International Journal of Pure and Applied Mathematics
Volume 88 No. 2 2013, 173-182
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v88i2.3

ijpam.eu

POSTULATION OF CURVES CONTAINED IN A UNION OF HYPERPLANES OF $\mathbb{P}^4$

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY

Abstract: Let $A \subset \mathbb{P}^4$ be a union of m distinct hyperplanes. In this note for many m, d we prove the existence of reduced, connected and nodal curves $C \subset A$ with $\deg(C) = d$, $p_a(C) = 0$ and maximal rank, i.e. $h^0(A, \mathcal{I}_{C,A}(t)) \cdot h^1(A, \mathcal{I}_{C,A}(t)) = 0$ for all $t \in \mathbb{N}$.

AMS Subject Classification: 14H50, 14N05
Key Words: postulation, reducible curve, reducible hypersurface, Hilbert function

1. The Statements

Let $A \subset \mathbb{P}^r$ be a reduced, but reducible hypersurface. For most quadruples $(d, g, r, \deg(A))$ there are reduced and connected and (say) nodal curve with degree d and arithmetic genus g which are contained in no irreducible hypersurface of degree $\deg(A)$ of $\mathbb{P}^r$. Hence allowing reducible hypersurface it is quite easy to construct reducible curve $C \subset A$ with prescribed degree and genera and some other properties (e.g. good postulation) (see [3], [2]). Here " good postulation " means that C has *maximal rank* in A, i.e. for each integer $t > 0$ the restriction map $r_{A,Y,t} : H^0(A, \mathcal{O}_A(t)) \to H^0(C, \mathcal{O}_C(t))$ has maximal rank as a linear map (i.e. it is either injective or surjective). Since A is arithmetically Cohen-

Received: June 1, 2013

Macaulay, C has maximal rank if and only if $h^0(A, \mathcal{I}_C(t)) \cdot h^1(A, \mathcal{I}_C(t)) = 0$ for all $t \in \mathbb{Z}$. In this paper we take $r = 4$ and as A a reduced union of m hyperplanes.

Let $A \subset \mathbb{P}^4$ be a reduced union of $m \geq 2$ hyperplanes. For each integer $k > m$ we have $h^0(A, \mathcal{O}_A(k)) = \binom{k+4}{4} - \binom{k-m+4}{4}$. This equality explains the integers appearing in the statements of Theorems 1 and 2.

Theorem 1. *Fix integers $k \geq 2m \geq 4$ and $d \geq 4$ such that $kd + 1 \leq \binom{k+4}{4} - \binom{k-m+4}{4}$. Let $A = H_1 \cup \cdots \cup H_m$ be a union of m hyperplanes such that $H_i \neq H_j$ for all $i \neq j$, the intersection of any 3 of them is a line, the intersection of any 4 of them is a point and no 5 of them has a common point. Then there exists a reduced, connected and nodal curve $C \subset A$ such that $\deg(C) = d$, $p_a(C) = 0$ and $h^1(A, \mathcal{I}_{C,A}(k)) = 0$.*

Theorem 2. *Fix integers $k \geq 2m \geq 4$ and $d \geq 4$ such that $kd + 1 \geq \binom{k+4}{4} - \binom{k-m+4}{4}$. Let $A = H_1 \cup \cdots \cup H_m$ be a union of m hyperplanes such that $H_i \neq H_j$ for all $i \neq j$, the intersection of any 3 of them is a line, the intersection of any 4 of them is a point and no 5 of them has a common point. Then there exists a reduced, connected and nodal curve $C \subset A$ such that $\deg(C) = d$, $p_a(C) = 0$ and $h^0(A, \mathcal{I}_{C,A}(k)) = 0$.*

Fix integers $m, k, , d$ such that $m \geq k + 2 \geq 4$, $d \geq 4$, $kd + 1 \leq \binom{k+4}{4} - \binom{k-m+4}{4}$ and $(k-1)d + 1 > \binom{k+3}{4} - \binom{k-m+3}{4}$ and A as in Theorems 1 and 2. We do not claim the existence of $C \subset A$ as in Theorem 1 with the additional condition $h^0(A, \mathcal{I}_{C,A}(k-1)) = 0$, i.e. we do not claim that C has maximal rank, i.e. the same curve may be used for Theorem 1 and for the case $k' := k - 1$ of Theorem 2. In most cases this is a byproduct of our proof, but for numerical reasons we are unable to prove it in all cases. We prove in the case $m = 2$, i.e. we prove the following result.

