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SUBSPACE MIXING

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Abstract: In this paper we introduce subspace-mixing and certain set $L(T, M, x)$, which we call it *M-limit set* of x under T and construct an example of a subspace-mixing operator. In the sequel, considering $L(T, M, x)$, the answer of this question:" does there exist an operator T such that T and $T^{\perp}$ are M-mixing and $M^{\perp}$-mixing, respectively?" is negative.

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1. Introduction

Let X be a Banach space. In what follows, the symbol T stands for a bounded linear operator acting on X and M will be a nonzero closed subspace of X. Consider any subset D of X. The symbol $Orb(T, D)$ denotes the orbit of D under T, i. e. $Orb(T, D) = \{T^n x : x \in D, n = 0, 1, 2, ...\}$. If $D = \{x\}$ is a singleton and the orbit $Orb(T, x)$ is dense in X, then the operator T is called hypercyclic and the vector x is a hypercyclic vector for T. If $D = \{\lambda x : \lambda \in C\}$ and the set $Orb(T, D)$ is dense in X, then the operator T is called supercyclic and the vector x is a supercyclic vector for T. Observe that in the case where the operator is hypercyclic the underlying Banach space X should

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be separable. Then it is well known and easy to show that an operator T is hypercyclic if and only if T is topologically transitive, to be precise, for every pair of nonempty open sets U, V of X there exists a non-negative integer n such that $T^{\mathsf{n}}(U) \cap V \neq \emptyset$, [2]. See the recent books [1], [5] and the survey article [6] and the references therein for some details on hypercyclicity and supercyclicity and related properties.

As in [4], [3] we say a bounded linear operator T on X is topologically mixing if for any nonempty open sets U, V there exists some non-negative integer N such that $T^{\mathsf{n}}(U) \cap V \neq \emptyset$ for all $n \geq N$. And We say that T is weakly mixing if $T \oplus T$is topologically transitive on $X \times X$, where

$$T \oplus T : X \times X \longrightarrow X \times X, \quad (T \oplus T)(x_1, x_2) = (Tx_1, Tx_2)$$

Recently, B. F. Madore and R. A. Martinez-Avendano in [8] introduced the concept of subspace-hypercyclicity. An operator T is subspace-hypercyclic (or M-hypercyclic) for a subspace M of X if there exists a vector x such that the intersection of its orbit and M is dense in M. They also introduces the notion of subspace-transitivity. An operator T is subspace-transitive (or *M-transitive*) for a nonzero closed subspace M of X if for any nonempty open subsets U, V of M there exists a non-negative integer n such that $T^{-n}(U) \cap V$ is a relatively open nonempty subset of M. They proved subspace-transitivity implies subspace- hypercyclicity, and C. M. Le in [7] constructed an operator T such that it is subspace-hypercyclic but it is not subspace-transitive. After that in [12], authors introduced subspace-chaos and subspaceweakly mixing and they investigated its related properties. Another nice article about subspace-hypercyclicity is [9].

In this paper we introduce subspace-mixing and give an example of a subspace-mixing operator, then we show that subspace-mixing implies subspace- weakly mixing.

In [11], Salas answered positively this question:" does there exist an operator T such that both T and T^* are hypercyclic?". This is natural to ask the analogous question:" does there exist an operator T such that T and T^* are M-mixing and M^*-mixing, respectively?". In the sequel, we introduce certain set *M-limit set* of x under T, $L(T, M, x)$, which leads to negative answer to above question.

2. Preliminaries and Basic Notions

Definition 2.1. An operator $T \in B(X)$ is called topologically mixing or mixing if for any nonempty open subsets U, V of T there exists a non-negative integer N such that $T^{-n}(U) \cap V \neq \emptyset$ for all $n \geq N$.

Definition 2.2. Let $T \in B(X)$. We say that T is *M-hypersyclic* if there exists $x \in X$ such that $Orb(T, x) \cap M$ is dense in M. Such a vector x is called an *M-hypersyclic* vector for T.

Definition 2.3. Let $T \in B(X)$. We say that T is *M-transitive* if for any nonempty subsets U and V, both relatively open in M, there exists $n \geq 0$ such that $T^{-n}(U) \cap V$ is a relatively open nonempty subset of M.

Definition 2.4. Let $T \in B(X)$. Then T is M-weakly mixing if the operator $T \oplus T$ on $X \times X$ is $(M \times M)$-transitive.

Definition 2.5. Let $T \in B(X)$. We say that T is *M-mixing* if for any nonempty sets $U \subseteq M$ and $V \subseteq M$, both relatively open, there exists $N \geq 0$ such that for all $n \geq N$, $T^{-n}(U) \cap V$ is a relatively open nonempty subset of M.

The following theorem charactrize the notion of M-mixing.

Theorem 2.6. *Let $T \in B(X)$. Then the following conditions are equivalent:*

(i) The operator T is M-mixing.

