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SOME GENERALIZED COMPANIONS OF PERTURBED OSTROWSKI-LIKE TYPE INEQUALITIES

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Abstract: In this paper, based on the generalized quadratic kernel function with three sections which was motivated by Liu, Zhu and Park[20], some companions of perturbed Ostrowski type inequalities for some several cases are obtained. The special cases of these results offer better estimation than the trapezoidal formula and the midpoint formula.

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1. Introduction

1938, Ostrowski [25] established the following interesting integral inequality for differentiable mappings with bounded derivatives:

Theorem 1.1. *Let $f : [a, b] \to R$ be a differentiable mapping on (a, b) whose derivative is bounded on (a, b) and denote $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then for all $x \in [a, b]$ we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a)\|f'\|_\infty. \qquad (1)$$

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The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

The first generalization of Ostrowski's type inequality was given by Milovanović and Pecarić in [23]. Some new types of these inequalities are formed, such as inequalities of Ostrowski-Grüss type, inequalties of Ostrowski-Chebyshev type and etc. The first inequality of Ostrowski-Grüss type was given by Dragomir and Wang in [12]. It was generalized and improved by Matić, Pecarić, Ujević, Alomari, Liu and Sarikaya in [1, 2, 3, 4, 16, 19, 18, 23, 24, 26, 27, 28, 29, 30, 31].

In [14], Guessab and Schmeisser proved the following companion of Ostrowski's inequality:

Theorem 1.2. *Let $f : [a, b] \to R$ be satisfying the Lipschitz condition, i.e., $|f(t) - f(s)| \leq M|t - s|$. Then for all $x \in [a, \frac{a+b}{2}]$ we have*

$$\left|\frac{f(x) + f(a + b - x)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \left[\frac{1}{8} + 2\Big(\frac{x - \frac{3a+b}{4}}{b-a}\Big)^2\right](b - a)M. \tag{2}$$

The constant $\frac{1}{8}$ is sharp in the sense that it can not be replaced by a smaller one. In (2), the point $x = \frac{3a+b}{4}$ gives the best estimator.

Motivated by [14], Dragomir [9] proved some companions of Ostrowski's inequality for absolutely continuous functions. Recently, Alomari [2] studied the companion of Ostrowski inequality (2) for differentiable bounded mappings. In [18], Liu established some companions of an Ostrowski type integral inequality for functions whose first derivatives are absolutely continuous and second derivatives belong to L^p spaces for $1 \leq p \leq \infty$.

In[22], Liu, Zhu and Park established some another companions of perturbed Ostrowski type inequalities for the case when $f'' \in L^1[a, b]$, $f''' \in L^2[a, b]$ and $f'' \in L^2[a, b]$, respectively:

Theorem 1.3. *Let $f : [a, b] \to R$ be such that f' is absolutely continuous on $[a, b]$. If $f'' \in L^1[a, b]$ and $\gamma \leq f''(x) \leq \Gamma$ for $x \in [a, b]$, then, for all $x \in [a, \frac{a+b}{2}]$ we have*

$$\Big|I\Big| \equiv \Big|\frac{1}{b-a}\int_a^b f(t)dt - \Big\{\frac{f(x) + f(a + b - x)}{2}\Big\}$$

$$+ \Big(x - \frac{3a+b}{4}\Big)\Big\{\frac{f'(x) - f'(a + b - x)}{2}\Big\}$$

$$- \left\{ \frac{f'(b) - f'(a)}{b-a} \right\} \left[\frac{1}{2} \left(x - \frac{3a+b}{4} \right)^2 + \frac{(b-a)^2}{96} \right] \Bigg|$$
$$\leq (S - \gamma)(b-a) \left[\frac{b-a}{48} + \frac{1}{4} \left| x - \frac{3a+b}{4} \right| \right] \tag{3}$$

and

$$\Big| I \Big| \leq (\Gamma - S)(b-a) \left[\frac{b-a}{48} + \frac{1}{4} \left| x - \frac{3a+b}{4} \right| \right], \tag{4}$$

where

$$S = \frac{f'(b) - f'(a)}{b-a}. \tag{5}$$

Theorem 1.4. *Let $f : [a,b] \to R$ be such that f' is absolutely continuous on $[a,b]$. If $f''' \in L^2[a,b]$, then, for all $x \in [a, \frac{a+b}{2}]$ we have*

$$\Big| I \Big| \leq \frac{\| f''' \|_2}{\pi} \left[\frac{1}{320} \big(a + b - 2x \big)^5 + \frac{1}{10} (x-a)^5 \right.$$
$$\left. -(b-a) \left\{ \frac{1}{2} (x - \frac{3a+b}{4})^2 + \frac{(b-a)^2}{96} \right\}^2 \right]^{\frac{1}{2}}, \tag{6}$$

where I is defined as in Theorem 1.3.

Theorem 1.5. *Let $f : [a,b] \to R$ be such that f' is absolutely continuous on $[a,b]$. If $f'' \in L^2[a,b]$, then, for all $x \in [a, \frac{a+b}{2}]$ we have*

$$\Big| I \Big| \leq \frac{\sqrt{\sigma(f'')}}{b-a} \left[\frac{1}{320} \big(a + b - 2x \big)^5 + \frac{1}{10} (x-a)^5 \right.$$
$$\left. -(b-a) \left\{ \frac{1}{2} \left(x - \frac{3a+b}{4} \right)^2 + \frac{(b-a)^2}{96} \right\}^2 \right]^{\frac{1}{2}}, \tag{7}$$

where I and S are defined as in Theorem 1.4 and $\sigma(f'')$ is defined by

$$\sigma(f'') = \| f'' \|_2^2 - \frac{\big(f'(b) - f'(a)\big)^2}{b-a} = \| f'' \|_2^2 - S^2(b-a). \tag{8}$$

For other related results, the reader may be refer to [3, 4, 5, 7, 8, 10, 11, 13, 15, 16, 19, 21, 17, 24, 26, 27, 28, 29, 30, 31, 32] and the references therein.

In [20], Liu and Park established a generalization of the companion of Ostrowski-like type integral inequality for mapping whose second derivatives belong to L^∞-spaces. Their results in special cases not only recapture known results, but also give smaller estimators than those of the known results.

The main aim of this paper is to establish some generalizations of the companions of perturbed Ostrowski type inequalities Theorem 1.3, Theorem 1.4 and Theorem 1.5 for the case when $f'' \in L^1[a,b]$, $f''' \in L^2[a,b]$ and $f'' \in L^2[a,b]$, respectively. For our purpose, we will use the generalized quadratic kernel function with three sections (see (9) below) which was defined by Liu and Park in [20, 18]. The special cases of the results we get offer better estimation than conventional trapezoidal formula and the midpoint formula. These results can apply to composite quadrature rules in numerical integration and probability density functions.

