International Journal of Pure and Applied Mathematics
Volume 87 No. 3 2013, 405-413
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v87i3.4
ijpam.eu

ON 3-DIMENSIONAL (□,δ)-TRANS-SASAKIAN STRUCTURE

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Abstract: The object of present paper is to study 3-dimensional (ε, δ)-trans-Sasakian manifold admitting Ricci solitons and K-torse forming vector fields. We prove the conditions for the Ricci solitons to be shrinking, expanding and steady. Further, we have obtained a condition for the vector field ξ in a generalized recurrent 3-dimensional (ε, δ)-trans-Sasakian manifold to be co-symplectic. We have also shown that such a manifold M is of constant scalar curvature.

AMS Subject Classification: 53D10, 53D15
Key Words: (ϵ, δ)trans-Sasakian manifold, Ricci soliton, shrinking, steady, expanding, K-torseforming vector field

1. Introduction

The concept of (ϵ)-Sasakian manifolds was introduced by A. Bejancu and K.L. Duggal [1] and further investigation was taken up by Xufend and Xiaoli[13] and Rakesh kumar et al [7]. De and Sarkar [3] introduced and studied conformally flat,Weyl semisymmetric, φ-recurrent (ε)-Kenmotsu manifolds. In [1], the authors obtained Riemannian curvature tensor of (ϵ)-Sasakian manifolds and established relations among different curvatures. H.G. Nagraja et al [4] have studied (ϵ, δ)-trans-Sasakian structures which generalizes both (ϵ)-manifolds and (ϵ)-Kenmotsu manifolds.

Received: April 13, 2013

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2. Preliminaries

Let (M,g) be an almost contact metric manifold of dimension n equipped with an almost contact metric structure (φ,ξ,η,g) consisting of a (1,1) tensor field φ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \tag{2.1}$$

$$\eta(\xi) = 1, \tag{2.2}$$

$$\varphi\xi = 0, \eta \circ \varphi = 0. \tag{2.3}$$

An almost contact metric manifold M is called an (ϵ)-almost contact metric manifold if

$$g(\xi,\xi) = \varepsilon, \tag{2.4}$$

$$\eta(X) = \varepsilon g(X,\xi), \tag{2.5}$$

$$g(\phi X, \phi Y) = g(X,Y) - \varepsilon\eta(X)\eta(Y), \forall X, Y \in TM, \tag{2.6}$$

where $\varepsilon = g(\xi,\xi) = \pm 1$. An (ε)-almost contact metric manifold is called an (ε,δ)-trans-Sasakian manifold if

$$(\nabla_X \phi)Y = \alpha[g(X,Y)\xi - \varepsilon\eta(Y)X] + \beta[g(\phi X,Y)\xi - \delta\eta(Y)\phi X], \tag{2.7}$$

holds for some smooth functions α and β on M and $\varepsilon = \pm 1$, $\delta = \pm 1$. For $\beta = 0$,$\alpha = 1$, an (ε,δ)-trans-Sasakian manifold reduces to an (ε)-Sasakian and for $\alpha = 0, \beta = 1$ it reduces to a (δ)-Kenmotsu manifold.

Let (M,g) be a (ε,δ)-trans-Sasakian manifold. Then from (2.7), it is easy to see that

$$(\nabla_X \xi) = -\varepsilon\alpha\phi X - \beta\delta\phi^2 X, \tag{2.8}$$

$$(\nabla_X \eta)Y = -\alpha g(Y,\phi X) + \varepsilon\delta\beta g(\phi X,\phi Y), \tag{2.9}$$

$$\xi\alpha + 2\alpha\beta = 0. \tag{2.10}$$

In a 3-dimensional (ε,δ)-trans-Sasakian manifold, the curvature tensor R and Ricci tensor S are given by[5]

$$\begin{aligned} R(X,Y)Z =& (2A - \frac{r}{2})(g(Y,Z)X - g(X,Z)Y) + B(g(Y,Z)\eta(X) - g(X,Z)\eta(Y))\xi \\ &+ B\eta(Z)(\eta(Y)X - \eta(X)Y), \end{aligned} \tag{2.11}$$

$$R(X,Y)\xi = \varepsilon[(\alpha^2 - \beta^2) + (\frac{2-\varepsilon}{2})r][\eta(Y)X - \eta(X)Y], \tag{2.12}$$

$$S(X,Y) = Ag(X,Y) + B\eta(X)\eta(Y), \tag{2.13}$$

where $\varepsilon\delta = 1$, $A = (\frac{r}{2} - (\alpha^2 - \beta^2))$, $B = (3(\alpha^2 - \beta^2) - \varepsilon\frac{r}{2})$ and r is the scalar curvature.

