}}^{x} \big\{t - (a + \lambda\frac{b-a}{2})\big\}\alpha(t)dt + \int_{x}^{b-\lambda\frac{b-a}{2}} \big\{t - (b - \lambda\frac{b-a}{2})\big\}\beta(t)dt$$

$$\leq (1-\lambda)(b-a)f(x) - \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} f(t)dt \tag{20}$$
$$\leq \int_{a+\lambda\frac{b-a}{2}}^{x} \{t-(a+\lambda\frac{b-a}{2})\}\beta(t)dt$$
$$+ \int_{x}^{b-\lambda\frac{b-a}{2}} \{t-(b-\lambda\frac{b-a}{2})\}\alpha(t)dt.$$

Proof. By the definition $K_2(\cdot,\cdot)$, we have

$$\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} K_2(x,t)f'(t)dt = (1-\lambda)(b-a)f(x) - \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} f(t)dt. \tag{21}$$

By referring to the kernel $K_2(\cdot,\cdot)$ and the identity (21) we have

$$\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} K_2(x,t)\Big\{f'(t) - \frac{\alpha(t)+\beta(t)}{2}\Big\}dt$$
$$= (1-\lambda)(b-a)f(x) - \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} f(t)dt$$
$$-\frac{1}{2}\Big[\int_{a+\lambda\frac{b-a}{2}}^{x} \{t-(a+\lambda\frac{b-a}{2})\}\{\alpha(t)+\beta(t)\}dt$$
$$+\int_{x}^{b-\lambda\frac{b-a}{2}} \{t-(b-\lambda\frac{b-a}{2})\}\{\alpha(t)+\beta(t)\}dt\Big]. \tag{22}$$

On the other hand, by the fact that $\alpha(x) \leq f'(x) \leq \beta(x)$ for any $\alpha, \beta \in C[a+\lambda\frac{b-a}{2}, b-\lambda\frac{b-a}{2}]$ and for $\lambda \in [0,1]$, we have

$$\Big|f'(t) - \frac{\alpha(t)+\beta(t)}{2}\Big| \leq \Big|\beta(t) - \frac{\alpha(t)+\beta(t)}{2}\Big| = \frac{\beta(t)-\alpha(t)}{2}. \tag{23}$$

Therefore, by (22) and (23) one can conclude that

$$\Big|(1-\lambda)(b-a)f(x) - \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} f(t)dt$$
$$-\frac{1}{2}\Big[\int_{a+\lambda\frac{b-a}{2}}^{x} \{t-(a+\lambda\frac{b-a}{2})\}\{\alpha(t)+\beta(t)\}dt$$
$$+\int_{x}^{b-\lambda\frac{b-a}{2}} \{t-(b-\lambda\frac{b-a}{2})\}\{\alpha(t)+\beta(t)\}dt\Big]\Big|$$

$$
\begin{aligned}
&\le \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} \Big|K_2(x,t)\Big|\Big|f'(t) - \frac{\alpha(t)+\beta(t)}{2}\Big|dt \\
&\le \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} \Big|K_2(x,t)\Big|\Big(\frac{\beta(t)-\alpha(t)}{2}\Big)dt \\
&= \frac{1}{2}\Big[\int_{a+\lambda\frac{b-a}{2}}^{x} \big\{t-(a+\lambda\frac{b-a}{2})\big\}\big\{\beta(t)-\alpha(t)\big\}dt \\
&\quad + \int_{x}^{b-\lambda\frac{b-a}{2}} \big\{(b-\lambda\frac{b-a}{2})-t\big\}\big\{\beta(t)-\alpha(t)\big\}dt.
\end{aligned}
$$

which completes the proof.