2. Main Results

To prove our main results, we need the following lemma:

Lemma 1. *Let $f : [a,b] \to R$ be such that f' is absolutely continuous on $[a,b]$. Denote by $K(x,t) : [a,b] \to R$ the kernel given by*

$$K(x,t) = \begin{cases} \frac{1}{2}\{t - (a + \lambda\frac{b-a}{2})\}^2, & t \in [a,x], \\ \frac{1}{2}\{t - (\frac{a+b}{2})\}^2, & t \in (x, a+b-x], \\ \frac{1}{2}\{t - (b - \lambda\frac{b-a}{2})\}^2, & t \in (a+b-x, b] \end{cases} \tag{9}$$

for $\lambda \in [0,1]$, then the following identity holds:

$$\begin{aligned} &\frac{1}{b-a}\int_a^b K(x,t) f''(t)dt \\ =&\frac{1}{b-a}\int_a^b f(t)dt - (1-\lambda)\left\{\frac{f(x)+f(a+b-x)}{2}\right\} \\ &-\lambda\left\{\frac{f(a)+f(b)}{2}\right\} + \frac{\lambda^2}{4}\left\{\frac{f'(b)-f'(a)}{2}\right\}(b-a) \end{aligned} \tag{10}$$

$$+ (1-\lambda)\left\{x - \frac{(3-\lambda)a + (1+\lambda)b}{4}\right\}\left\{\frac{f'(x) - f'(a+b-x)}{2}\right\}.$$

Proof. From the definition (9) of $K(x,t)$ we have

$$\frac{1}{b-a}\int_a^b K(x,t) f''(t)dt$$

$$
\begin{aligned}
=&\frac{1}{2}\int_a^x \left\{t-\left(a+\lambda\frac{b-a}{2}\right)\right\}^2 f''(t)dt\\
&+\frac{1}{2}\int_x^{a+b-x} \left\{t-\frac{a+b}{2}\right\}^2 f''(t)dt\\
&+\frac{1}{2}\int_{a+b-x}^{b} \left\{t-\left(b-\lambda\frac{b-a}{2}\right)\right\}^2 f''(t)dt\\
&= I_1+I_2+I_3 \text{ (Say)}.
\end{aligned}
$$

Integrating by parts, we have

$$
\begin{aligned}
I_1 =&\frac{1}{2}\left\{x-\left(a+\lambda\frac{b-a}{2}\right)\right\}^2 f'(x)-\left\{x-\left(a+\lambda\frac{b-a}{2}\right)\right\}f(x)\\
&-\frac{\lambda^2}{8}(b-a)^2 f'(a)-\frac{\lambda}{2}(b-a)f(a)+\int_a^x f(t)dt,\\
I_2 =&\frac{1}{2}\left\{x-\frac{a+b}{2}\right\}^2 f'(a+b-x)+\left\{x-\frac{a+b}{2}\right\}f(a+b-x)\\
&-\left\{x-\frac{a+b}{2}\right\}^2 f'(x)+\left\{x-\frac{a+b}{2}\right\}f(x)+\int_x^{a+b-x} f(t)dt,\\
I_3 =&-\frac{1}{2}\left\{x-\left(a+\lambda\frac{b-a}{2}\right)\right\}^2 f'(a+b-x)+\frac{\lambda^2}{8}(b-a)^2 f'(b)\\
&-\left\{x-\left(a+\lambda\frac{b-a}{2}\right)\right\}f(a+b-x)\\
&-\frac{\lambda}{2}(b-a)f(b)+\int_{a+b-x}^{b} f(t)dt.
\end{aligned}
$$

By adding $I_1,\ I_2,\ I_3$ and dividing by $b-a$, we get the desired result (10).

2.1. The Case when $f'' \in L^1[a,b]$ and is Bounded

Theorem 2.1. *Let $f:[a,b]\to R$ be such that f' is absolutely continuous on $[a,b]$. If $f''\in L^1[a,b]$ and $\gamma\le f''(x)\le\Gamma$ for all $x\in[a,b]$, then for all $x\in\left[a+\lambda\frac{b-a}{2},\frac{a+b}{2}\right]$ we have*

$$
\begin{aligned}
&\left|R_n\right|\\
\equiv&\left|\frac{1}{b-a}\int_a^b f(t)dt-(1-\lambda)\left\{\frac{f(x)+f(a+b-x)}{2}\right\}\right.\\
&+(1-\lambda)\left\{x-\frac{(3-\lambda)a+(1+\lambda)b}{4}\right\}\left\{\frac{f'(x)-f'(a+b-x)}{2}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&-\lambda\Big\{\frac{f(a)+f(b)}{2}\Big\}-\Big\{\frac{f'(b)-f'(a)}{b-a}\Big\}\Big\{\big(\frac{1-\lambda}{2}\big)\\
&\times\big(x-\frac{(3-\lambda)a+(1+\lambda)b}{4}\big)^2+\frac{(b-a)^2}{96}\left(1-3\lambda-9\lambda^2+3\lambda^3\right)\Big\}\Big|\\
\le&(S-\gamma)\Big[\big(\frac{\lambda}{2}\big)\Big\{(x-\frac{(3-\lambda)a+(1+\lambda)b}{4}\Big\}^2+\big(\frac{(b-a)^2}{96}\big)\\
&\times\left(2-3\lambda-3\lambda^2\right)+\big(\frac{1-\lambda}{4}\big)(b-a)\Big|x-\frac{(3-\lambda)a+(1+\lambda)b}{4}\Big|\Big]
\end{aligned}
\tag{11}
$$

and

$$
\begin{aligned}
\Big|R_n\Big|\le&(\Gamma-S)\Big[\big(\frac{\lambda}{2}\big)\Big\{x-\frac{(3-\lambda)a+(1+\lambda)b}{4}\Big\}^2\\
&+\frac{(b-a)^2}{96}\left(2-3\lambda-3\lambda^2\right)\\
&+\big(\frac{1-\lambda}{4}\big)(b-a)\left|x-\frac{(3-\lambda)a+(1+\lambda)b}{4}\right|\Big],
\end{aligned}
\tag{12}
$$

where S is defined as in Theorem 1.3.

Proof. From (10) in Lemma 1 and the facts

$$
\frac{1}{b-a}\int_a^b f''(t)dt=\frac{f'(b)-f'(a)}{b-a}
\tag{13}
$$

and

$$
\begin{aligned}
&\frac{1}{b-a}\int_a^b K(x,t)dt\\
=&\big(\frac{1-\lambda}{2}\big)\Big\{x-\frac{(3-\lambda)a+(1+\lambda)b}{4}\Big\}^2\\
&+\frac{(b-a)^2}{96}\left(1-3\lambda+3\lambda^2+3\lambda^3\right),
\end{aligned}
\tag{14}
$$

it follows that

$$
\begin{aligned}
&\frac{1}{b-a}\int_a^b K(x,t)f''(t)dt-\frac{1}{(b-a)^2}\int_a^b K(x,t)dt\int_a^b f''(t)dt\\
=&\frac{1}{b-a}\int_a^b f(t)dt+(1-\lambda)\Big\{x-\frac{(3-\lambda)a+(1+\lambda)b}{4}\Big\}\\
&\times\left\{\frac{f'(x)-f'(a+b-x)}{2}\right\}-(1-\lambda)\left\{\frac{f(x)+f(a+b-x)}{2}\right\}
\end{aligned}
$$

$$- \lambda \Big\{ \frac{f(a)+f(b)}{2} \Big\} - \Big\{ \frac{f'(b)-f'(a)}{b-a} \Big\} \Big\{ (\frac{1-\lambda}{2}) \tag{15}$$
$$\times \big(x - \frac{(3-\lambda)a+(1+\lambda)b}{4} \big)^2 + \frac{(b-a)^2}{96} \left(1-3\lambda-9\lambda^2+3\lambda^3\right) \Big\}.$$