Theorem 3. *Fix hyperplanes $H_i \subset \mathbb{P}^4$, $i = 1, 2$ such that $H_1 \neq H_2$. Set $A := H_1 \cup H_2$. For each integer $d > 0$ there is a connected and nodal curve $C \subset A$ such that $\deg(C) = d$, $p_a(C) = 0$ and C has maximal rank.*

The curve C in Theorems 1, 2 and 3 satisfies $h^1(C, \mathcal{O}_C(1)) = 0$. It seems obvious how to extend these theorems to curves with arithmetic genus g at least if $d \gg g$. This in left to any interested reader (the only problems should be numerical, the most annoying ones being the ones related to Lemma 1 and Claim 3 of the proof of Theorem 1).

2. The Proofs

For any hyperplane $H \subset \mathbb{P}^4$ and all integers $a \geq 0$ and $b \geq 0$ Let $Z(H, a, b)$ denote the set of all smooth curve $E \subset H$ which are the disjoint union of a smooth rational curve of degree a and b lines. $Z(H, a, b)$ is a quasi-projective variety of dimension $4a + 4b$.

Let $H \subset \mathbb{P}^4$ be any hyperplane. Let $Z \subset H$ be any closed subscheme with $\dim(Z) \leq 1$. We do not assume that Z is equidimensional, because we need the concepts which are discussing here for the scheme $E_i \cap H_i$ which is the disjoint union of a very nice curve and finitely many points. If either $\dim(Z) = 0$ or each connected component of the one-dimensional part of Z has degree 1, then set $e(Z) = -1$. In all other cases let $e(Z)$ be the first integer $t \geq 0$ such that $h^1(Z, \mathcal{O}_Z(t)) = 0$. We have $h^1(Z, \mathcal{O}_Z(t)) = 0$ for all $t \geq e(Z)$ and either $\dim(Z) = 0$ or $h^1(Z, \mathcal{O}_Z(t)) > 0$ for each $t < e(Z)$. Since each connected component of C_i is a reduced curve with arithmetic genus 0 and one connected component of C_i is not a line, we have $e(C_i) = 0$. Since $E_i \cap H_i$ is the disjoint union of C_i and finitely many points, we have $e(E_i \cap H_i) = 0$. Let $c(Z)$ be the minimal integer $t > e(Z)$ such that $h^0(Z, \mathcal{O}_Z(t)) \leq \binom{t+3}{3}$. Fix an integer $x > e(Z)$ and assume $h^1(H, \mathcal{I}_{Z,H}(x)) = 0$. Since $x > e(Z)$, we have $h^2(H, \mathcal{I}_{Z,H}(x-1)) = h^1(Z, \mathcal{O}_Z(x-1)) = 0$. Hence Castelnuovo-Mumford's lemma gives $h^1(H, \mathcal{I}_{Z,H}(t)) = 0$ for all $t \geq x$. Hence $c(Z)$ is the first integer $t > e(Z)$ such that $h^1(H, \mathcal{I}_{Z,H}(t)) = 0$. Assume (as in our cases with C_i or $E_i \cap H$ as Z) that $h^0(Z, \mathcal{O}_Z(e(Z)) \geq \binom{e(Z)+3}{3}$. We say that Z has maximal rank if $h^1(H, \mathcal{I}_{Z,H}(c(Z)) = 0$ and $h^1(H, \mathcal{I}_{Z,H}(c(Z)-1))$. In our set-up C_i has maximal rank, while $E_i \cap H$ has maximal rank if and only if $h^1(H_i, \mathcal{I}_{E_i \cap H_i, H_i}(k+i-1)) = 0$.