Let (M, g) be a Riemannian manifold with metric g. The metric g is called a Ricci soliton if[8]

$$L_V g + 2S + 2\lambda g = 0, \tag{2.14}$$

where L is the Lie derivative, S is a Ricci tensor, V is a complete vector field on M and λ is a constant. So Ricci soliton is a generalization of Einstein metric. In[9],Ramesh sharma started the study of the Ricci solitons in contact geometry. Later Mukutmani Tripathi[11], Cornelia Livia Bejan and Mircea Crasmareanu [2] and others extensively studied Ricci solitons in contact metric manifolds. The Ricci soliton is said to be shrinking, steady and expanding according as λ is positive, zero and negative respectively.

We also study K-torse forming vector fields in 3-dimensional (ε, δ)-trans-Sasakian manifold. Torse forming vector fields were introduced by K.Yano [6],

$$\nabla_X \rho = aX + \pi(X)\rho, \tag{2.15}$$

where ρ is a vector field and π is a non-zero 1-form. Further, a complex analogue of a torse forming vector field is called K-torse forming vector field and it was introduced by S.Yamaguchi[10],

$$\nabla_X \rho = aX + b\varphi X + B(X)\rho + D(X)\varphi\rho, \tag{2.16}$$

where ρ is a vector field, a and b are functions, B(X) and D(X) are 1-forms.

A 3-dimensional (ε, δ)-trans-Sasakian manifold (M, g) is called generalized recurrent[12], if its curvature tensor R satisfies the condition

$$(\nabla_X R)(Y,Z)(W) = A(X)R(Y,Z)W + B(X)[g(Z,W)Y - g(Y,W)Z)], \tag{2.17}$$

where A and B are two 1-forms and B is non zero.

3. Ricci Soliton

Let M be a 3-dimensional (ε, δ)-trans-Sasakian manifold with metric g. A Ricci soliton is a generalization of an Einstein metric and defined on a Riemannian manifold (M, g) by (2.14).

Let V be pointwise collinear with ξ ie.,$V = b\xi$,Then (2.14) implies

$$g(\nabla_X b\xi, Y) + g(\nabla_Y b\xi, X) + 2S(X,Y) + 2\lambda g(X,Y) = 0, \tag{3.1}$$

which is reduced to

$$bg(\nabla_X \xi, Y) + (Xb)\eta(Y) + bg(\nabla_Y \xi, X) + (Yb)\eta(X) + 2S(X,Y) + 2\lambda g(X,Y) = 0. \tag{3.2}$$

Using (2.8) in (3.2), we obtain

$$2b\beta\delta g(X,Y) - 2b\beta\eta(X)\eta(Y) + (Xb)\eta(Y) + (Yb)\eta(X) + 2S(X,Y) + 2\lambda g(X,Y) = 0. \tag{3.3}$$

Replace Y by ξ in (3.3) and use (2.13) to get

$$2b\beta\delta\varepsilon\eta(X) - 2b\beta\eta(X) + (Xb) + (\xi b)\eta(X) + 2A\varepsilon\eta(X) + 2B\eta(X) + 2\lambda\varepsilon\eta(X) = 0. \tag{3.4}$$

Again putting $X = \xi$ in (3.4), we obtain

$$\xi b = -b\beta\delta\varepsilon + b\beta - A\varepsilon - B - \lambda\varepsilon. \tag{3.5}$$

By using (3.5) in (3.4), we get

$$Xb = [-b\beta\delta\varepsilon + b\beta - A\varepsilon - B - \lambda\varepsilon]\eta(X), \tag{3.6}$$

or

$$db = [-b\beta\delta\varepsilon + b\beta - A\varepsilon - B - \lambda\varepsilon]\eta, \tag{3.7}$$

where we have put $\nabla_X = d$.