We denote

$$\begin{aligned} R_n(x) =& \frac{1}{b-a} \int_a^b K(x,t) f''(t)dt \\ & - \frac{1}{(b-a)^2} \int_a^b K(x,t)dt \int_a^b f''(t)dt. \end{aligned} \tag{16}$$

If $C \in R$ is an arbitrary constant, then we have

$$R_n(x) = \frac{1}{b-a} \int_a^b (f''(t)-C) \Big\{ K(x,t) - \frac{1}{b-a} \int_a^b K(x,s)ds \Big\} dt. \tag{17}$$

Furthermore, we have

$$\begin{aligned} & |R_n(x)| \\ \le & \frac{1}{b-a} \max_{t\in[a,b]} \Big| K(x,t) - \frac{1}{b-a} \int_a^b K(x,s)ds \Big| \int_a^b \Big| f''(t)-C \Big| dt. \end{aligned} \tag{18}$$

Let us compute:

$$\max_{t\in[a,b]} \Big| K(x,t) - \frac{1}{b-a} \int_a^b K(x,s)ds \Big| = \max \left\{ |y_1|, |y_2|, |y_3| \right\}, \tag{19}$$

where

$$\begin{aligned} y_1 =& \frac{\lambda}{2} x^2 - \frac{1}{4} \Big\{ (1+2\lambda-\lambda^2)a + (-1+2\lambda+\lambda^2)b \Big\} x \\ & + \frac{1}{24} \Big\{ (5-3\lambda^2)a^2 + 4(3\lambda-1)ab + (3\lambda^2-1)b^2 \Big\}, \end{aligned}$$

$$\begin{aligned} y_2 =& \frac{\lambda}{2} x^2 - \frac{1}{4} \Big\{ (-1+4\lambda-\lambda^2)a + (1+\lambda^2)b \Big\} x \\ & + \frac{1}{12} \Big\{ (-2+6\lambda-3\lambda^2)a^2 + (1+3\lambda^2)ab + b^2 \Big\}, \\ y_3 =& \Big(\frac{1-\lambda}{2} \Big) \Big\{ x - \frac{(3-\lambda)a+(1+\lambda)b}{4} \Big\}^2 \\ & + \frac{(b-a)^2}{96} \left(1-3\lambda+3\lambda^2+3\lambda^3\right). \end{aligned}$$

If we choose $y_1 = 0$ and $\lambda = 0$, then we get $x_1 = \frac{5a+b}{6}$. If we choose $y_2 = 0$ and $\lambda = 0$, then we get $x_2 = \frac{2a+b}{3}$.

A direct computation gives that

$$\begin{cases} y_2 \geq \max\{y_1, y_3\}, & x \in \big[a + \lambda\dfrac{b-a}{2}, \dfrac{(3-\lambda)a+(1+\lambda)b}{4}\big], \\ y_1 > \max\{y_2, y_3\}, & x \in \big(\dfrac{(3-\lambda)a+(1+\lambda)b}{4}, \dfrac{a+b}{2}\big]. \end{cases} \tag{20}$$

Therefore, we get

$$\begin{aligned} &\max_{t\in[a,b]} \left|K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)ds\right| \\ =&\max\{y_1, y_2\} \\ =&\frac{1}{2}\{y_1 + y_2 + |y_1 - y_2|\} \\ =&\big(\frac{\lambda}{2}\big)\Big\{x - \frac{(3-\lambda)a+(1+\lambda)b}{4}\Big\}^2 + \frac{(b-a)^2}{96}\left(2 - 3\lambda - 3\lambda^2\right) \\ &+ \big(\frac{1-\lambda}{4}\big)(b-a)\Big|x - \frac{(3-\lambda)a+(1+\lambda)b}{4}\Big|. \end{aligned} \tag{21}$$

We also have

$$\int_a^b |f''(t) - \gamma|dt = \int_a^b \big(f''(t) - \gamma\big)\,dt = (S-\gamma)(b-a) \tag{22}$$

and

$$\int_a^b |f''(t) - \Gamma|dt = \int_a^b \big(\Gamma - f''(t)\big)\,dt = (\Gamma - S)(b-a). \tag{23}$$

Therefore, we obtain (11) and (12) by using (15)-(18), (21)-(23) and choosing $C = \gamma$ and $C = \Gamma$ in (18), respectively.

Remark 1. The above theorem is a generalization of Theorem 1.3.

Corollary 2.1. *Under the assumptions of Theorem 2.1,*
(1) *choosing* $x = \frac{(3-\lambda)a+(1+\lambda)b}{4}$, *then we have*

$$\begin{aligned} (a)\ &\bigg|\frac{1}{b-a}\int_a^b f(t)dt - \big(\frac{1-\lambda}{2}\big)\Big\{f\big(\frac{(3-\lambda)a+(1+\lambda)b}{4}\big) \\ &+ f\big(\frac{(1+\lambda)a+(3-\lambda)b}{4}\big)\Big\} - \lambda\left\{\frac{f(a)+f(b)}{2}\right\} \\ &- \Big\{f'(b) - f'(a)\Big\}\frac{(b-a)}{96}\left(1 - 3\lambda - 9\lambda^2 + 3\lambda^3\right)\bigg| \end{aligned} \tag{24}$$

$$\leq \frac{(b-a)^2}{96}(S-\gamma)\left(2-3\lambda-3\lambda^2\right).$$

$$(b)\ \left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1-\lambda}{2}\Big\{f\big(\frac{(3-\lambda)a+(1+\lambda)b}{4}\big) + f\big(\frac{(1+\lambda)a+(3-\lambda)b}{4}\big)\Big\} - \lambda\Big\{\frac{f(a)+f(b)}{2}\Big\} - \Big\{f'(b)-f'(a)\Big\}\frac{(b-a)}{96}\left(1-3\lambda-9\lambda^2+3\lambda^3\right)\right| \leq \frac{(b-a)^2}{96}(\Gamma-S)\left(2-3\lambda-3\lambda^2\right). \tag{25}$$

(2) *choosing* $x=a$*, then we have*

$$(a)\ \left|\frac{1}{b-a}\int_a^b f(t)dt + \frac{(1-\lambda)^2}{4}(b-a)\Big\{\frac{f'(b)-f'(a)}{2}\Big\} - \lambda\Big\{\frac{f(a)+f(b)}{2}\Big\} - \Big\{f'(b)-f'(a)\Big\}\frac{(b-a)}{24}\left(1-3\lambda^2\right)\right| \leq \frac{(b-a)^2}{96}(S-\gamma)\left(8-3\lambda^2+3\lambda^3\right).$$

$$(b)\ \left|\frac{1}{b-a}\int_a^b f(t)dt + \frac{(1-\lambda)^2}{4}(b-a)\Big\{\frac{f'(b)-f'(a)}{2}\Big\} - \lambda\Big\{\frac{f(a)+f(b)}{2}\Big\} - \Big\{f'(b)-f'(a)\Big\}\frac{(b-a)}{24}\left(1-3\lambda^2\right)\right| \leq \frac{(b-a)^2}{96}(\Gamma-S)\left(8-3\lambda^2+3\lambda^3\right).$$