Lemma 1. *Let $H \subset \mathbb{P}^4$ be a hyperplane. Fix a reduced curve $D \subset H$ such that each connected component of D has arithmetic genus 0. Set $d := \deg(D)$ and let u be the number of the connected components of D. Assume that D has maximal rank. Fix integers $b \geq 0$ and $f > 0$. Take a finite set $B \subset H \setminus D$ such that $\sharp(B) = b$ and $h^0(H, \mathcal{I}_{D \cup B}(t)) = \max\{0, h^0(H, \mathcal{I}_D(t)) - b\}$ for all $t > 0$. Set $w := c(D \cup B)$ and $a := h^0(H, \mathcal{I}_{D \cup B}(w)) = \binom{w+3}{3} - wd - u - b$. If $f > a$, then assume $f + d \leq \binom{w+2}{2}$. Fix a plane $M \subset H$ such that $B \cap M = \emptyset$ and no irreducible component of D is contained in M.Then $D \cup B$ and $D \cup B \cup F$ has maximal rank.*

Proof. Obviously $D \cup B$ has maximal rank. The integer $c(D \cup B)$ is the first positive integer t such that $td + u + b \leq \binom{t+3}{3}$. We order the points $P_1, \ldots, P_f$ of F. For each integer x set $S_x := \cup_{i \leq x} P_i$. Notice that $S_0 = \emptyset$ and

$S_f = F$. Since $D \cup B$ has maximal rank, we have $h^0(H, \mathcal{I}_{D\cup B}(w)) = a$ and $h^0(H, \mathcal{I}_{D\cup B}(w-1)) = 0$. First assume $f \le a$ (this is the case if and only if $c(D \cup B \cup F) = w$). Since $h^0(H, \mathcal{I}_{D\cup B}(w-1)) = 0$, we have $h^0(H, \mathcal{I}_{D\cup B\cup F}(w-1)) = 0$. Since $e(D \cup B \cup F) = e(D) \le 0$, Castelnuovo-Mumford's lemma shows that $D\cup B\cup F$ has maximal rank, it is sufficient to prove $h^1(H, \mathcal{I}_{D\cup B\cup F}(w)) = 0$. Let y be the maximal integer $\le m$ such $h^1(H, \mathcal{I}_{D\cup B\cup S_y}(w)) = 0$. Assume $y < f$. Since $h^1(H, \mathcal{I}_{D\cup B\cup S_y}(w)) = 0$, we have $h^0(H, \mathcal{I}_{D\cup B\cup S_y}(w)) = a - y$. Since $h^1(H, \mathcal{I}_{D\cup B\cup S_{y+1}}(w)) > 0$ and P_{y+1} is a general in M, M is in the base locus of $|\mathcal{I}_{D\cup B\cup S_y}(w)|$. Since M contains no irreducible component of D and $B\cap M = \emptyset$, we get $h^0(H, \mathcal{I}_{D\cup B}(w-1)) \ge a-y > 0$, a contradiction. Now assume $f > a$. Take any $F' \subset F$ such that $\sharp(F') = a$. The first part of the proof gives $h^0(H, \mathcal{I}_{D\cup B\cup F'}(w)) = 0$. Hence $h^0(H, \mathcal{I}_{D\cup B\cup F}(w)) = 0$. Hence to prove that $D\cup B\cup F$ has maximal rank it is sufficient to prove that $h^1(H, \mathcal{I}_{D\cup B\cup F}(w+1)) = 0$. Let z be the maximal integer $\le f$ such that $h^1(H, \mathcal{I}_{D\cup B\cup S_z}(w+1)) = 0$. Assume $z < f$. We have $h^0(H, \mathcal{I}_{D\cup B\cup S_z}(w+1)) = \binom{w+4}{3} - (w+1)d - u - b - z$. Since P_{z+1} is a general point of M, we get that M is contained in the base locus of $|\mathcal{I}_{D\cup B\cup S_z}(w+1)|$. Hence $h^0(H, \mathcal{I}_{D\cup B\cup S_z}(w+1)) = h^0(H, \mathcal{I}_{D\cup B}(w))$, i.e. $\binom{w+4}{3} - (w+1)d - u - b - z = \binom{w+3}{3} - wd - u - b$. Hence $m > z = \binom{w+2}{2} - d$, a contradiction. □

Proof of Theorem 1. For all integers $k > m \ge 1$ define the integers $d_{k,m}$ and $a_{k,m}$ by the following relations

$$kd_{k,m} + 1 + a_{k,m} = \binom{k+4}{4} - \binom{k+4-m}{4}, \ 0 \le a_{k,m} \le k-1 \tag{1}$$