(ii) For any nonempty sets $U \subseteq M$ and $V \subseteq M$, both relatively open, there exists $N \geq 0$ such that for all $n \geq N$, $T^{-n}(U) \cap V$ contains a relatively open nonempty subset of M.

(iii) For any nonempty sets $U \subseteq M$ and $V \subseteq M$, both relatively open, there exists $N \geq 0$ such that for all $n \geq N$, $T^{-n}(U) \cap V$ is nonempty and $T^n(M) \subseteq M$.

Proof. We first prove that *(ii)* implies *(iii)*. Let U and V be nonempty relatively open subsets of M. By hypothesis, it follows that;

$$\exists N \geq 0 \quad such\ that \quad \forall n \geq N, \quad T^{-n}(U) \cap V,$$

contains a relatively open nonempty set, say W_n. Now let $x \in M$, $n \geq N$. Since $W_n \subseteq T^{-n}(U)$, it follows that $T^n(W_n) \subseteq M$. Take $x_n \in W_n$, then for $r_n > 0$ small enough, we have $x_n + r_n x \in W_n$, and hence $T^n(x_n + r_n x) \in M$. Since $r_n > 0$ and $T^n(x_n) \in M$, it follows that $T^n x \in M$, showing $T^n(M) \subseteq M$.

Since the implication *(i)* $\Longrightarrow$ *(ii)* is obvious, so it is sufficient to prove that *(iii)* implies *(i)*. Let U and V be relatively open nonempty subsets of U . By assumption, it follows that there exists a non-negative integer N such that for all $n \geq N$;

$$T^n(M) \subseteq M \quad and \quad T^{-n}(U) \cap V \neq \emptyset.$$

Take $n \geq N$. Hence $(T^n)_{|_M} : M \longrightarrow M$ is continuous and consequentely $(T^n)_{|_M}(U)$ is relatively open in M. Therefore $T^{-n}(U) \cap V$ is a relatively open nonempty subset of M. □

Definition 2.7. Let $T \in B(X)$. Then for any subsets $A, B \subseteq M$, the *return set from A to B* defined as

$$N_{(T,M)}(A, B) = \{n \geq 0 : T^{-n}(A) \cap B \ \ is\ nonempty\ open\ subset\ of\ \ M\}$$

In this notation, T is M-transitive and M-mixing if and only if, for any nonempty sets $U \subseteq M$ and $V \subseteq M$, both relatively open, the return set $N_{(T,M)}(U, V)$ is nonempty and cofinite, respectively.

Theorem 2.8. *Let T be an M-mixing operator. Then T is M-weakly mixing.*

Proof. Let U_i, V_i, $i = 1, 2$ be relatively open nonempty subsets of M. Then

$$\exists N_i \geq 0 \quad such \ \ that \ \ \forall n \geq N_i; \quad n \in N_{(T,M)}(U_i, V_i), \ \ i = 1, 2.$$

Set $N = max\{N_1,\ N_2\}$ and take $n \geq N$. Thus

$$(T \oplus T)^{-n}(U_1 \times U_2) \cap (V_1 \times V_2) = (T^{-n}(U_1) \cap V_1) \times (T^{-n}(U_2) \cap V_2)$$

is a relatively open nonempty subset of $M \times M$. Therefore $T \oplus T$ is subspace-transitive with respect to $M \times M$. □

Definition 2.9. Let T be an operator. For every $x \in M$ the set

$$L(T, M, x) = \{y \in X : there\ exists\ a\ strictly\ increasing\ sequence\ of\ positive\ integers\ \{k_n\}\ such\ that\ T^{k_n}x \longrightarrow y\ and\ for\ every\ n,\ T^{k_n}(M) \subseteq M\}$$

denote the (*M-limit set*) or *subspace-limit set* of x under T with respect to M.

Remark 2.10. Observe that for all x the set $L(T, M, x)$ is a closed subset of M.

3. An Example and Main Results

Example 3.1. Let $T = 2B$ where B is the backward shift on l^2, i. e. for every $x = (x_0, x_1, x_2, ...) \in l^2$, $B(x_0, x_1, x_2, ...) = (x_1, x_2, ...)$. $T \oplus I : l^2 \times l^2 \longrightarrow l^2 \times l^2$ is M-hypercyclic where $M = l^2 \times \{0\}$, but $T \oplus I$ is not hypercyclic, see [8]. Note that

$$T^n(x_0, x_1, x_2, ...) = 2^n(x_n, x_{n+1}, ...).$$

Let $U \subseteq l^2$, $V \subseteq l^2$, both relatively open, and $x = (x_0, x_1, x_2, ...) \in U$, $y = (y_0, y_1, y_2, ...) \in V$. Then $U \times \{0\}$ and $V \times \{0\}$ are two elements in basis of M. If

$$z_n = (x_0, x_1, ..., x_{n-1}, \frac{y_0}{2^n}, \frac{y_1}{2^n}, ..., \frac{y_{n-1}}{2^n}, \frac{y_0}{2^{2n}}, \frac{y_1}{2^{2n}}, ..., \frac{y_{n-1}}{2^{2n}}, ...),\ n = 1, 2, ...$$

then $z_n \longrightarrow x$ and $T^n z_n \longrightarrow y$. Hence there exists a non-negative integer N such that;

$$\forall n \geq N, \qquad (T \oplus I)^n(U \times \{0\}) \cap (V \times \{0\}) \neq \emptyset.$$