(3) *choosing* $x=\frac{a+b}{2}$*, then we have*

$$(a)\ \left|\frac{1}{b-a}\int_a^b f(t)dt - (1-\lambda)f\big(\frac{a+b}{2}\big) - \lambda\Big\{\frac{f(a)+f(b)}{2}\Big\} - \Big\{f'(b)-f'(a)\Big\}\frac{(b-a)}{24}\left(1-3\lambda\right)\right| \leq \frac{(b-a)^2}{96}(S-\gamma)\left(8-3\lambda^2+3\lambda^3\right).$$

$$(b)\ \left|\frac{1}{b-a}\int_a^b f(t)dt - (1-\lambda)f\big(\frac{a+b}{2}\big) - \lambda\Big\{\frac{f(a)+f(b)}{2}\Big\} - \Big\{f'(b)-f'(a)\Big\}\frac{(b-a)}{24}\left(1-3\lambda\right)\right|$$

$$\leq \frac{(b-a)^2}{96}(\Gamma - S)\left(8 - 3\lambda^2 + 3\lambda^3\right).$$

Corollary 2.2. *Let f as in Theorem 2.1. Additionally, if f is symmetric about $x = \frac{a+b}{2}$, then for all $x \in \left[a + \lambda\frac{b-a}{2}, \frac{a+b}{2}\right]$ we have*

$$(a)\ \Bigg|\frac{1}{b-a}\int_a^b f(t)dt - (1-\lambda)f(x) - \lambda\Big\{\frac{f(a)+f(b)}{2}\Big\}$$
$$-(1-\lambda)\big(x - \frac{(3-\lambda)a+(1+\lambda)b}{4}\big)f'(x)$$
$$-\Big\{\frac{f'(b)-f'(a)}{b-a}\Big\}\Big\{\big(\frac{1-\lambda}{2}\big)\big(x - \frac{(3-\lambda)a+(1+\lambda)b}{4}\big)^2$$
$$+\frac{(b-a)^2}{96}\big(1 - 3\lambda - 9\lambda^2 + 3\lambda^3\big)\Big\}\Bigg|$$
$$\leq (S-\gamma)(b-a)\left\{\big(\frac{\lambda}{2}\big)\big(x - \frac{(3-\lambda)a+(1+\lambda)b}{4}\big)^2\right.$$
$$+\frac{(b-a)^2}{96}(2 - 3\lambda - 3\lambda^2)$$
$$\left.+\big(\frac{1-\lambda}{4}\big)(b-a)\Big|x - \frac{(3-\lambda)a+(1+\lambda)b}{4}\Big|\right\}.$$

$$(b)\ \Bigg|\frac{1}{b-a}\int_a^b f(t)dt - (1-\lambda)\big(x - \frac{(3-\lambda)a+(1+\lambda)b}{4}\big)f'(x)$$
$$-(1-\lambda)f(x) - \lambda\Big\{\frac{f(a)+f(b)}{2}\Big\} - \Big\{\frac{f'(b)-f'(a)}{b-a}\Big\}$$

$$\times\Big\{\big(\frac{1-\lambda}{2}\big)\big(x - \frac{(3-\lambda)a+(1+\lambda)b}{4}\big)^2$$
$$+\frac{(b-a)^2}{96}\Big(1 - 3\lambda - 9\lambda^2 + 3\lambda^3\Big)\Big\}\Bigg|$$
$$\leq (\Gamma - S)(b-a)\left\{\big(\frac{\lambda}{2}\big)\big(x - \frac{(3-\lambda)a+(1+\lambda)b}{4}\big)^2\right.$$
$$+\frac{(b-a)^2}{96}(2 - 3\lambda - 3\lambda^2)$$
$$\left.+\big(\frac{1-\lambda}{4}\big)(b-a)\Big|x - \frac{(3-\lambda)a+(1+\lambda)b}{4}\Big|\right\}.$$

2.2. The Case when $f''' \in L^2[a,b]$

Theorem 2.2. *Let $f : [a,b] \to R$ be a thrice continuously differentiable mapping in (a,b) with $f''' \in L^2[a,b]$. Then for all $\lambda \in [0,1]$ and $x \in \left[a+\lambda\frac{b-a}{2}, \frac{a+b}{2}\right]$ we have*

$$\begin{aligned}
&\frac{\pi^2}{\| f''' \|_2^2} |R_n|^2 \\
&\le \frac{1}{320}\Big\{(b-a)^5\lambda^5 + (a+b-2x)^5 + \big\{2x - ((2-\lambda)a + \lambda b)\big\}^5\Big\} \\
&\quad - (b-a)\left\{ \Big(\frac{1-\lambda}{2}\Big)\Big(x - \frac{(3-\lambda)a+(1+\lambda)b}{4}\Big)^2 \right. \\
&\qquad \left. + \frac{(b-a)^2}{96}\left(1-3\lambda+3\lambda^2+3\lambda^3\right)\right\}^2, \qquad (26)
\end{aligned}$$

where R_n is defined as in Theorem 2.1.

Proof. Let $R_n(x)$ be defined by (16). From (15), we get

$$\begin{aligned}
R_n(x) = &\frac{1}{b-a}\int_a^b f(t)dt - (1-\lambda)\left\{\frac{f(x)+f(a+b-x)}{2}\right\} \\
&- \lambda\Big\{\frac{f(a)+f(b)}{2}\Big\} + (1-\lambda)\Big\{x - \frac{(3-\lambda)a+(1+\lambda)b}{4}\Big\} \\
&\times \left\{\frac{f'(x)-f'(a+b-x)}{2}\right\} - \left\{\frac{f'(b)-f'(a)}{b-a}\right\} \\
&\times \left\{ \Big(\frac{1-\lambda}{2}\Big)\Big(x - \frac{(3-\lambda)a+(1+\lambda)b}{4}\Big)^2 \right. \\
&\qquad \left. + \frac{(b-a)^2}{96}\big(1-3\lambda-9\lambda^2+3\lambda^3\big)\right\}.
\end{aligned}$$

If we choose $C = f''((a+b)/2)$ in (18) and use the Cauchy inequality, then we get

$$\begin{aligned}
\Big|R_n(x)\Big| \le &\frac{1}{b-a}\int_a^b \Big|f''(t) - f''\Big(\frac{a+b}{2}\Big)\Big| \\
&\qquad \times \Big|K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)ds\Big| dt \\
\le &\frac{1}{b-a}\left[\int_a^b \Big(f''(t) - f''\Big(\frac{a+b}{2}\Big)\Big)^2 dt\right]^{1/2} \qquad (27)
\end{aligned}$$

$$\times \left[\int_a^b \Big(K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)ds\Big)^2 dt\right]^{1/2}.$$

We can use the Diaz-Metcalf inequality (see [24, p. 83] or [31, p. 424]) to get

$$\int_a^b \Big(f''(t) - f''(\frac{a+b}{2})\Big)^2 dt \le \frac{(b-a)^2}{\pi^2}\|f'''\|_2^2.$$

We also have

$$\begin{aligned}
&\int_a^b \Big\{K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)ds\Big\}^2 dt \\
=&\int_a^b K(x,t)^2 dt - \frac{1}{b-a}\Big(\int_a^b K(x,s)ds\Big)^2 \\
=&\frac{1}{320}\Big\{(b-a)^5\lambda^5 + (a+b-2x)^5 + \big(2x - \{(2-\lambda)a + \lambda b\}\big)^5\Big\} \qquad (28)\\
&\quad - (b-a)\Big\{(\frac{1-\lambda}{2})\big(x - \frac{(3-\lambda)a + (1+\lambda)b}{4}\big)^2 \\
&\quad + \frac{(b-a)^2}{96}\big(1 - 3\lambda + 3\lambda^2 + 3\lambda^3\big)\Big\}^2.
\end{aligned}$$

Therefore, using the above relations (27)-(28), we obtain (26).

