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SHORT NOTE ON THE DISTRIBUTIONAL DIFFRACTION FRESNEL SINE (COSINE) TRANSFORM

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Abstract: The diffraction Fresnel sine and diffraction Fresnel cosine transforms are extended to spaces of distributions of compact support. The convolution Theorem of the transforms has been established in a generalized sense. Certain theorems are also discussed.

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1. Introduction

The diffraction Fresnel transform of a function $f(t)$ is defined by [5]

$$\mathrm{F}_d f(\xi) = \int_{\mathbb{R}} K(\alpha_1, \gamma_1, \gamma_2, \alpha_2; \xi, t)\, f(t)\, dt. \tag{1}$$

$K(\alpha_1, \gamma_1, \gamma_2, \alpha_2; \xi, t)$ being the transform kernel function given by

$$K(\alpha_1, \gamma_1, \gamma_2, \alpha_2; \xi, t) = \frac{1}{\sqrt{2\pi i \gamma_1}} \exp\left(\frac{i}{2\gamma_1}\left(\alpha_1 t^2 - 2\xi t + \alpha_2 \xi^2\right)\right),$$

where the real parameters, $\alpha_1, \gamma_1, \gamma_2$ and the prarameter α_2 are defined so that

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$\alpha_1\alpha_2 - \gamma_1\gamma_2 = 1$.

If the parameters $\alpha_1, \gamma_1, \gamma_2$ and α_2 are related by the matrix

$$\begin{pmatrix} \alpha_1 & \gamma_1 \\ \gamma_2 & \alpha_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

then the diffraction Fresnel transform becomes a fractional Fourier transform.

Let $\mathcal{S}$ denote the space of all complex valued functions $\phi(t)$ that are infinitely smooth and satisfies the infinite set of inequalities

$$\left|t^m \phi^{(k)}(t)\right| \leq C_{m,k}, t \in \mathtt{R},$$

where m and k traverse the set of nonegative integers $[5, (2.1)]$. Members of $\mathcal{S}$ are testing functions of rapid descent. The strong dual $\mathcal{S}'$ of $\mathcal{S}$ defines a space of distributions of slow growth (the space of tempered distributions). See [7,6,4].

The Parseval's relation of the diffraction Fresnel transform was established in $[5, \text{Theorem } 2.2]$ as

$$\int_{\mathtt{R}} f(x)\,\mathrm{F}_d g(x)\,dx = \int_{\mathtt{R}} \mathrm{F}_d f(x)\,g(x)\,dx \tag{2}$$

where $\mathrm{F}_d g$ and $\mathrm{F}_d f$ are the respective diffraction Fresnel transforms of f and g.

2. Distribution Spaces and F_d Analysis

When discussing the distribution spaces $\mathcal{S}'$; see [7], the extended transform $\overrightarrow{\mathrm{F}}_d$ of a slow growth distribution $f \in \mathcal{S}'$ is expressed as $[5, (2.10)]$

$$\left\langle \overrightarrow{\mathrm{F}}_d f, \phi \right\rangle = \langle f, \mathrm{F}_d \phi \rangle, \tag{3}$$

for every $\phi \in \mathcal{S}$.

Let $f \in \mathcal{E}'$ (the space of distributions of compact supports) then the distributional transform of f is extended in the same citation as

$$\overrightarrow{\mathrm{F}}_d f(\xi) = \frac{1}{\sqrt{2\pi i \gamma_1}} \left\langle f(t), \exp\left(\frac{i\left(\alpha_1 t^2 - 2t\xi + \alpha_2 \xi^2\right)}{2\gamma_1}\right)\right\rangle; \tag{4}$$

see $[5, (2.10)]$.

In this note, let (1) be factored into the components:

$$\mathrm{F}_d f(\xi) = \mathrm{F}_c f(\xi) + i\mathrm{F}_s f(\xi)$$

where

$$\mathrm{F}_s f(\xi) = \frac{e^{\frac{i\alpha_2\xi^2}{2\gamma_1}}}{\sqrt{2\pi i\gamma_1}} \int_{\mathtt{R}} f(t) \sin\left(\frac{\alpha_1 t^2 - 2\xi t}{2\gamma_1}\right) dt \tag{5}$$

and

$$\mathrm{F}_c f(\xi) = \frac{e^{\frac{i\alpha_2\xi^2}{2\gamma_1}}}{\sqrt{2\pi i\gamma_1}} \int_{\mathtt{R}} f(t) \cos\left(\frac{\alpha_1 t^2 - 2\xi t}{2\gamma_1}\right) dt. \tag{6}$$

then the integral equations (5) and (6) interpretted to present a diffraction Fresnel sine and diffraction Fresnel cosine transforms, respectively.