Theorem 3.2. *Let $f : I \to R$ be a differentiable function in the interior I^0 of an interval I in R and $a, b \in I^0$ with $a < b$. If $\alpha(x) \le f'(x)$ for any $\alpha \in C[a+\lambda\frac{b-a}{2}, b-\lambda\frac{b-a}{2}]$ and $x \in [a+\lambda\frac{b-a}{2}, b-\lambda\frac{b-a}{2}]$ for $\lambda \in [0,1]$, then the following inequalities hold:*

$$
\begin{aligned}
&\Big[\int_{a+\lambda\frac{b-a}{2}}^{x} \big\{t-(a+\lambda\frac{b-a}{2})\big\}\alpha(t)dt \\
&\quad + \int_{x}^{b-\lambda\frac{b-a}{2}} \big\{t-(b-\lambda\frac{b-a}{2})\big\}\alpha(t)dt\Big] \\
&\quad - \max\Big\{x-(a+\lambda\frac{b-a}{2}), (b-\lambda\frac{b-a}{2})-x\Big\} \\
&\quad \times \Big\{f(b-\lambda\frac{b-a}{2}) - f(a+\lambda\frac{b-a}{2}) - \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} f(t)dt\Big\} \\
&\le (1-\lambda)(b-a)f(x) - \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} f(t)dt \\
&\le \Big[\int_{a+\lambda\frac{b-a}{2}}^{x} \big\{t-(a+\lambda\frac{b-a}{2})\big\}\beta(t)dt \\
&\quad + \int_{x}^{b-\lambda\frac{b-a}{2}} \big\{t-(b-\lambda\frac{b-a}{2})\big\}\alpha(t)dt\Big] \\
&\quad + \max\Big\{x-(a+\lambda\frac{b-a}{2}), (b-\lambda\frac{b-a}{2})-x\Big\} \\
&\quad \times \Big\{f(b-\lambda\frac{b-a}{2}) - f(a+\lambda\frac{b-a}{2}) - \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} f(t)dt\Big\}.
\end{aligned}
$$

Proof. By the definition $K_2(\cdot,\cdot)$, we have

$$\begin{aligned}
&\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} K_2(x,t)\big(f'(t)-\alpha(t)\big)dt \\
&= (1-\lambda)(b-a)f(x) - \Big[\int_{a+\lambda\frac{b-a}{2}}^{x} \big\{t-(a+\lambda\frac{b-a}{2})\big\}\alpha(t)dt \\
&\quad + \int_{x}^{b-\lambda\frac{b-a}{2}} \big\{t-(b-\lambda\frac{b-a}{2})\big\}\alpha(t)dt\Big] - \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} f(t)dt,
\end{aligned}$$

we have

$$\begin{aligned}
&\Big|(1-\lambda)(b-a)f(x) - \Big[\int_{a+\lambda\frac{b-a}{2}}^{x} \big\{t-(a+\lambda\frac{b-a}{2})\big\}\alpha(t)dt \\
&\quad + \int_{x}^{b-\lambda\frac{b-a}{2}} \big\{t-(b-\lambda\frac{b-a}{2})\big\}\alpha(t)dt\Big] - \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} f(t)dt\Big| \\
&\leq \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} \Big|K_2(x,t)\Big|\Big|f'(t)-\alpha(t)\Big|dt \\
&= \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} \Big|K_2(x,t)\Big|\big(f'(t)-\alpha(t)\big)dt \\
&\leq \max_{t\in[a+\lambda\frac{b-a}{2},\, b-\lambda\frac{b-a}{2}]} \Big|K_2(x,t)\Big| \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} \big(f'(t)-\alpha(t)\big)dt \\
&= \max\Big\{x-(a+\lambda\frac{b-a}{2}),\, (b-\lambda\frac{b-a}{2})-x\Big\} \\
&\quad \times \Big\{f(b-\lambda\frac{b-a}{2}) - f(a+\lambda\frac{b-a}{2}) - \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} \alpha(t)dt\Big\},
\end{aligned}$$

which completes the proof.