Since $(T \oplus I)(M) \subseteq M$, *Theorem 2.6* implies $N_{(T \oplus I, M)}(U \times \{0\}, V \times \{0\})$ is cofinite and consequently $T \oplus I$ is M-mixing.

The subspace-mixing property consists in demanding the cofiniteness of the return sets $N_{(T,M)}(U, V)$ for each pair U, V, relatively open nonempty subsets of M. This requirement can be weakened as follows.

As usual, we set $A + B = \{a + b :\ a \in A,\ b \in B\}$ for subsets A, B of a Banach space. In this notation, let U be a relatively open nonempty subset of M. Then there is a relatively open nonempty subset U_1 of M and a relatively open neighbourhood W of zero such that $U_1 + W \subset U$, see [10] .

Theorem 3.2. *An operator T is M-mixing if and if for any nonempty relatively open set $U \subseteq M$ and any relatively open neighbourhood $W \subseteq M$ of zero, the return sets*

$$N_{(T,M)}(U,W) \ \textit{and} \ N_{(T,M)}(W,U)$$

are cofinite.

Proof. It is suffices to show sufficiency of the condition. Let $U \subseteq M$ and $V \subseteq M$, both relatively open nonempty sets. There are relatively open nonempty sets U_1, V_1 and a relatively open neighbourhood $W \subseteq M$ of zero, such that $U_1 + W \subseteq U$ and $V_1 + W \subseteq V$. By hypothesis, there exists a non-negative integer N such that for any $n \geq N$,

$$T^{-n}(U_1) \cap W \quad \textit{and} \quad T^{-n}(W) \cap V_1$$

are nonempty and $T^n(M) \subseteq M$. Now if $n \geq N$, then there exists $w \in W$ such that $T^{-n}w \in V_1$, and there exists $u \in U_1$ such that $T^{-n}u \in W$. Thus

$$T^{-n}u + T^{-n}w \in V \quad \textit{and} \quad T^n(T^{-n}u + T^{-n}w) \in U.$$

Consequently $V \cap T^{-n}(U)$ is nonempty and $T^n(M) \subseteq M$. Finally, *Theorem 2.6* concludes that $N_{(T,M)}(U,V)$ is cofinite.

□

In what follows, we show that there is not any M-mixing operator whose adjoint is M^*-mixing.

Remark 3.3. Let $T \in B(X)$ and $x \in M$. Observe that $L(T,M,x) = M$ if and only if T is an M-hypercyclic operator and x is an M-hypercyclic vector for T.

Theorem 3.4. *Let $T \in B(X)$ be an M-mixing operator. Then*

$$\forall x^* \in M^*, \quad L(T^*, M^*, x^*) = \emptyset.$$

Proof. If not, then there exists a non-zero $x^* \in M^*$ such that

$$L(T^*, M^*, x^*) \neq \emptyset$$

or equivalently there exists a strictly increasing sequence of positive integers $\{k_n\}$ such that the sequence $\{(T^*)^{k_n} x^*\} \subset M^*$ is convergent and consequently bounded. Hence there exists $L > 0$ such that

$$\| (T^*)^{k_n} x^* \| \leq L, \quad for\ every\ n = 1, 2, \dots\ .$$

Since x^* is a continuous map, so

$$U = \{x \in M : \| x \| < 1\} \quad \textit{and} \quad V = M \cap (x^*)^{-1}[= B(0,L)]^c$$

are open subsets of M. By assumption, there exists $N > k_1$ such that

$$\forall n \geq N, \quad n \in N_{(T,M)}(U,V).$$

Take $n \geq N$, then there exists an $x \in U$ such that $T^{k_n}x \in V$. Hence

$$L < |< T^{k_n}x, x^* >| = |< x, (T^*)^{k_n}x^* >| \leq \| (T^*)^{k_n}x^* \|$$

which is a contradiction. □

Theorem 3.5. *There is no M-mixing operator whose adjoint is M^*-mixing.*

Proof. If T is M-mixing and T^* is M^*-mixing, then *Remark 3.3* implies that for some nonzero vector $x^* \in M^*$, $L(T^*, M^*, x^*) = M^*$ which is contradiction with previous theorem. □

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