Applying d on (3.7), we get

$$[b\beta\delta\varepsilon - b\beta + A\varepsilon + B + \lambda\varepsilon]d\eta = 0. \tag{3.8}$$

Since $d\eta \neq 0$ we have

$$b\beta\delta\varepsilon - b\beta + A\varepsilon + B + \lambda\varepsilon = 0. \tag{3.9}$$

Using (3.9) in (3.7) then $db = 0$ and hence b is a constant. Therefore from (3.3) it follows

$$S(X,Y) = -(\lambda + b\beta\delta)g(X,Y) + b\beta\eta(X)\eta(Y), \tag{3.10}$$

which implies that M is of constant scalar curvature provided $\beta = constant$. This leads to the following:

Theorem 3.1. *Let M be a 3-dimensional (ε, δ)-trans-Sasakian manifold in which the metric tensor g is Ricci soliton with the vector field V as pointwise collinear with ξ, then M is a space of constant scalar curvature provided β is a constant.*

Now let $V = \xi$. Then the (2.14) becomes

$$L_\xi g + 2S + 2\lambda g = 0. \tag{3.11}$$

Using (2.8), we get

$$L_\xi g(X,Y) = 2\beta\delta g(X,Y) - 2\beta\delta\varepsilon\eta(X)\eta(Y), \tag{3.12}$$

and therefore,

$$(L_\xi g + 2S)(X,Y) = 2\beta\delta[g(X,Y) - \varepsilon\eta(X)\eta(Y)] - 2[(\frac{r}{2} - (\alpha^2 - \beta^2))g(X,Y) + (3(\alpha^2 - \beta^2) - \varepsilon\frac{r}{2})\eta(X)\eta(Y)]. \tag{3.13}$$

Using (3.13) in (3.11), we obtain

$$2\left[-\frac{r}{2} + \beta\delta + (\alpha^2 - \beta^2) + \lambda\right] g(X,Y) - 2\left[\beta\delta\varepsilon + 3(\alpha^2 - \beta^2) - \varepsilon\frac{r}{2}\right]\eta(X)\eta(Y) = 0. \tag{3.14}$$

Take $X = Y = \xi$ in (3.14). We get

$$\lambda = \frac{(\beta^2 - \alpha^2)(\varepsilon + 3) - r(\varepsilon - 1)}{2\varepsilon}. \tag{3.15}$$

Hence we state the following

Theorem 3.2. *Let M be a 3-dimensional (ε, δ)-trans-Sasakian manifold. Then a Ricci soliton (g, ξ, λ) in (M, g) is:*

(i) shrinking for $(\beta^2 - \alpha^2)$ $(\varepsilon + 3) > r(\varepsilon\text{-}1)$,

(ii) expanding for $(\beta^2 - \alpha^2)$ $(\varepsilon + 3) < r(\varepsilon\text{-}1)$.

Corollary 3.1. *In a 3-dimensional (ε, δ)-trans-Sasakian manifold M Ricci soliton is:*

(i) shrinking if $\beta > \alpha$ and $\varepsilon = 1$,

(ii) expanding if $\beta < \alpha$ and $\varepsilon = 1$,

(iii) steady if $\alpha = \beta$ and $\varepsilon = 1$.

Suppose (M, g) is a 3-dimensional (ε, δ)-trans-Sasakian manifold and (g, V, λ) is a Ricci soliton in (M, g). If V is a conformal killing vector field, then

$$L_V g = \rho g, \tag{3.16}$$

for some scalar function ρ. From (2.14)

$$S(X,Y) = -[\lambda g(X,Y) + \frac{1}{2} L_V g(X,Y)]. \tag{3.17}$$

Now from (3.16) it follows that

$$S(X,Y) = -(\lambda + \frac{\rho}{2}) g(X,Y), \tag{3.18}$$

$$QX = -(\lambda + \frac{\rho}{2}) X, \tag{3.19}$$

$$r = -3(\lambda + \frac{\rho}{2}). \tag{3.20}$$

Put $X = Z = \xi$ and use (3.20) in (2.11) to obtain

$$R(\xi, Y)\xi = (\alpha^2 - \beta^2)(3 - 2\varepsilon)[\eta(Y)\xi - Y]. \tag{3.21}$$

Put $X = \xi$ in (2.12) to obtain

$$R(\xi, Y)\xi = \varepsilon \left[(\alpha^2 - \beta^2) + \left(\frac{2 - \varepsilon}{2} \right) r \right] [\eta(Y)\xi - Y]. \tag{3.22}$$