Set $x_{m,k,m} := d_{k-m+1,1}$ and $y_{m,k,m} := a_{k,k-m+1}$. Fix an integer i such that $1 \le i < m$ and assume defined the integers $x_{j,k,m}$ and $y_{j,k,m}$ for all $j \in \{i+1, \ldots, m\}$. Define the integers $x_{i,k,m}$ and $y_{i,k,m}$ by the relations

$$\begin{aligned}(k+1-i)x_{i,k,m} + 1 + y_{i,k,m} - y_{i+1,k,m} + \sum_{j=i+1}^{m} x_{i,k,m} \\ = \binom{k-i+1}{3}, \ 0 \le y_{i,k,m} \le k-i\end{aligned} \tag{2}$$

Hence $d_{k,m} = \sum_{i=1}^{m} x_{i,k,m}$

Claim 1. *$x_{i,k,m} > 0$ and $x_{i,k,m} \le (k+4-i)(k+3-i)(k+2-j)/6(k+1-i)$ for all i.*

Proof of Claim 1. We have $x_{m,k,m} = \lfloor (\binom{k+4-m}{3} - 1)/k \rfloor > 0$. Hence $0 < x_{m,k,m} \le (k+4-m)(k+3-m)(k+2-m)/6(k+1-m)$. Fix an integer $i \in \{1, \dots, m\}$ and assume $0 < x_{j,k,m} \le (k+4-j)(k+3-j)/6$ for all $j \in \{i+1, \dots, m\}$. Since $x_{j,k,m} > 0$ for all $j > 0$, we have $x_i \le (k+4-i)(k+3-i)(k+2-j)/6(k+1-i)$. Since $(m-i)(k+3-i)(k+2-i)/6(k-i) + 1 + (k+1-i) + (k-i) \le \binom{k-i+1}{3}$, we get $b_{i,k,m} > 0$, concluding the proof of Claim 1.

Claim 2. Assume $k \ge m+2$. Then $x_{i,k,m} \ge x+1-i$.

Proof of Claim 2. Assume $x_{i,k,m} \le x - i$. First assume $i = m$. By the case $m = 1$ and $k' := k - m + 1$ of (1) we get $(k+1-m)^2 + 1 > \binom{k+4-m}{3}$, i.e. $6(k+1-m)^2 \ge (k+4-m)(k+3-m)(k+2-m)$, contradicting the assumption $k > m$. Now assume $i < m$ and that i is the maximal positive integer for which Claim 2 fails. Claim 1 gives $x_{j,k,m} \le (k+4-j)(k+3-j)(k+2-j)/6(k+1-j)$ for all j. We have $y_{i,k,m} \le k - i$ and $y_{i+1,k,m} \ge 0$. Hence from (2) we get

$$(k+1-i)(k+1-i) + \sum_{j=i+1}^{m} (k+4-j)(k+3-j)(k+2-j)/6(k+1-j) \le \binom{k+1-i}{3} \quad (3)$$

Hence

$$6(k+1-i)(k+1-i)(k-i) + (m-i)(k+3-i)(k+2-i)(k+1-i) \le (k+4-i)(k+3-i)(k+2-i)(k-i) \quad (4)$$

with strict inequality if $i \ne m-1$. First assume $i \ne m-1$. Since $k \ge m+2$ and $i < m$, we have $k+4-i \ge m+6-i$. Hence we get a contradiction. Now assume $i = m-1$. From (4) we get $6(k+2-m)(k+2-m)(k+1-m) + (k+4-m)(k+3-m)(k+2-m) \le (k+5-m)(k+4-m)(k+3-m)(k+1-m)$, a contradiction. Let $C_m \subset H_m$ be a general union of $y_{m,k,m}$ lines and a smooth rational curve of degree $x_{m,k,m} - y_{m,k,m}$. By [2], C_m has maximal rank in H_m. By our definition of the integers $x_{m,k,m}$ and $y_{m,k,m}$ we have $h^j(H_m, \mathcal{I}_{C_m,H_m}(k-m+1)) = 0$, $j = 1, 2$. Let N_{C_m,H_m} denote the normal bundle of C_m in H_m. Since each connected component of C_m is smooth and rational and the tangent bundle of H_m is a quotient of $\mathcal{O}_{H_m}(1)^{\oplus 4}$ by the Euler's sequence, we have $h^1(C_m, N_{C_m,H_m}(-1)) = 0$. By [4] we get that for general C_m each set $C_m \cap H_i$, $1 \le i < m$, is formed by $x_{m,k,m}$ general points of H_i. Fix an integer $i \in \{1, \dots, m-1\}$ and assume defined the curves $C_j \subset H_j$, $i+1 \le j \le m$, with the following properties:

