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NUMERICAL RADIUS PRESERVING LINEAR MAPS ON BANACH ALGEBRAS

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Abstract: Let A and B be unital complex Banach algebras, and φ be a unital surjective numerical radius preserving linear map from A into B. We discuss a Nagasawa type theorem for this maps and show that φ is a Jordan isomorphism, if A and B are commutative.

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1. Introduction

Linear preserver problems(LPP)have a relatively long history and different kinds. Some of the most popular linear preserver problems are linear maps preserving properties related to invertibility or spectrum. This subject goes back to 1897, when G. Frobenius describing the structure of determinant preserving linear maps. In 1970 Kaplansky [6] asked the following question:

Let $\varphi : A \longrightarrow B$ be a unital invertibility preserving linear map between unital Banach algebras A and B. Is φ a Jordan homomorphism?

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The above question of Kaplansky is too general and answer to it is negative when φ is not surjective or A and B are not semi-simple, thus it was reduced to the following conjecture

Conjecture: Suppose A and B are unital semisimple Banach algebras and $\varphi : A \longrightarrow B$ is a unital surjective linear map preserving invertibility. Is φ a Jordan homomorphism?

This problem remains unsolved. A number of partial positive results have been found, especially in the case where $A = \mathbf{B}(X)$, $B = \mathbf{B}(Y)$ [9] and when A and B are Von Neumann algebra [2].

A similar conjecture is as follows:

Let A and B be unital semisimple Banach algebras and $\varphi : A \longrightarrow B$ is a unital surjective spectral isometry. Is φ a Jordan homomorphism?

A partial positive answers were given to this question by Mathieu and Sourour when A and B are finite-dimensional [7] and by Nagasawa when A and B are commutative [1, Theorem 4.1.17]. We answer the above conjecture for the numerical radius preserving maps.

2. Preliminaries

Let A be a complex unital normed algebra, and

$$D(A,1) = \{f \in A', f(1) = \parallel f \parallel = 1\},$$

where A' is the dual space of A. The elements of $D(A,1)$ are called normalized states on A. For $a \in A$ let,

$$V(a) = \{f(a) : f \in D(A,1)\}, v(a) = sup\{|\lambda| : \lambda \ \in V(A)\}.$$

$V(a)$ and $v(a)$ are called the numerical range and numerical radius of a respectively. The set of all singular elements of A is denoted by $sing(A)$. The spectrum and spectral radius of a is denoted by $Sp(a)$ and $r(a)$ respectively, and defined by

$$Sp(a) = \{\lambda \in C : \lambda - a \in sing(A)\},$$

$$r(a) = \inf\{\|a^n\|^{\frac{1}{n}} : n = 1, 2, 3, ..\} = \sup\{|\lambda| : \lambda \in sp(a)\}.$$

The convex hull of $Sp(x)$ is denoted by $coSp(x)$. A non-zero linear functional φ on Banach algebra A is called multiplicative if $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in A$.

Let A and B are complex unital normed algebras, a linear map $\varphi : A \to B$ is called numerical radius preserving if $v(\varphi(a)) = v(a)$, and Jordan homomorphism if $\varphi(a^2) = (\varphi(a))^2$, and spectral radius preserving if $r(\varphi(a)) = r(a)$ for every $a \in A$. We say that φ preserves commutativity in both directions if $ab = ba$ if and only if $\varphi(a)\varphi(b) = \varphi(b)\varphi(a)$ for every $a, b \in A$.

Let A be a commutative Banach algebra. We say that a linear functional f on A is a spectral state if it satisfies $f(1) = 1$ and $\|f(a)\| \leq r(a)$ for all $a \in A$.

Theorem 1. *Let A and B be complex Banach algebras and φ be a linear map from A onto B.*

(1) If φ preserves numerical radius, then φ^{-1} preserves numerical radius.

(2) If φ preserves numerical radius and commutativity in both directions, then φ^{-1} preserves commutativity.

Proof. Let $a \in A$ and $\varphi(a) = 0$, then $v(a) = v(\varphi(a)) = v(0) = 0$, and by [4, Theorem 1.4.1] $a = 0$, so φ is injective therefore invertible.