Remark 2. The above theorem is a generalization of Theorem 1.4.

Corollary 2.3. *Under the assumptions of Theorem 2.1,*
(1) *choosing* $x = \frac{(3-\lambda)a+(1+\lambda)b}{4}$*, then we have*

$$\begin{aligned}
&\Big|\frac{1}{b-a}\int_a^b f(t)dt - (\frac{1-\lambda}{2})\Big\{f(\frac{(3-\lambda)a + (1+\lambda)b}{4}) \\
&+f(\frac{((1+\lambda)a + (3-\lambda)b}{4})\Big\} - \frac{\lambda}{2}\Big\{f(a) + f(b)\Big\} \\
&-\Big\{f'(b) - f'(a)\Big\}\frac{(b-a)}{96}\Big(1 - 3\lambda - 9\lambda^2 + 3\lambda^3\Big)\Big| \\
&\le \frac{\| f''' \|_2}{\pi}(b-a)^{\frac{5}{2}} \\
&\quad \times \Big\{\frac{1}{46080}\big(4 - 15\lambda + 15\lambda^2 - 30\lambda^3 + 90\lambda^5 - 45\lambda^6\big)\Big\}^{\frac{1}{2}}. \qquad (29)
\end{aligned}$$

(2) *choosing $x = a$, then we have*

$$\Big|\frac{1}{b-a}\int_a^b f(t)dt + \big(\frac{(1-\lambda)^2}{4}\big)(b-a)\Big\{\frac{f'(b)-f'(a)}{2}\Big\}$$
$$-\lambda\Big\{\frac{f(a)+f(b)}{2}\Big\} - \Big\{f'(b)-f'(a)\Big\}\frac{(b-a)}{24}\left(1-3\lambda^2\right)\Big|$$
$$\leq \frac{(b-a)^{\frac{5}{2}}}{12\pi\sqrt{5}} \parallel f''' \parallel_2 .$$

(3) *choosing $x = \frac{a+b}{2}$, then we have*

$$\Big|\frac{1}{b-a}\int_a^b f(t)dt - (1-\lambda)f(\frac{a+b}{2}) - \lambda\frac{f(a)+f(b)}{2}$$
$$-\Big\{f'(b)-f'(a)\Big\}\frac{(b-a)}{24}\left(1-3\lambda\right)\Big|$$
$$\leq \frac{(b-a)^{\frac{5}{2}}}{12\pi\sqrt{5}} \parallel f''' \parallel_2 .$$

Corollary 2.4. *Let f as in Theorem 2.1. Additionally, if f is symmetric about $x = \frac{a+b}{2}$, then for all $x \in \left[a+\lambda\frac{b-a}{2}, \frac{a+b}{2}\right]$ we have*

$$\Big|\frac{1}{b-a}\int_a^b f(t)dt - (1-\lambda)f(x) - \lambda\Big\{\frac{f(a)+f(b)}{2}\Big\}$$
$$-(1-\lambda)\big(x - \frac{(3-\lambda)a+(1+\lambda)b}{4}\big)f'(x)$$
$$-\Big\{\frac{f'(b)-f'(a)}{b-a}\Big\}\Big\{\big(\frac{1-\lambda}{2}\big)\big(x - \frac{(3-\lambda)a+(1+\lambda)b}{4}\big)^2$$
$$+\frac{(b-a)^2}{96}\big(1-3\lambda-9\lambda^2+3\lambda^3\big)\Big\}\Big|$$

$$\leq \frac{\parallel f'' \parallel_2}{\pi}\Bigg[\frac{1}{320}\Big\{(b-a)^5\lambda^5 + (a+b-2x)^5$$
$$+(2x - \{(2-\lambda)a+\lambda b\})^5\Big\}$$
$$-(b-a)\Big\{\big(\frac{1-\lambda}{2}\big)\big(x - \frac{(3-\lambda)a+(1+\lambda)b}{4}\big)^2$$
$$+\frac{(b-a)^2}{96}\big(1-3\lambda+3\lambda^2+3\lambda^3\big)\Big\}^2\Bigg]^{1/2}.$$

2.3. The Case when $f'' \in L^2[a,b]$

The following theorem is a generalization of Theorem 1.5:

Theorem 2.3. *Let $f : [a,b] \to R$ be a thrice continuously differentiable mapping in (a,b) with $f'' \in L^2[a,b]$. Then for all $\lambda \in [0,1]$ and $x \in \left[a+\lambda\frac{b-a}{2}, \frac{a+b}{2}\right]$ we have*

$$
\begin{aligned}
|R_n| \leq \frac{\sigma^{\frac{1}{2}}(f'')}{b-a}\Bigg[&\frac{1}{320}\Big\{(b-a)^5\lambda^5+(a+b-2x)^5 \\
&+(2x-\{(2-\lambda)a+\lambda b\})^5\Big\} \\
&-(b-a)\Big\{\big(\frac{1-\lambda}{2}\big)\big(x-\frac{(3-\lambda)a+(1+\lambda)b}{4}\big)^2 \\
&+\frac{(b-a)^2}{96}\big(1-3\lambda+3\lambda^2+3\lambda^3\big)\Big\}^2\Bigg]^{1/2},
\end{aligned} \tag{30}
$$

where R_n and S are defined as in Theorem 2.1 and $\sigma(f'')$ is defined by

$$
\sigma(f'') = \| f'' \|_2^2 - \frac{(f'(b)-f'(a))^2}{b-a} = \| f'' \|_2^2 - S^2(b-a).
$$

Proof. Let $R_n(x)$ be defined by (16). If we choose $C = \frac{1}{b-a}\int_a^b f''(s)ds$ in (18) and use the Cauchy inequality, then we get

$$
\begin{aligned}
|R_n(x)| \leq &\frac{1}{b-a}\int_a^b \left| f''(t) - \frac{1}{b-a}\int_a^b f''(s)ds \right| \\
&\times \left| K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)ds \right| dt
\end{aligned}
$$

$$
\begin{aligned}
\leq &\frac{1}{b-a}\left\{\int_a^b \left(f''(t) - \frac{1}{b-a}\int_a^b f''(s)ds\right)^2 dt\right\}^{1/2} \\
&\times \left\{\int_a^b \left(K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)ds\right)^2 dt\right\}^{1/2}.
\end{aligned} \tag{31}
$$