(a) $C_j \subset H_j$ is a disjoint union of $y_{j,k,m}$ lines and a smooth rational curve of degree $x_{y,k,m} - y_{j,k,m}$;

(b) Each curve $E_j := \cup_{h=j}^m C_h$ is nodal, with exactly $y_{j,k,m}$ connected component, each of them with arithmetic genus 0;

(c) Each C_j is transversal to H_h for all $h \neq j$;

(d) no C_j contains a point common to 3 of the planes H_h;

(e) for each h, j such that $1 \le h < j \le m$ the set $E_j \cap H_h$ is general in H_h.

Part (b) implies that each irreducible component of E_j is smooth and rational. Since $x_{i,k,m} \ge k+1-i$ we may find a disjoint union $C_i \subset H_i$ of $x_{i,k,m} - y_{i,k,m}$ lines and a smooth rational curve, such that no point of C_i is contained in two other hyperplanes H_h, $h \neq i$, and $E_i := E_{i-1} \cup C_i$ satisfies properties (b) and (c) above. Since for a general $S \subset H_i \cap H_{i-1}$ with $\sharp(S) = x_{i,k,m}$ there is $E \in Z(H_i, x_{i,k,m} - y_{i,k,m}, y_{i,k,m})$ containing it, we may also assume that C_i is a general element of $Z(H_i, x_{i,k,m} - y_{i,k,m}, y_{i,k,m})$ (seen as an abstract curve in H_i, independently of the curve E_{i-1} constructed before). Hence C_i has maximal rank as a curve as a curve of H_i. We may also assume that $E_i := E_{i+1} \cup C_i$ satisfies condition (d). Set $E_m := C_m$. Since $E_m = C_m \subset H_m$, we have $C_m = E_m \cap H_m$ (scheme-theoretic intersection). For any $i \in \{1, \dots, m-1\}$ the set $E_i \cap H_i$ is a disjoint union of C_i and the points of $E_{i+1} \cap H_i$ not contained in C_i. Each of these points corresponds to a reduced connected component of the scheme-theoretic intersection $E_i \cap H_i$, because E_i is a nodal curve. Now fix $P \in C_i \cap E_{i-1}$ (if any). We have $P \in H_i \cap H_{i-1}$ and $P \in C_{i-1}$. Since E_i is nodal and the tangent line to C_{i-1} is not contained in H_i, $E_i \cap H_i$ and C_i coincide in a neighborhood of P. Hence the scheme-theoretic and the set-theoretic intersections of E_i and H_i are the same. We again remark that this construction make sense, because $\sum_{j=i+1}^m b_j - y_{i-1,k,m} \ge 0$; indeed, $y_{i-1,k,m} \le (k-i) \le b_{i-1,k,m} \le \sum_{j=i+1}^m b_{j,k,m}$, the second inequality being true by Claim 2. For all $i \in \{1, \dots, m-1\}$ the set $E_i \cap H_i \setminus C_i$ has exactly $\sum_{j=i+1}^m b_{j,k,m} - y_{i-1,k,m}$ points.

Claim 3. *For each $i \in \{1, \dots, m\}$ we have $h^j(H_i, \mathcal{I}_{E_i \cap H_i, H_i}(k+i-1)) = 0$, $j = 0, 1$.*

Proof of Claim 3. The case $i = m$ is true by our choice of the curve C_m. Now assume $i \in \{1, \dots, m-1\}$. The scheme $E_i \cap H_i$ is the disjoint union of C_i and finitely many points. We have $h^0(C_i, \mathcal{O}_{C_i}(k+i-1)) = (k+i-1)x_{i,k,m} + 1 + y_{i,k,m}$. By (3) we have $h^0(E_i \cap H_i, \mathcal{O}_{E_i \cap H_i}(k+1-i)) = \binom{k+4-i}{3}$. Hence $h^0(H_i, \mathcal{I}_{E_i \cap H_i, H_i}(k+i-1)) = h^0(H_i, \mathcal{I}_{E_i \cap H_i}(k+i-1))$. Thus it is sufficient to show that either $h^0(H_i, \mathcal{I}_{E_i \cap H_i}(k+i-1)) = 0$ or $h^1(H_i, \mathcal{I}_{E_i \cap H_i}(k+i-1)) = 0$. In our set-up we have $H := H_i$, $D := C_i$ (which have maximal rank) and

$f := \sharp(S) = -y_{i+1.k,m} + \sum_{j=i+1}^{m} x_{j,k,m}$. However, S is not general in H_i, since we prescribed the values of the cardinalities of the sets $S_j := S \cap H_j$, $i+1 \le j \le m$ and $S \subset H_{j+1} \cup \cdots \cup H_m$. We apply $m-i+1$ times Lemma 1.