From (4) we have the following definition :

Definition 2.1. Let $f \in \mathcal{E}'(\mathtt{R})$ then the extended diffraction Fresnel sine and diffraction Fresnel cosine transforms of f are defined respectively as

$$\overrightarrow{\mathrm{F}_s} f(\xi) = e^{\frac{i\alpha_2\xi^2}{2\gamma_1}} \left\langle f(t), \sin\frac{(\alpha_1 t^2 - 2\xi t)}{2\gamma_1} \right\rangle, \tag{7}$$

and

$$\overrightarrow{\mathrm{F}_c} f(\xi) = e^{\frac{i\alpha_2\xi^2}{2\gamma_1}} \left\langle f(t), \cos\frac{(\alpha_1 t^2 - 2\xi t)}{2\gamma_1} \right\rangle. \tag{8}$$

Analyticity of $\overrightarrow{\mathrm{F}_s}$ and $\overrightarrow{\mathrm{F}_c}$ can be expressed to mean :

Theorem 2.2. *Let $f \in \mathcal{E}'(\mathtt{R})$ then $\overrightarrow{\mathrm{F}_s}$ and $\overrightarrow{\mathrm{F}_c}$ are analytic and*

$$\mathcal{D}_\xi \overrightarrow{\mathrm{F}_s} f(\xi) = \left\langle f(t), \mathcal{D}_\xi e^{\frac{i\alpha_2\xi^2}{2\gamma_1}} \sin\left(\frac{\alpha_1 t^2 - 2\xi t}{2\gamma_1}\right) \right\rangle, \tag{9}$$

and

$$\mathcal{D}_\xi \overrightarrow{\mathrm{F}_c} f(\xi) = \left\langle f(t), \mathcal{D}_\xi e^{\frac{i\alpha_2\xi^2}{2\gamma_1}} \cos\left(\frac{\alpha_1 t^2 - 2\xi t}{2\gamma_1}\right) \right\rangle. \tag{10}$$

Proof of this theorem is analogous to that found in the literature; see the corresponding theorem in [5]

Theorem 2.3. *The transforms* (7) *and* (8) *are linear.*

The proof of theorem follows from simple computations in $\mathcal{E}'$.

Let f and g be in $\mathcal{E}'$ then the generalized convolution of f and g is defined by [7, P.123, (2)]

$$\langle f * g, \phi \rangle = \langle f(t) \times g(\tau), \phi(t+\tau) \rangle = \langle f(t), \langle g(\tau), \phi(t+\tau) \rangle \rangle.$$

We have the following theorem.

Theorem 2.4. (Convolution Theorem) *Let* $f \in \mathcal{E}^{'}, g \in \mathcal{E}^{'}$ *then*

$$\overrightarrow{\mathrm{F}_s}\,(f*g)\,(s) = \left\langle f\,(t)\sin\frac{st}{\gamma_1}, \overrightarrow{\mathrm{F}_c}\,(g\,(x))\,(t)\right\rangle + \left\langle f\,(t)\cos\frac{st}{\gamma_1}, \overrightarrow{\mathrm{F}_s}\,(g\,(x))\,(t)\right\rangle.$$

Proof. Since the diffraction Fresnel sine transform is defined by

$$(\mathrm{F}_s f)\,(s) = \frac{e^{\frac{i}{2}\left(\frac{\alpha_2}{\gamma_1}\right)s^2}}{\sqrt{2\pi i\gamma_1}}\int_{\mathbb{R}}\sin\left(\frac{s}{\gamma_1}t\right)e^{\frac{i}{2}\left(\frac{\alpha_1}{\gamma_1}\right)t^2}f\,(t)\,dt,$$

it acts on $*$ as

$$\begin{aligned}\overrightarrow{\mathrm{F}_s}\,(f*g)\,(s) &= \frac{e^{\frac{i}{2}\left(\frac{\alpha_2}{\gamma_1}\right)s^2}}{\sqrt{2\pi i\gamma_1}}\left\langle f*g\,(t)\,, e^{\frac{i}{2}\left(\frac{\alpha_1}{\gamma_1}\right)t^2}\sin\left(\frac{s}{\gamma_1}t\right)\right\rangle \\ &= \frac{e^{\frac{i}{2}\left(\frac{\alpha_2}{\gamma_1}\right)s^2}}{\sqrt{2\pi i\gamma_1}}\left\langle f\,(t)\,, \left\langle g\,(x)\,, e^{\frac{i}{2}\left(\frac{\alpha_1}{\gamma_1}\right)(t+x)^2}\sin\left(\frac{s}{\gamma_1}(t+x)\right)\right\rangle\right\rangle.\end{aligned}$$