For (1) let $x \in B$, then exists $a \in A$ such that $y = \varphi(a)$. Then $v(y) = v(\varphi(a)) = v(a) = v(\varphi^{-1}(y))$. For (2) Let $x, y \in B$ and $xy = yx$, then exists $a, b \in A$ such that $x = \varphi(a)$ and $y = \varphi(b)$ and $\varphi(a)\varphi(b) = \phi(b)\phi(a)$. Since φ preserves commutativity in both directions, $ab = ba$ therefore $\varphi^{-1}(x)\varphi^{-1}(y) = \varphi^{-1}(y)\varphi^{-1}(x)$. $\square$

Theorem 2. *Let A be a unital complex Banach algebra, and f be a linear functional on A. Then f is a normalized state on A if and only if $f(x) \in coSp(x)$ for all $x \in A$.*

Proof. If $f(x) \in coSp(x)$ for all $x \in A$, then $f(1) \in coSp(1) = \{1\}$, so $f(1) = 1$. Moreover by [4, Theorem 1.4.1] and [4, Theorem 1.2.6], $|f(x)| \leq \max\{|z| : z \in coSp(x)\} \leq v(x) \leq \|x\|$, so $\|f\| \leq 1$ thus $\|f\| = 1$, therefore f is a normalized state.

Conversely, suppose f is a normalized state on A, and $x \in A$. If $\lambda \in C$ such that $\lambda 1_A \neq x$, then $v(x - \lambda 1_A) \neq 0$, and we have $|f(x) - \lambda| = |f(x - \lambda)| \leq v(x - \lambda)$, which means that $f(x)$ is in the closed disk centered at λ with radius $v(x - \lambda)$. So $f(x) \in \cap B(\lambda, v(x - \lambda))$ for all λ where $\lambda 1_A \neq x$. But $coSp(x)$ is the intersection of all such disks, because if $\lambda \in C$ such that $\lambda 1_A \neq x$ and $\mu \in coSp(x)$, then $\mu = \Sigma \alpha_i \lambda_i$ where $0 \leq \alpha_i \leq 1$ and $\sum \alpha_i = 1, \lambda_i \in Sp(x)$, so $|\mu - \lambda| = |\sum(\alpha_i \lambda_i - \lambda)| \leq \sum \alpha_i |\lambda_i - \lambda|$, but by [3, Proposition 1.10.4] $\lambda_i - \lambda \in V(x) - \lambda = V(x - \lambda)$ thus $|\mu - \lambda| \leq v(x - \lambda)$, this implies that $coSp(x) \subseteq B(\lambda, v(x - \lambda))$ for all $\lambda \in C$ where $\lambda 1_A \neq x$. Also, $coSp(x)$ is convex and compact, so by [1, Exercise IV.2.8] $coSp(x) = \cap B(\lambda, v(x-\lambda))$, for all $\lambda \in C$ such that $\lambda 1_A \neq x$. Therefore $f(x) \in coSp(x)$. $\square$

Theorem 3. *Let A be a unital commutative complex Banach algebra. If f is an extreme normalized state on A, then it is multiplicative.*

Proof. Since A is commutative, by Theorem 2 and [1, Lemma 4.1.15] every normalized state is a spectral state, so by [1, Lemma 4.1.16] f is a multiplicative. $\square$

Theorem 4. *If A is a complex unital normed algebras, then $ext(D(A,1))$ separates the points of A.*

Proof. Let $a \neq 0 \in A$, by [3, Corollary 1.10.15] there exists $f \in D(A,1)$ such that $f(a) \neq 0$. Also, by Krein-Milman Theorem [5, Theorem 5.7.4] and [3, Lemma 1.10.3] $D(A,1) = \overline{co}(ext(D(A,1)))$, which implies that there exists $g \in extD(A,1)$ such that $g(a) \neq 0$. Otherwise, for every g belong to $\overline{co}(ext(D(A,1)))$ we have $g(a) = 0$. $\square$

3. Maine Result

Theorem 5. *Let A and B be unital commutative complex Banach algebras. If φ is a unital numerical radius preserving linear map from A onto B, then φ is an isomorphism.*

Proof. If $a \in$A and $\varphi(a) = 0$, then $v(a) = v(\varphi(a)) = v(0) = 0$, and by [4, Theorem 1.4.1] $a = 0$, so φ is injective therefore φ is invertible. By [4, Theorem 1.4.1] $\frac{1}{e}\|\varphi(x) \leq (\varphi(x)) = v(x) \leq \|x\|$ so $\|\varphi(x)\| \leq e\|x\|$, therefore φ is continuous.

Let f be an extreme normalized states on B and let $g = f\varphi$. Then g is linear, continuous, $|g(x)| = |f(\varphi(x))| \leq \ v(\varphi(x)) = v(x) \leq \|x\|$ for all $x \in A$, so $\|g\| \leq 1$, also $g(1) = (f\varphi)(1) = 1$, it follows that $\|g\| = 1$. Therefore g is a normalized states on A. It is easy to see that g is an extreme point. Indeed, assume that $g_1, g_2 \in D(A,1)$ such that $g = \frac{1}{2}(g_1 + g_2)$. We shall verify that $g = g_1 = g_2$. Since $f = g\varphi^{-1}$, we have $f = \frac{1}{2}(g_1\varphi^{-1} + g_2\varphi^{-1})$.