Claim 4. *For each* $i \in \{1, \dots, m\}$ *have* $h^j(H_i \cup \cdots \cup H_m, \mathcal{I}_{E_i, H_i \cup \cdots \cup H_m}(k)) = 0$, $j = 0, 1$.

Proof of Claim 4. Since $E_m = C_m$, Claim 4 is true for $i = m$. Now assume $i < m$ and that Claim 4 is true for the integer $i+1$. Since the set-theoretic and the scheme-theoretic intersection of H_{i-1} and E_i are the same, we have an exact sequence of sheaves on $H_i \cup \cdots \cup H_m$:

$$0 \to \mathcal{I}_{E_{i-\mathbf{1}}, H_{i-\mathbf{1}} \cup \cdots \cup H_m}(k-i) \to \mathcal{I}_{E_i, H_i \cup \cdots \cup H_m}(k)) \to \mathcal{I}_{E_i \cap H_i, H_i}(k+1-i) \to 0 \quad (5)$$

The inductive assumption gives $h^j((H_{i+1}, i \cup \cdots \cup H_m, \mathcal{I}_{E_{i-\mathbf{1}}, H_{i-\mathbf{1}} \cup \cdots \cup E_m}(k-i)) = 0$, $j = 0, 1$. Claim 3 gives $h^j(H_i, \mathcal{I}_{E_i \cap H_i}(k+i-1)) = 0$, $j = 0, 1$. Apply (5).

Claim 5. *We have* $h^1(A, \mathcal{I}_{E_{\mathbf{1}}, A}(k)) = 0$.

Proof of Claim 5. This is the case $i = m$ of Claim 4.

Recall that $d \le d_{k,m} = \sum_{i=1}^{m} x_{i,k,m}$. Let c be the minimal integer $\le m$ such that $d \le \sum_{i=c}^{m} x_{c,k,m}$. First assume $c = m$. In this case we may take as C a general element of $Z(H_m, d, 0)$. Since $d \le x_{k+1-m,k,m}$ and C has maximal rank ([3]), we have $h^1(H_m, \mathcal{I}_{C,H_m}(k+1-m)) = 0$. Hence $h^1(H_m, \mathcal{I}_{C,H_m}(k)) = 0$. Since the restriction map $H^0(A, \mathcal{O}_A(k)) \to H^0(H_m, \mathcal{O}_{H_m}(k))$ is surjective, we get $h^1(A, \mathcal{I}_{C,A}(k)) = 0$. Hence we may assume $1 \le c < m$. First assume $d \ge \sum_{i=c+1}^{m} x_{i,k,m} + y_{c+1,k,m} + 1$. Take as C the union of E_{c-1} with a general $F \in Z(H_c, d - x_{c+1,k,m}, 0)$ with the only restriction that $F \cap H_{c-1}$ contains exactly one point of each of the connected components of E_{c-1}. Since $d \ge x_{c-1,k,m} + y_{c-1,k,m} + 1$, F may be considered as a general smooth rational curve of degree $d - x_{c+1,k,m}$ of H_c. Hence it has maximal rank ([3]). As in the proofs of Claims 3 and 4 we get $h^1(H_c \cup \cdots \cup C_m, \mathcal{I}_{C, H_c \cup \cdots \cup C_m}(k+1-c)) = 0$. Hence $h^1(H_c \cup \cdots \cup C_m, \mathcal{I}_{C, H_c \cup \cdots \cup C_m}(k)) = 0$. Hence $h^1(A, \mathcal{I}_{C,A}(k)) = 0$. Now assume $d \le \sum_{i=c+1}^{m} x_{i,k,m} + y_{c-1,k,m}$. Instead of E_{c+1} we take the following curve F_{c+1}. We start with E_{c+2} (with the convention $E_{c+2} = \emptyset$ if $c = m-1$. Let D_{c+1} be a general element of $Z(H_{c+1}, x_{c+1,k,m}, 0)$ with the only condition that $D_{c+1} \cap H_{c+2}$ contains exactly one point of each of the $y_{c+2,k,m} + 1$ connected components of E_{c+2}. □