Using the fact that

$$\sin\left(\frac{st}{\gamma_1}+\frac{sx}{\gamma_1}\right) = \sin\frac{st}{\gamma_1}\cos\frac{sx}{\gamma_1} + \sin\frac{sx}{\gamma_1}\cos\frac{st}{\gamma_1}$$

we get

$$\begin{aligned}&\overrightarrow{\mathrm{F}_s}\,(f*g)\,(s) = \frac{e^{\frac{i}{2}\left(\frac{\alpha_2}{\gamma_1}\right)s^2}}{\sqrt{2\pi i\gamma_1}}\times \\ &\quad\left\langle f\,(t)\,, \left\langle g\,(x)\,, e^{\frac{i}{2}\left(\frac{\alpha_1}{\gamma_1}\right)(t+x)^2}\left(\sin\frac{st}{\gamma_1}\cos\frac{sx}{\gamma_1} + \sin\frac{sx}{\gamma_1}\cos\frac{st}{\gamma_1}\right)\right\rangle\right\rangle \\ &= \frac{e^{\frac{i}{2}\left(\frac{\alpha_2}{\gamma_1}\right)s^2}}{\sqrt{2\pi i\gamma_1}}\left\langle f\,(t)\,, \left\langle g\,(x)\,, e^{\frac{i}{2}\left(\frac{\alpha_1}{\gamma_1}\right)(t+x)^2}\sin\frac{st}{\gamma_1}\cos\frac{sx}{\gamma_1}\right\rangle\right\rangle \\ &\quad + \frac{e^{\frac{i}{2}\left(\frac{\alpha_2}{\gamma_1}\right)s^2}}{\sqrt{2\pi i\gamma_1}}\left\langle f\,(t)\,, \left\langle g\,(x)\,, e^{\frac{i}{2}\left(\frac{\alpha_1}{\gamma_1}\right)(t+x)^2}\sin\frac{sx}{\gamma_1}\cos\frac{st}{\gamma_1}\right\rangle\right\rangle \\ &= \left\langle e^{\frac{i}{2}\left(\frac{\alpha_1}{\gamma_1}\right)t^2}f\,(t)\sin\frac{st}{\gamma_1}, \left\langle e^{\frac{i}{2}\left(\frac{\alpha_1}{\gamma_1}\right)(2tx)}g\,(x)\,, \frac{e^{\frac{i}{2}\left(\frac{\alpha_2}{\gamma_1}\right)s^2}e^{\frac{i}{2}\left(\frac{\alpha_2}{\gamma_1}\right)x^2}}{\sqrt{2\pi i\gamma_1}}\cos\frac{sx}{\gamma_1}\right\rangle\right\rangle \\ &\quad + \left\langle e^{\frac{i}{2}\left(\frac{\alpha_1}{\gamma_1}\right)t^2}f\,(t)\cos\frac{st}{\gamma_1}, \left\langle e^{\frac{i}{2}\left(\frac{\alpha_1}{\gamma_1}\right)(2tx)}g\,(x)\,, \frac{e^{\frac{i}{2}\left(\frac{\alpha_2}{\gamma_1}\right)s^2}e^{\frac{i}{2}\left(\frac{\alpha_2}{\gamma_1}\right)x^2}}{\sqrt{2\pi i\gamma_1}}\sin\frac{sx}{\gamma_1}\right\rangle\right\rangle\end{aligned}$$

$$= \left\langle e^{\frac{i}{2}\left(\frac{\alpha_1}{\gamma_1}\right)t^2} f(t) \sin \frac{st}{\gamma_1}, \overrightarrow{\mathrm{F}_c} \left(e^{\frac{i}{2}\left(\frac{\alpha_1}{\gamma_1}\right)(2tx)} g(x) \right)(t) \right\rangle$$
$$+ \left\langle e^{\frac{i}{2}\left(\frac{\alpha_1}{\gamma_1}\right)t^2} f(t) \cos \frac{st}{\gamma_1}, \overrightarrow{\mathrm{F}_s} \left(e^{\frac{i}{2}\left(\frac{\alpha_1}{\gamma_1}\right)(2tx)} g(x) \right)(t) \right\rangle .$$

This completes the proof of the theorem.