Theorem 3.3. *Let $f: I \to R$ be a differentiable function in the interior I^0 of an interval I in R and $a, b \in I^0$ with $a < b$. If $f'(x) \leq \beta(x)$ for any $\beta \in C[a+\lambda\frac{b-a}{2}, b-\lambda\frac{b-a}{2}]$ and $x \in [a+\lambda\frac{b-a}{2}, b-\lambda\frac{b-a}{2}]$ for $\lambda \in [0,1]$, then the following inequalities hold:*

$$\begin{aligned}
&\Big[\int_{a+\lambda\frac{b-a}{2}}^{x} \big\{t-(a+\lambda\frac{b-a}{2})\big\}\beta(t)dt \\
&\quad + \int_{x}^{b-\lambda\frac{b-a}{2}} \big\{t-(b-\lambda\frac{b-a}{2})\big\}\beta(t)dt\Big]
\end{aligned}$$

$$
\begin{aligned}
&- \max\Big\{x-(a+\lambda\frac{b-a}{2}),(b-\lambda\frac{b-a}{2})-x\Big\} \\
&\quad \times \Big\{f(a+\lambda\frac{b-a}{2})-f(b-\lambda\frac{b-a}{2})+\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}\beta(t)dt\Big\} \\
&\le (1-\lambda)(b-a)f(x)-\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}f(t)dt \\
&\le \Big[\int_{a+\lambda\frac{b-a}{2}}^{x}\big\{t-(a+\lambda\frac{b-a}{2})\big\}\beta(t)dt \\
&\quad +\int_{x}^{b-\lambda\frac{b-a}{2}}\big\{t-(b-\lambda\frac{b-a}{2})\big\}\beta(t)dt\Big] \\
&\quad + \max\Big\{x-(a+\lambda\frac{b-a}{2}),(b-\lambda\frac{b-a}{2})-x\Big\} \\
&\quad \times \Big\{f(a+\lambda\frac{b-a}{2})-f(b-\lambda\frac{b-a}{2})+\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}\beta(t)dt\Big\}.
\end{aligned}
$$

Proof. By the definition $K_2(\cdot,\cdot)$, we have

$$
\begin{aligned}
&\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}K_2(x,t)\big(f'(t)-\beta(t)\big)dt \\
&= (1-\lambda)(b-a)f(x)-\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}f(t)dt \\
&\quad -\Big[\int_{a+\lambda\frac{b-a}{2}}^{x}\big\{t-(a+\lambda\frac{b-a}{2})\big\}\beta(t)dt \\
&\qquad +\int_{x}^{b-\lambda\frac{b-a}{2}}\big\{t-(b-\lambda\frac{b-a}{2})\big\}\beta(t)dt\Big].
\end{aligned}
$$

So, we have

$$
\begin{aligned}
&\Big|(1-\lambda)(b-a)f(x)-\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}f(t)dt \\
&-\Big[\int_{a+\lambda\frac{b-a}{2}}^{x}\big\{t-(a+\lambda\frac{b-a}{2})\big\}\beta(t)dt \\
&\quad +\int_{x}^{b-\lambda\frac{b-a}{2}}\big\{t-(b-\lambda\frac{b-a}{2})\big\}\beta(t)dt\Big]\Big|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} \Big|K_2(x,t)\Big|\Big|\beta(t)-f'(t)\Big|dt \\
&\leq \max_{t\in[a+\lambda\frac{b-a}{2},b-\lambda\frac{b-a}{2}]} \Big|K_2(x,t)\Big| \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} \big(\beta(t)-f'(t)\big)dt \\
&= \max\Big\{x-(a+\lambda\frac{b-a}{2}),(b-\lambda\frac{b-a}{2})-x\Big\} \\
&\quad \times \Big\{f(a+\lambda\frac{b-a}{2})-f(b-\lambda\frac{b-a}{2})+\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} \beta(t)dt\Big\},
\end{aligned}
$$

which completes the proof.

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