Proof of Theorem 2. Take the proof just given for the integer $k' := k+1$ and make minimal modifications. □

Proof of Theorem 3. Take $A = H_1 \cup H_2 \subset \mathbb{P}^4$ with H_1 and H_2 hyperplanes and $H_1 \ne H_2$. For all integers $k \ge 3$ we have $h^0(A, \mathcal{O}_A(k)) = \binom{k+4}{4} - \binom{k+2}{4} =$

$2\binom{k+3}{3} - \binom{k+2}{2} = (k+2)(k+1)(2k+3)/6$. We have $h^0(A, \mathcal{O}_A(2)) = 14$ and $h^0(A, \mathcal{O}_A(1)) = 4$. Set $d_1 := 4$, $\alpha_1 := 0$, $d_2 := 6$ and $\alpha_2 := 1$. For all integer $k \geq 3$ define the integers d_k and α_k by the relations

$$kd_k + \alpha_k = \binom{k+4}{4} - \binom{k+2}{4},\ 0 \leq \alpha_k \leq k-1 \tag{6}$$

(a) We have $d_k := \lfloor ((k+2)(k+1)(2k+3) - 6)/6k \rfloor = \lfloor (2k^2 + 9k + 13)/6$. Hence

$$(2k^2 + 9k + 13)/6 - 1 \leq d_k \leq (2k^2 + 9k + 13)/6 \tag{7}$$

Claim 1. *$d_k > d_{k-1}$ for all $k \geq 2$.*

Proof of Claim 1. Since $d_3 = 10$, we may assume $k \geq 4$. Subtracting the case $k' := k-1$ of (6) from (6) we get

$$(k-1)(d_k - d_{k-1}) + d_k + \alpha_k - \alpha_{k-1} = (k+1)^2 \tag{8}$$

Assume $d_k \leq d_{k-1}$, Since $\alpha_k \leq k-1$, step (a) gives a contradiction.

Fix an integer $d > 0$ and let k be the minimal positive integer such that $d \leq d_k$. Claim 1 gives that k is the only integer such that $d_{k-1} < d \leq d_k$. To prove Theorem 3 for the integer d it is sufficient to prove the existence of a reduced, connected and nodal curve $C \subset A$ such that $\deg(C) = d$, $p_a(C) = 0$, $h^0(A, \mathcal{I}_{C,A}(k-1)) = 0$ and $h^1(A, \mathcal{I}_{C,A}(t)) = 0$ for all $t \geq k$. Since $h^1(C, \mathcal{O}_C) = 0$ for any reduced and connected curve with arithmetic genus zero, Castelnuovo-Mumford's lemma gives that it is sufficient to prove the existence a reduced, connected and nodal curve $C \subset A$ such that $\deg(C) = d$, $p_a(C) = 0$, $h^0(A, \mathcal{I}_{C,A}(k-1)) = 0$ and $h^1(A, \mathcal{I}_{C,A}(t)) = 0$. The case $k = 1$, i.e. $d \leq 4$, is obvious.

(b) Assume $k = 2$, i.e. assume $5 \leq d \leq 7$. Take as $C = A_1 \cup A_2$ the union of rational normal curve $A_1 \subset H_1$ and a general $A_2 \in Z(H_2, d-3, 0)$ meeting C_1 at one point.

(c) From now on we assume $k \geq 3$. For all integers $t > 0$, define the integers u_t and γ_t by the relations

$$tu_t + 1 + \gamma_t = \binom{t+3}{3},\ 0 \leq \gamma_t \leq t-1 \tag{9}$$

The explicit values of the integers u_t and γ_t given in [3] show that $u_t > \gamma_t$ for all $t \geq 2$. Let $C_1 \subset H_1$ be a general union of a smooth rational curve of degree $u_{k-1}-\gamma_{k-1}$ and γ_{k-1} lines. By [2] we have $h^i(H_1,\mathcal{I}_{C_1,H_1}(k-1))=0$, $i=0,1$. Let $M_1\subset H_1$ be a general union of a smooth rational curve of degree $u_{k-2}-\gamma_{k-2}$ and γ_{k-2} lines. By [2] we have $h^i(H_1,\mathcal{I}_{M_1,H_1}(k-2))=0$, $i=0,1$.