Also since $g_1 \in D(A,1)$, $g_1\varphi^{-1}$ is continuous, and $(g_1\varphi^{-1})(1) = 1$, and for all $y \in B$ we have $g_1(\varphi^{-1}(y)) \in V(\varphi^{-1}(y))$, so $|g_1\varphi^{-1}(y)| \leq v(\varphi^{-1}(y)) = v(y) \leq \|y\|$ thus $\|g_1\varphi^{-1}\| \leq 1$, therefore $g_1\varphi^{-1} \in D(B,1)$. A similarly proof implies that $g_2\varphi^{-1} \in D(B,1)$, but f is an extreme point, it follows that $g_1\varphi^{-1} = g_2\varphi^{-1} = f$ so $g = g_1 = g_2$.

Since f and $f\varphi$ are extreme normalized states on B and A receptivity, by Theorem 3 f and $f\varphi$ are multiplicative, so $f(\varphi(xy)) = f(\varphi(x))f(\varphi(y)) = f(\varphi(x)\varphi(y))$ for all $x, y \in A$. Since f is an extreme normalized states on B by Theorem 4 $ext(D(B,1))$ separates point of B, therefore $\varphi(x)\varphi(y) = \varphi(xy)$. □

Theorem 6. *Let A and B be unital complex Banach algebras. If a unital linear map φ from A onto B preserves numerical radius and commutativity in both directions, then φ is a Jordan isomorphism.*

Proof. φ is invertible by Theorem 1. Suppose $a \in A$ and let $A_1 = \langle a, 1\rangle$ be the closed sub-algebra of A generated by a and 1. Define a linear map $\varphi_1 : A_1 \to \varphi(A_1)$ by $\varphi_1(x) = \varphi(x)$ for all $x \in A_1$. suppose $\varphi(A_1)$ is a sub-algebra of B, since A_1 is communicative, and φ preserves commutativity so $\varphi(A_1)$ is communicative. Also, φ_1 preserves numerical radius, therefore φ_1 is an isomorphism by Theorem 5, so $\varphi(a^2) = \varphi(a)^2$. Otherwise, let $B_1 = \langle \varphi(A_1)\rangle$ and define a linear map $\psi_1 : B_1 \to \varphi^{-1}(B_1)$ by $\psi_1(y) = \varphi^{-1}(y)$ for all $y \in B_1$. Suppose $\varphi^{-1}(B_1)$ is a subalgebra of A, since B_1 is communicative, and φ^{-1} preserves commutativity by Theorem 1, so $\varphi^{-1}(B_1)$ is communicative. Also φ^{-1} is numerical radius preserving by Theorem 1, therefore ψ_1 is an isomorphism by Theorem 5 and hence $\psi_1(y^2) = \psi_1(y)^2$, then $\psi_1(\varphi(a)^2) = \psi_1(\varphi(a))^2 = a^2$, therefore $\varphi(a)^2 = \varphi(a^2)$. Otherwise, let $A_2 = \langle \varphi^{-1}(B_1)\rangle$ and define a linear map $\varphi_2 : A_2 \to \varphi(A_2)$ by $\varphi_2(x) = \varphi(x)$ for all $x \in A_2$. By continuing this process we obtain sequences $\{A_n\}$ and $\{B_n\}$ commutative subalgebras of A and B respectively such that $A_1 = \langle\ a, 1\rangle$, $A_n = \langle \varphi^{-1}(B_{n-1})\rangle\, B_n = \langle \varphi(A_n)\rangle\, A_1 \subseteq A_2 \subseteq ... \subseteq A$ and $B_1 \subseteq B_2 \subseteq .... \subseteq B$. Define $A' = \cup\ A_n$ and $B' = \cup\ B_n$ and $\varphi' : A' \to\ \ B'$ by $\varphi'(x) = \varphi(x)$ for every $x \in A'$. A' and B' are commutative, and φ' is a unital surjective and numerical radius preserving, so by Theorem 5 φ' is isomorphism and hence $\varphi'(a^2) = \varphi'(a)^2$, therefore $\varphi(a^2) = \varphi(a)^2$. □

Remark 7. *In particular if both A and B are communicative unital $C^* - algebras$ and φ is a unital surjective numerical radius preserving linear map from A to B, then by Gelfand Theorem [8, Theorem 2.1.10] $r(a) = \|a\|$ and $r(\varphi(a)) = \|\varphi(a)\|$, so by [4, Theorem 1.4.1] and [4, Theorem 1.2.6] $r(a) = \|a\| = v(a)$ and $r(\varphi(a)) = \|\varphi(a)\| = v(\varphi(a))$ for all $a \in A$, therefore $r(a) = r(\varphi(a))$ and we can use the Nagasawa Theorem[1, Theorem 4.1.17].*

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