Theorem 2.5. *Let* $f \in \mathcal{E}'(\mathtt{R})$ *then we have*

$$\left(\overrightarrow{\mathrm{F}_s} \sin(ut) f(t)\right)(s) = \frac{e^{-\frac{i}{2}\left(\alpha_2 u^2 \gamma_1\right)}}{2} e^{-i\alpha_2 su} \left(\overrightarrow{\mathrm{F}_c} f\right)(s + \gamma_1 u) .$$

Proof. Let $f \in \mathcal{E}'(\mathtt{R})$ then

$$\left(\overrightarrow{\mathrm{F}_s} \sin(ut) f(t)\right)(s) = \frac{e^{\frac{i}{2}\left(\frac{\alpha_2}{\gamma_1}\right)s^2}}{\sqrt{2\pi i \gamma_1}} \left\langle \sin(ut) f(t), e^{\frac{i}{2}\left(\frac{\alpha_1}{\gamma_1}\right)t^2} \sin\left(\frac{s}{\gamma_1} t\right) \right\rangle$$
$$= \frac{e^{\frac{i}{2}\left(\frac{\alpha_2}{\gamma_1}\right)s^2}}{\sqrt{2\pi i \gamma_1}} \left\langle f(t), e^{\frac{i}{2}\left(\frac{\alpha_1}{\gamma_1}\right)t^2} \sin(ut) f(t) \sin\left(\frac{s}{\gamma_1} t\right) \right\rangle . \quad (13)$$

Using the identity

$$2 \sin x \sin y = \cos(x - y) - \cos(x + y) ,$$

(13) becomes

$$\left(\overrightarrow{\mathrm{F}_s} \sin(ut) f(t)\right)(s)$$
$$= \frac{e^{\frac{i}{2}\left(\frac{\alpha_2}{\gamma_1}\right)s^2}}{\sqrt{2\pi i \gamma_1}} \left\langle f(t), e^{\frac{i}{2}\left(\frac{\alpha_1}{\gamma_1}\right)t^2} \frac{\left(\cos\left(\frac{s}{\gamma_1}t - ut\right) - \cos\left(\frac{s}{\gamma_1}t + ut\right)\right)}{2} \right\rangle . \quad (14)$$

Calculations on (14) together with the equation

$$e^{\frac{i}{2}\left(\frac{\alpha_2}{\gamma_1}\right)s^2} = e^{-\frac{i}{2}\left(\alpha_2 u^2 \gamma_1\right)} . e^{\frac{i}{2}\left(\frac{\alpha_2}{\gamma_1}\right)\left(s - u\gamma_1^2\right)} . e^{i\alpha_2 us}$$

and that

$$e^{\frac{i}{2}\left(\frac{\alpha_2}{\gamma_1}\right)s^2} = e^{-\frac{i}{2}\left(\alpha_2 u^2 \gamma_1\right)} . e^{\frac{i}{2}\left(\frac{\alpha_2}{\gamma_1}\right)\left(s + u\gamma_1^2\right)} . e^{-i\alpha_2 us},$$

when employed to (14), give

$$\left(\overrightarrow{\mathrm{F}_s} \sin(ut) f(t)\right)(s) \quad = \quad \left(\frac{1}{2} e^{-\frac{i}{2}\left(\alpha_2 u^2 \gamma_1\right)} . e^{i\alpha_2 su}\right) \frac{e^{\frac{i}{2}\left(\frac{\alpha_2}{\gamma_1}\right)(s - u\gamma_1)^2}}{\sqrt{2\pi i \gamma_1}}$$

$$\times\left\langle f\left(t\right),e^{\frac{i}{2}\left(\frac{\alpha_1}{\gamma_1}\right)t^2}\cos\left(\tfrac{s-u\gamma_1}{\gamma_1}\right)t\right\rangle$$
$$-\left(\frac{1}{2}e^{-\frac{i}{2}\left(\alpha_2u^2\gamma_1\right)}e^{-i\alpha_2su}\right)$$
$$\times\frac{e^{\frac{i}{2}\left(\frac{\alpha_2}{\gamma_1}\right)(s-u\gamma_1)^2}}{\sqrt{2\pi i\gamma_1}}\left\langle f\left(t\right),e^{\frac{i}{2}\left(\frac{\alpha_1}{\gamma_1}\right)t^2}\cos\left(\tfrac{s+u\gamma_1}{\gamma_1}\right)t\right\rangle.$$

which produces our desired result.

Theorem 2.6. Let $f\in\mathcal{E}'\left(\mathtt{R}\right)$ then

$$\left(\overrightarrow{\mathrm{F}}_c\cos\left(ut\right)f\left(t\right)\right)(s)=\frac{e^{-\frac{i}{2}\left(\alpha_2u^2\gamma_1\right)}}{2}\left(\begin{array}{c}e^{-i\alpha_2su}\left(\overrightarrow{\mathrm{F}}_sf\right)(s+\gamma_1u)+\\e^{i\alpha_2su}\left(\overrightarrow{\mathrm{F}}_sf\right)(s-\gamma_1u)\end{array}\right).$$

Proof of this theorem is quite analogous to that of Theorem 2.5, thus avoided.

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