Comparing (3.21)and (3.22), we obtain

$$r = \frac{6(1 - \varepsilon)(\alpha^2 - \beta^2)}{(2\varepsilon - 1)}. \tag{3.23}$$

By using (3.20) and (3.23), we get

$$\lambda = \frac{2(\varepsilon - 1)(\alpha^2 - \beta^2)}{(2\varepsilon - 1)} - \frac{\rho}{2}. \tag{3.24}$$

Theorem 3.3. *Let M be a 3-dimensional (ε, δ)-trans-Sasakian manifold admitting Ricci soliton (g, V, λ), where V is conformal killing vector field. Then (g, V, λ) is:*

i) expanding for $\rho > \frac{4(\varepsilon-1)(\alpha^2-\beta^2)}{(2\varepsilon-1)}$,

ii) shrinking for $\rho < \frac{4(\varepsilon-1)(\alpha^2-\beta^2)}{(2\varepsilon-1)}$

iii) steady for $\rho = \frac{4(\varepsilon-1)(\alpha^2-\beta^2)}{(2\varepsilon-1)}$.

4. K-Torse Forming Vector Field

In 3-dimensional (ε, δ)-trans-Sasakian manifold, ξ is always a K-torse forming vector field. Take $\rho = \xi$ in (2.16) and compare the result with (2.8) to obtain $a = \beta\delta$, $b = -\varepsilon\alpha$, $B(X) = -\beta\eta(X)$ and D(X)=0. Then it implies

$$\nabla_X \xi = \beta\delta X - \varepsilon\alpha\varphi X - \beta\eta(X)\xi, \tag{4.1}$$

$$(\nabla_X \eta)Y = \beta\delta g(X,Y) - \varepsilon\alpha g(\varphi X, Y) - \varepsilon\beta\eta(X)\eta(Y), \tag{4.2}$$

$$R(X,Y)\xi = (a\beta + b\alpha)[\eta(X)Y - \eta(Y)X], \tag{4.3}$$

$$R(\xi, X)Y = (a\beta + b\alpha)[\eta(Y)X - \varepsilon g(X,Y)\xi], \tag{4.4}$$

and

$$S(Y,\xi) = -2(a\beta + b\alpha)\eta(Y), \tag{4.5}$$

where α and β are constants.

Taking $Y = W = \xi$ in (2.17),we obtain

$$(\nabla_X R)(\xi, Z)(\xi) = A(X)R(\xi, Z)\xi + B(X)[\varepsilon\eta(Z)\xi - \varepsilon Z]. \tag{4.6}$$

By the definition of covariant derivative, we have

$$(\nabla_X R)(\xi, Z)(\xi) = \nabla_X R(\xi, Z)\xi - R(\nabla_X \xi, Z)\xi - R(\xi, \nabla_X Z)\xi - R(\xi, Z)\nabla_X \xi. \tag{4.7}$$

Using (4.1),(4.2) and (4.4) in (4.7), we get

$$\begin{aligned}(\nabla_X R)(\xi, Z)(\xi) &= d(a\beta + b\alpha)(X)[Z - \eta(Z)\xi] + (a\beta + b\alpha)[(1-\delta)\beta g(X,Z)\xi \\ &+ (\varepsilon - 1)\alpha g(\varphi X, Z)\xi + (\varepsilon - 1)\beta\eta(X)\eta(Z)\xi + 2\eta(\nabla_X Z)\xi - 2\beta\eta(X)Z + 2\varepsilon\beta\eta(X)Z].\end{aligned} \tag{4.8}$$

From (4.6) and (4.8), we have

$$\begin{aligned}&d(a\beta + b\alpha)(X)[Z - \eta(Z)\xi] + (a\beta + b\alpha)[(1-\delta)\beta g(X,Z)\xi + (\varepsilon - 1)\alpha g(\varphi X, Z)\xi \\ &+ (\varepsilon - 1)\beta\eta(X)\eta(Z)\xi + 2\eta(\nabla_X Z)\xi - 2\beta\eta(X)Z + 2\varepsilon\beta\eta(X)Z] \\ &= A(X)[(a\beta + b\alpha)(Z - \eta(Z)\xi)] + B(X)[\varepsilon\eta(Z)\xi - \varepsilon Z].\end{aligned} \tag{4.9}$$