Note that

$$
\int_a^b \left(f''(t) - \frac{1}{b-a}\int_a^b f''(s)ds\right)^2 dt
$$

$$\leq \int_a^b (f''(t))^2 dt - 2S\int_a^b f''(t)dt + S^2\int_a^b dt$$
$$= \| f'' \|_2^2 - S^2(b-a).$$

We also have

$$\int_a^b \Big(K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)ds\Big)^2 dt$$
$$= \frac{1}{320}\Big\{(b-a)^5\lambda^5 + (a+b-2x)^5 + (2x - \{(2-\lambda)a + \lambda b\})^5\Big\}$$
$$- (b-a)\Bigg\{\Big(\frac{1-\lambda}{2}\Big)\Big(x - \frac{(3-\lambda)a+(1+\lambda)b}{4}\Big)^2 \tag{32}$$
$$+\frac{(b-a)^2}{96}\big(1-3\lambda+3\lambda^2+3\lambda^3\big)\Bigg\}^2.$$

Therefore, using the above relations (31)-(32), we obtain (30).

Remark 3. The above theorem is a generalization of Theorem 1.5.

Corollary 2.5. *Under the assumptions of Theorem 2.1,*
(1) *choosing* $x = \frac{(3-\lambda)a+(1+\lambda)b}{4}$, *then we have*

$$\Bigg|\frac{1}{b-a}\int_a^b f(t)dt - \Big(\frac{1-\lambda}{2}\Big)\Big\{f\Big(\frac{(3-\lambda)a+(1+\lambda)b}{4}\Big)$$
$$+f\Big(\frac{((1+\lambda)a+(3-\lambda)b}{4}\Big)\Big\} - \Big(\frac{\lambda}{2}\Big)\Big\{f(a)+f(b)\Big\}$$
$$-\Big\{f'(b)-f'(a)\Big\}\frac{(b-a)}{96}\big(1-3\lambda-9\lambda^2+3\lambda^3\big)\Bigg|$$
$$\leq \sqrt{\sigma(f'')}(b-a)^{\frac{3}{2}}\Big\{\frac{4-15\lambda+15\lambda^2-30\lambda^3+90\lambda^5-45\lambda^6}{46080}\Big\}^{\frac{1}{2}}. \tag{33}$$

(2) *choosing* $x = a$, *then we have*

$$\Bigg|\frac{1}{b-a}\int_a^b f(t)dt + \frac{(1-\lambda)^2}{4}(b-a)\left\{\frac{f'(b)-f'(a)}{2}\right\}$$
$$-\lambda\Big\{\frac{f(a)+f(b)}{2}\Big\} - \Big\{f'(b)-f'(a)\Big\}\frac{(b-a)}{24}\big(1-3\lambda^2\big)\Bigg|$$
$$\leq \sqrt{\sigma(f'')}\frac{(b-a)^{\frac{3}{2}}}{12\sqrt{5}}.$$

(3) *choosing* $x = \frac{a+b}{2}$, *then we have*

$$\Big|\frac{1}{b-a}\int_a^b f(t)dt - (1-\lambda)f(\frac{a+b}{2}) - \lambda\Big\{\frac{f(a)+f(b)}{2}\Big\}$$
$$-\Big\{f'(b)-f'(a)\Big\}\frac{(b-a)}{24}(1-3\lambda)\Big|$$
$$\leq \sqrt{\sigma(f'')}\frac{(b-a)^{\frac{3}{2}}}{12\sqrt{5}}.$$

Corollary 2.6. *Let f as in Theorem 2.1. Additionally, if f is symmetric about $x = \frac{a+b}{2}$, then for all $x \in \big[a + \lambda\frac{b-a}{2}, \frac{a+b}{2}\big]$ we have*

$$\Big|\frac{1}{b-a}\int_a^b f(t)dt - (1-\lambda)f(x) - \lambda\Big\{\frac{f(a)+f(b)}{2}\Big\}$$
$$-(1-\lambda)\Big(x - \frac{(3-\lambda)a+(1+\lambda)b}{4}\Big)f'(x)$$
$$-\Big\{\frac{f'(b)-f'(a)}{b-a}\Big\}\Big\{(\frac{1-\lambda}{2})\big(x - \frac{(3-\lambda)a+(1+\lambda)b}{4}\big)^2$$
$$+\frac{(b-a)^2}{96}\big(1-3\lambda-9\lambda^2+3\lambda^3\big)\Big\}\Big|$$
$$\leq \frac{\sqrt{\sigma(f'')}}{b-a}\Big[\frac{1}{320}\Big\{(b-a)^5\lambda^5 + (a+b-2x)^5$$
$$+\big(2x - \{(2-\lambda)a+\lambda b\}\big)^5\Big\}$$
$$-(b-a)\Big\{(\frac{1-\lambda}{2})\big(x - \frac{(3-\lambda)a+(1+\lambda)b}{4}\big)^2$$
$$+\frac{(b-a)^2}{96}\big(1-3\lambda+3\lambda^2+3\lambda^3\big)\Big\}^2\Big]^{1/2}.$$

3. Application to Composite Quadrature Rules

Let $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a partition of the interval $[a,b]$ with $h_i = x_{i+1} - x_i$ $(i = 0,1,2,\cdots,n-1)$ and $\lambda \in [0,1]$.

Consider the perturbed composite quadrature rules

$$Q_n(I_n, f, \lambda)$$

$$= (\frac{1-\lambda}{2}) \sum_{i=0}^{n-1} \Big[f(\frac{(3-\lambda)x_i + (1+\lambda)x_{i+1}}{4})$$
$$+ f(\frac{(1+\lambda)x_i + (3-\lambda)x_{i+1}}{4})\Big] h_i - \frac{\lambda}{2} \sum_{i=0}^{n-1} \Big\{ f(x_i) + f(x_{i+1} \Big\}$$
$$+ \frac{(1-3\lambda-9\lambda^2+3\lambda^3)}{96} \sum_{i=0}^{n-1} \Big\{ f'(x_{i+1}) - f'(x_i) \Big\} h_i^2. \tag{34}$$

The following results hold:

Theorem 3.1. *Let $f : [a,b] \to R$ be such that f' is absolutely continuous on $[a,b]$. If $f'' \in L^1[a,b]$ and $\gamma \le f''(x) \le \Gamma$ for all $x \in [a,b]$, then for all $x \in \big[a + \lambda \frac{b-a}{2}, \frac{a+b}{2}\big]$ we have*

$$\int_a^b f(t)dt = Q_n(I_n, f, \lambda) + R_n(I_n, f, \lambda),$$

where $Q_n(I_n, f, \lambda)$ is defined by formula (34), and the remainder $R_n(I_n, f, \lambda)$ satisfies the estimate

$$\Big| R_n(I_n, f, \lambda) \Big| \le \frac{2-3\lambda-3\lambda^2}{96} (S-\gamma) \sum_{i=0}^{n-1} h_i^3 \tag{35}$$

and

$$\Big| R_n(I_n, f, \lambda) \Big| \le \frac{2-3\lambda-3\lambda^2}{96} (S-\gamma) \sum_{i=0}^{n-1} h_i^3. \tag{36}$$