(d) In this step we assume

$$(k-1)(d-u_{k-1})+1+u_{k-1}-\gamma_{k-1}\geq \binom{k+2}{3} \tag{10}$$

Claim 2. *Assume (10). Then* $d-u_{k-1}\geq \gamma_{k-1}+1$.

Proof of Claim 2. Assume $d-u_{k-1}\leq \gamma_{k-1}$. From (10) we get $(k-2)\gamma_{k-1}+1+u_{k-1}\geq \binom{k+2}{3}$. Since $u_{k-1}\leq \binom{k+2}{3}/(k-1)$ and $\gamma_{k-1}\leq k-2$, we get a contradiction.

Let C_1 be a general element of $Z(H_1,d-u_{k-1},0)$, with the only restriction that $C_1\cap H_2$ contains exactly one point of each connected component of C_2. Set $C:=C_1\cup C_2$. Since $d\leq d_k$, Lemma 1 gives $h^1(H_1,\mathcal{I}_{C\cap H_1,H_1}(k))=0$. Since $h^1(H_2,\mathcal{I}_{C_2,H_2}(k-1))=0$, we get $h^1(A,\mathcal{I}_{C,A}(k))=0$. From (10) and the generality of the set $C_2\cap H_1$ in the plane $H_1\cap H_2$, we get $h^0(H_1,\mathcal{I}_{C\cap H_1,H_1}(k-1))=0$. Since $h^0(H_2,\mathcal{I}_{C_2,H_2}(k-1))=0$, we get $h^0(A,\mathcal{I}_{C,A}(k-1))=0$.

(e) In this step we assume $d\leq d_k-u_{k-1}+u_{k-2}$ and

$$(k-1)(d-u_{k-1})+u_{k-1}-\gamma_{k-1}\leq \binom{k+2}{3} \tag{11}$$

Let $M_1\subset H_1$ be a general element of $Z(H_1,d-u_{k-2},0)$, with the only restriction that $M_1\cap H_2$ contains exactly one point of each of the components of M_2; since $d>d_{k-1}$ and $d_{k-1}\geq u_{k-2}+k-3$, this is obviously possible. In this case we take $C:=M_1\cup M_2$. We first check that $h^0(A,\mathcal{I}_{C,A}(k-1))=0$. Since $d>d_{k-1}$, this is just a modification of step (b), taking $k-1$ instead of k. The scheme $C\cap H_1$ is a general union of a smooth rational curve of degree $d-u_{k-2}$ and $u_{k-2}-\gamma_{k-2}$ general points of H_2. The assumption " $d\leq d_k-u_{k-1}+u_{k-2}$ " gives $h^0(C\cap H_1,\mathcal{O}_{C\cap H_1}(k))\leq \binom{k+3}{3}$. Hence $h^1(H_1,\mathcal{I}_{C_1\cap H_1,H_1}(k))=0$ (Lemma 1). Hence $h^1(A,\mathcal{I}_{C,A}(k))=0$.

(g) In this step we assume $d>d_k-u_{k-1}+u_{k-2}$ and prove that (10) is satisfied. Indeed, we have $(k-1)(d-u_{k-1})+1+u_{k-1}-\gamma_{k-1}\geq (k-1)(d_k-u_{k-1})+1+u_{k-1}-\gamma_{k-1}+(k-1)u_{k-1}=(k-1)(d_k-u_{k-1})+1+u_{k-1}-\gamma_{k-1}+\binom{k+2}{3}-\gamma_{k-1}$. □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] E. Ballico, Ph. Ellia, On postulation of curves in $\mathbb{P}^4$, *Math. Z.*, **188** (1985), 215-223.

[2] R. Hartshorne, A. Hirschowitz, Courbes rationnelles et droites en position générale, *Ann. Inst. Fourier*, Grenoble, **35**, No. 4 (1985), 39-58.

[3] A. Hirschowitz, Sur la postulation générique des courbes rationnelles, *Acta Math.*, **146** (1981), 209-230.

[4] D. Perrin, Courbes passant par m points g'enéraux de $\mathbb{P}^3$, *Bull. Soc. Math. France*, Mémoire 28/29 (1987).