Put $Z = \xi$ and use (2.2),(2.4),(2.5) in (4.9) to obtain

$$(a\beta + b\alpha)[2(\varepsilon - 1)\beta\eta(X)\xi + \eta(\nabla_X \xi)\xi] = 0. \tag{4.10}$$

If $(a\beta + b\alpha) \neq 0$ and $\varepsilon = 1$, then (4.10) is

$$\nabla_X \xi = 0. \tag{4.11}$$

Thus we have

Theorem 4.4. *In a generalized recurrent 3-dimensional (ε,δ)-trans-Sasakian manifold M the vector field ξ is co-symplectic provided $(a\beta+b\alpha)\neq 0$ and $\varepsilon=1$.*

Suppose the Ricci tensor S is semi conjugate in 3-dimensional (ε,δ)-trans-Sasakian manifold M with respect to the vector field ξ, which is K-torse forming vector field in M. So $R(X,\xi).S(Y,Z)=0$.

Then we have

$$S(R(X,\xi)Y,Z)+S(Y,R(X,\xi)Z)=0. \tag{4.12}$$

Using (4.4) and (4.5) in (4.12), we get

$$\begin{aligned}(a\beta+b\alpha)[S(X,Z)\eta(Y)+2\varepsilon(a\beta+b\alpha)g(Y,X)\eta(Z)+S(Y,X)\eta(Z)\\ +2\varepsilon(a\beta+b\alpha)g(X,Z)\eta(Y)]=0.\end{aligned} \tag{4.13}$$

Put $Z=\xi$ and use (2.5),(4.5) in (4.13) to obtain

$$S(X,Y)=-2\varepsilon(a\beta+b\alpha)g(X,Y), \tag{4.14}$$

which gives

$$r=-6\varepsilon(a\beta+b\alpha), \tag{4.15}$$

where a and b are constants. Hence r is constant.

Theorem 4.5. *Suppose the ricci tensor S in a 3-dimensional (ε,δ)-trans-Sasakian manifold M is semi-conjugate with respect the vector field ξ.Then M is a space of constant scalar curvature.*

References

[1] Bejancu A. and Duggal K.L.,Real hypersurfaces of idefinite Kahler manifolds, Int.J.Math.Math.sci.,16,No.3(1993),545-556.

[2] Cornelia Livia Benjan and Mircea Crasmareanu, Ricci Solitons in manifolds with Quasi-constant curvature, Publ.Math.Debrecen, 78,No.1(2011),235-243.

[3] De.U.C. and Sarkar A.,On (ε)-Kenmotsu manifolds, Hadronic journal, 32 (2009),231-242.

[4] Halammanavar G.Nagaraja, Rangaswami C.Premalatha, Ganganna Somashekara, On an (ε,δ)-trans-Sasakian structure, Proceedings of the Estonian Academy of sciences,61,No.1(2012),20-28.

[5] H.G.Nagaraja and Somashekhara,Special Trans-Sasakian manifolds and curvature conditions, International journal of pure and applied Mathematics,81, 3 (2012),411-420.

[6] K.Yano, On the torse forming direction in Riemannian spaces, Proc.Imp.Acd.Tokyo., 20 (1944),340-345.

[7] R.Kumar, R.Rani and R.K.Nagaich,On sectional curvature of (ε)-Sasakian manifolds, Int.J.Math.Math.sci.,(2007),ID.93562.

[8] R.S.Hamilton, The Ricci flow on surfaces, Mathematics and general relativity(Santa Cruz, CA,1986),Contemp.Math.,71,237-262,American Math.soc.,(1988).

[9] Sinha B.B. and Ramesh sharma,On Para-A-Einstein manifolds, Publications De L'Institute Mathematique.Nouvelle Serie,34,No.48(1983),211-215.

[10] S.Yamaguchi, On Kaehlerian torse forming vector fields, Kodai Math.J.,2(1979),103-115.

[11] Tripathi M.M., Ricci solitons in contact metric manifolds, arXiv:0801.4222v1,[math.DG],28(2008).

[12] U.C.De, N.Guha and D.Kamilya, On generalized Ricci-recurrent manifolds, Tensor,N.S.,56(1995),312-317.

[13] X.Xufeng, C.Xiaoli, Two theorems on ε-Sasakian manifolds, Int.J.Math.Math.sci.,21,No.2(1998),249-254.