Proof. Applying inequality (24) and (25) to the intervals $[x_i, x_{i+1}]$, then we get

$$\Big| \int_{x_i}^{x_{i+1}} f(t)dt - (\frac{1-\lambda}{2}) \Big\{ f(\frac{(3-\lambda)x_i + (1+\lambda)x_{i+1}}{4})$$
$$+ f(\frac{(1+\lambda)x_i + (3-\lambda)x_{i+1}}{4}) \Big\} h_i - (\frac{\lambda}{2}) \Big\{ f(x_i) + f(x_{i+1}) \Big\} h_i$$

$$-(\frac{1-3\lambda-9\lambda^2+3\lambda^3}{96}) \Big\{ f'(x_{i+1}) - f'(x_i) \Big\} h_i^2 \Big|$$
$$\le (\frac{2-3\lambda-3\lambda^2}{96})(S-\gamma) h_i^3$$

and

$$\Big|\int_{x_i}^{x_{i+1}} f(t)dt - (\frac{1-\lambda}{2})\Big\{f(\frac{(3-\lambda)x_i+(1+\lambda)x_{i+1}}{4})$$
$$+f(\frac{(1+\lambda)x_i+(3-\lambda)x_{i+1}}{4})\Big\}h_i - \frac{\lambda}{2}\Big\{f(x_i)+f(x_{i+1})\Big\}$$
$$-(\frac{1-3\lambda-9\lambda^2+3\lambda^3}{96})\Big\{f'(x_{i+1})-f'(x_i)\Big\}h_i^2\Big|$$
$$\le (\frac{2-3\lambda-3\lambda^2}{96})(\Gamma - S)h_i^3$$

for $i = 0, 1, 2, \cdots, n-1$. Now summing over i from 0 to $n-1$ and using the triangle inequality, we get (35) and (36).

Theorem 3.2. *Let $f : [a, b] \to R$ be a thrice continuously differentiable mapping in (a, b) with $f''' \in L^2[a, b]$. Then for all $x \in \big[a + \lambda\frac{b-a}{2}, \frac{a+b}{2}\big]$ we have*

$$\int_a^b f(t)dt = Q_n(I_n, f, \lambda) + R_n(I_n, f, \lambda),$$

where $Q_n(I_n, f, \lambda)$ is defined by formula (34), and the remainder $R_n(I_n, f, \lambda)$ satisfies the estimate

$$\Big|R_n(I_n, f, \lambda)\Big|$$
$$\le \frac{\| f''' \|_2}{48\pi\sqrt{5}}\Big(4-15\lambda+15\lambda^2-30\lambda^3+90\lambda^5-45\lambda^6\Big)^{\frac{1}{2}}\sum_{i=0}^{n-1} h_i^{7/2}. \qquad (37)$$

Proof. Applying inequality (29) to the intervals $[x_i, x_{i+1}]$, then we get

$$\Big|\int_{x_i}^{x_{i+1}} f(t)dt - (\frac{1-\lambda}{2})\Big\{f(\frac{(3-\lambda)x_i+(1+\lambda)x_{i+1}}{4})$$
$$+f(\frac{(1+\lambda)x_i+(3-\lambda)x_{i+1}}{4})\Big\}h_i - (\frac{\lambda}{2})\Big\{f(x_i)+f(x_{i+1})\Big\}h_i$$
$$-(\frac{1-3\lambda-9\lambda^2+3\lambda^3}{96})\Big\{f'(x_{i+1})-f'(x_i)\Big\}h_i^2\Big|$$

$$\le \frac{\| f''' \|_2}{\pi}\left\{\frac{1}{46080}\big(4-15\lambda+15\lambda^2-30\lambda^3+90\lambda^5-45\lambda^6\big)\right\}^{\frac{1}{2}} h_i^{\frac{7}{2}}. \qquad (38)$$

for $i = 0, 1, 2, \cdots, n-1$. Now summing over i from 0 to $n-1$ and using the triangle inequality, we get the inequality (37).

Theorem 3.3. *Let $f : [a, b] \to R$ be such that f' is absolutely continuous on $[a, b]$ with $f'' \in L^2[a, b]$. Then for all $x \in \left[a + \lambda\frac{b-a}{2}, \frac{a+b}{2}\right]$ we have*

$$\int_a^b f(t)dt = Q_n(I_n, f, \lambda) + R_n(I_n, f, \lambda),$$

where $Q_n(I_n, f, \lambda)$ is defined by formula (34), and the remainder $R_n(I_n, f, \lambda)$ satisfies the estimate

$$\Bigg|\int_a^b f(t)dt - (\frac{1-\lambda}{2})\sum_{i=0}^{n-1}\Big\{f(\frac{(3-\lambda)x_i + (1+\lambda)x_{i+1}}{4}) + f(\frac{(1+\lambda)x_i + (3-\lambda)x_{i+1}}{4})\Big\}h_i - (\frac{\lambda}{2})\sum_{i=0}^{n-1}\Big\{f(x_i) + f(x_{i+1})\Big\}h_i - \sum_{i=0}^{n-1}\Big\{\frac{f'(x_{i+1}) - f'(x_i)}{96}\Big\}h_i^2\big(1 - 3\lambda - 9\lambda^2 + 3\lambda^3\big)\Bigg| \leq \sqrt{\sigma(f'')}\Big\{\frac{4 - 15\lambda + 15\lambda^2 - 30\lambda^3 + 90\lambda^5 - 45\lambda^6}{46080}\Big\}^{\frac{1}{2}}\sum_{i=0}^{n-1} h_i^{\frac{5}{2}}. \tag{39}$$

Proof. Applying inequality (33) to the intervals $[x_i, x_{i+1}]$, then we get

$$\Bigg|\int_{x_i}^{x_{i+1}} f(t)dt - (\frac{1-\lambda}{2})\Big[f(\frac{(3-\lambda)x_i + (1+\lambda)x_{i+1}}{4}) + f(\frac{(1+\lambda)x_i + (3-\lambda)x_{i+1}}{4})\Big]h_i - (\frac{\lambda}{2})\Big\{f(x_i) + f(x_{i+1}\Big\} - \big(1 - 3\lambda - 9\lambda^2 + 3\lambda^3\big)\frac{f'(x_{i+1}) - f'(x_i)}{96}h_i^2\Bigg| \leq \sqrt{\sigma(f'')}\Big\{\frac{4 - 15\lambda + 15\lambda^2 - 30\lambda^3 + 90\lambda^5 - 45\lambda^6}{46080}\Big\}^{\frac{1}{2}} h_i^{\frac{5}{2}}.$$

for $i = 0, 1, 2, \cdots, n-1$. Now summing over i from 0 to $n-1$ and using the triangle inequality, we get (39).

4. Application to Probability Density Functions

Now, let X be a random variable taking values in the finite interval $[a, b]$, with the probability density function $f : [a, b] \to [0, 1]$ and with the cumulative distribution function

$$F(x) = Pr(X \leq x) = \int_a^x f(t)dt.$$

The following results hold:

Theorem 4.1. *With the assumptions of Theorem 2.1, we have*

$$\Bigg|\frac{b-E(X)}{b-a}-(1-\lambda)\left\{\frac{F(X)+F(a+b-X)}{2}\right\}$$
$$+(1-\lambda)\left\{x-\frac{(3-\lambda)a+(1+\lambda)b}{4}\right\}\left\{\frac{f(x)-f(a+b-x)}{2}\right\}$$
$$-(1-\lambda)\left\{\frac{F(X)+F(a+b-X)}{2}\right\}-\lambda\left\{\frac{F(b)}{2}\right\}$$
$$-\left\{\frac{f(b)-f(a)}{b-a}\right\}\Big\{(\frac{1-\lambda}{2})\big(x-\frac{(3-\lambda)a+(1+\lambda)b}{4}\big)^2$$
$$+\frac{(b-a)^2}{96}\big(1-3\lambda-9\lambda^2+3\lambda^3\big)\Big\}\Bigg|$$
$$\le(S-\gamma)$$
$$\times\Bigg[(\frac{\lambda}{2})\Big\{x-\frac{(3-\lambda)a+(1+\lambda)b}{4}\Big\}^2+\frac{(b-a)^2}{96}\big(2-3\lambda-3\lambda^2\big)$$
$$+(\frac{1-\lambda}{4})\,(b-a)\left|x-\frac{(3-\lambda)a+(1+\lambda)b}{4}\right|\Bigg] \tag{40}$$

and

$$\Bigg|\frac{b-E(X)}{b-a}-(1-\lambda)\left\{\frac{F(X)+F(a+b-X)}{2}\right\}$$
$$+(1-\lambda)\left\{x-\frac{(3-\lambda)a+(1+\lambda)b}{4}\right\}\left\{\frac{f(x)-f(a+b-x)}{2}\right\}$$
$$-(1-\lambda)\left\{\frac{F(X)+F(a+b-X)}{2}\right\}-\lambda\left\{\frac{F(b)}{2}\right\}$$
$$-\left\{\frac{f(b)-f(a)}{b-a}\right\}\Big\{(\frac{1-\lambda}{2})\Big(x-\frac{(3-\lambda)a+(1+\lambda)b}{4}\Big)^2$$
$$+\frac{(b-a)^2}{96}\big(1-3\lambda-9\lambda^2+3\lambda^3\big)\Big\}\Bigg|$$
$$\le(\Gamma-S)$$
$$\times\Bigg[(\frac{\lambda}{2})\Big\{x-\frac{(3-\lambda)a+(1+\lambda)b}{4}\Big\}^2+\frac{(b-a)^2}{96}\big(2-3\lambda-3\lambda^2\big)$$
$$+(\frac{1-\lambda}{4})\,(b-a)\left|x-\frac{(3-\lambda)a+(1+\lambda)b}{4}\right|\Bigg] \tag{41}$$

for all $x \in [a+\lambda\frac{b-a}{2}, \frac{a+b}{2}]$, *where* $E(X)$ *is the expectation of* X.

Proof. By (11) and (12) on choosing $f = F$ and taking into account

$$E(X) = \int_a^b t dF(t) = b - \int_a^b F(t)dt,$$

we obtain (40) and (41).

Theorem 4.2. *With the assumptions of Theorem 2.2, we have*

$$\begin{aligned}
&\Big| \frac{b-E(X)}{b-a} + (1-\lambda)\Big\{x - \frac{(3-\lambda)a+(1+\lambda)b}{4}\Big\} - \lambda\Big\{\frac{F(b)}{2}\Big\} \\
&\quad \times\Big\{\frac{f(x)-f(a+b-x)}{2}\Big\} - (1-\lambda)\Big\{\frac{F(X)+F(a+b-X)}{2}\Big\} \\
&\quad -\Big\{\frac{f(b)-f(a)}{b-a}\Big\}\Big\{(\frac{1-\lambda}{2})\Big(x - \frac{(3-\lambda)a+(1+\lambda)b}{4}\Big)^2 \\
&\qquad + \frac{(b-a)^2}{96}(1-3\lambda-9\lambda^2+3\lambda^3)\Big\}\Big| \\
&\le \frac{\| f''' \|_2}{\pi}\Big[\frac{1}{320}\Big\{(b-a)^5\lambda^5 + (a+b-2x)^5 \\
&\qquad + (2x-((2-\lambda)a+\lambda b))^5\Big\} \\
&\quad - (b-a)\Big\{(\frac{1-\lambda}{2})\Big(x - \frac{(3-\lambda)a+(1+\lambda)b}{4}\Big)^2 \\
&\qquad + \frac{(b-a)^2}{96}(1-3\lambda+3\lambda^2+3\lambda^3)\Big\}^2\Big]^{1/2} \qquad (42)
\end{aligned}$$

for all $x \in [a+\lambda\frac{b-a}{2}, \frac{a+b}{2}]$, *where* $E(X)$ *is the expectation of* X.

Proof. By (26) on choosing $f = F$ and taking into account

$$E(X) = \int_a^b t dF(t) = b - \int_a^b F(t)dt,$$

we obtain (42).

Theorem 4.3. *With the assumptions of Theorem 2.3, we have*

$$\begin{aligned}
&\Big| \frac{b-E(X)}{b-a} - (1-\lambda)\Big\{\frac{F(X)+F(a+b-X)}{2}\Big\} - \lambda\Big\{\frac{F(b)}{2}\Big\} \\
&\quad +(1-\lambda)\Big\{x - \frac{(3-\lambda)a+(1+\lambda)b}{4}\Big\}\Big\{\frac{f(x)-f(a+b-x)}{2}\Big\}
\end{aligned}$$

$$
\begin{aligned}
&-\Big\{\frac{f(b)-f(a)}{b-a}\Big\}\Big\{\big(\frac{1-\lambda}{2}\big)\Big(x-\frac{(3-\lambda)a+(1+\lambda)b}{4}\Big)^2 \\
&\qquad +\frac{(b-a)^2}{96}\big(1-3\lambda-9\lambda^2+3\lambda^3\big)\Big\}\Big| \\
&\le \frac{\sigma^{\frac{1}{2}}(f'')}{b-a}\Big[\frac{1}{320}\Big\{(b-a)^5\lambda^5 \\
&\qquad + (a+b-2x)^5+(2x-((2-\lambda)a+\lambda b))^5\Big\} \\
&\qquad - (b-a)\Big\{\big(\frac{1-\lambda}{2}\big)\Big(x-\frac{(3-\lambda)a+(1+\lambda)b}{4}\Big)^2 \\
&\qquad + \frac{(b-a)^2}{96}\big(1-3\lambda+3\lambda^2+3\lambda^3\big)\Big\}^2\Big]^{1/2}
\end{aligned} \tag{43}
$$

for all $x \in [a+\lambda\frac{b-a}{2}, \frac{a+b}{2}]$, *where* $E(X)$ *is the expectation of* X.

Proof. By (30) on choosing $f = F$ and taking into account

$$E(X) = \int_a^b t dF(t) = b - \int_a^b F(t)dt,$$

we obtain (43).

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