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TRANSITION DENSITY FOR CIR PROCESS BY LIE SYMMETRIES AND APPLICATION TO ZCB PRICING

Francesco Cordoni[1], Luca Di Persio[2 §]

[1]Department of Mathematics
University of Trento
Via Sommarive, 14 Trento, ITALY

[2]Department of Computer Science
University of Verona
Via le Grazie, 14 Verona, ITALY

Abstract: Using a Lie algebraic approach we explicitly compute the transition density function for the solution of the stochastic differential equation defining the CIR process. Moreover we show how to use such a derivation to recover the well-known formula for the asscoiated ZCB fair price.

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1. Introduction and Preliminary Results

In this section, following [3, 4], we will give the basic results needed to characterize the group of symmetries for the solutions of a particular class of PDE which will be later used to compute the transition density function of the CIR process.

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[§]Correspondence author

This process is widely used in finance to model short term interest rate, see [2] for details, and it is also used to model stochastic volatility in the Heston model.

Theorem 1.1. *Let us consider the PDE*

$$u_\tau = \frac{\sigma^2}{2} x^\gamma u_{xx} + f(x) u_x - g(x) u \, , \; \gamma \neq 2 \tag{1}$$

and suppose that g *and* $h := x^{1-\gamma} f$ *satisfy the second Riccati equation*

$$\frac{\sigma^2}{2} x h' - \frac{\sigma^2}{2} h + \frac{1}{2} h^2 = \frac{A}{2(2-\gamma)^2} x^{4-2\gamma} + \frac{B}{2-\gamma} x^{2-\gamma} + C \, , \tag{2}$$

where $A > 0$ *and* B, C *are arbitrary constants. Let* u_0 *be a stationary, analytic solution of* (1)*, then one can define the following family of symmetric solutions depending on a parameter* ϵ *which is linked to the Lie algebraic representaion of solutions of* (1)*, see [3], Th. 2.5, for details:*

$$\begin{aligned}
\bar{U}_\epsilon(x,\tau) =& (1 + \epsilon^2(\cosh(\sqrt{A\tau} - 1) + 2\epsilon \sinh(\sqrt{A}))^{-c} \\
& \times \left| \frac{\cosh(\frac{\sqrt{A}\tau}{2}) + (1+2\epsilon)\sinh(\frac{\sqrt{A}\tau}{2})}{\cosh(\frac{\sqrt{A}\tau}{2}) - (1-2\epsilon)\sinh(\frac{\sqrt{A}\tau}{2})} \right|^{\frac{B}{\sigma^2\sqrt{A}(2-\gamma)}} e^{\left(\frac{1}{\sigma^2}F(x) - \frac{B\tau}{\sigma^2(2-\gamma)}\right)} \\
& \times \exp\left(\frac{-2\sqrt{A}\epsilon x^{2-\gamma}(\cosh(\sqrt{A}\tau) + \epsilon \sinh(\sqrt{A}\tau))}{\sigma^2(2-\gamma)^2(1+\epsilon^2(\cosh(\sqrt{A}\tau) - 1) + 2\epsilon\sinh(\sqrt{A}\tau))} \right) \\
& \times \exp\left(\frac{1}{\sigma^2} F\left(\frac{x}{(1+\epsilon^2(\cosh(\sqrt{A}\tau)-1) + 2\epsilon\sinh(\sqrt{A}\tau))^{\frac{1}{2-\gamma}}} \right) \right) \\
& \times u_0 \left(\frac{x}{(1+\epsilon^2(\cosh(\sqrt{A}\tau)-1) + 2\epsilon\sinh(\sqrt{A}\tau))^{\frac{1}{2-\gamma}}} \right) ,
\end{aligned}$$

with F *such that* $F'(x) = \frac{f(x)}{x^\gamma}$ *and* $c := \frac{1-\gamma}{2-\gamma}$.

Furthermore there exists a fundamental solution $p(x, y, \tau)$ *of* (1) *such that*

$$\int_0^\infty e^{-\lambda y^{2-\gamma}} u_0 p(x,y,\tau) dy = U_\lambda(x,\tau) \, , \tag{3}$$

where $U_\lambda(x,\tau) = \bar{U}_{\frac{\sigma^2(2-\gamma)^2\lambda}{2\sqrt{A}}}(x,\tau)$*, and* u_0 *is the stationary solution of* (1)*.*

Remark 1.1. We would like to note that the time variable τ in equation (1), will be later defined as $\tau := T - t$. The latter notation is motivated by standard financial arguments since τ actually refers to *time left to maturity* where T indicates, e.g., the expiration time of a certain financial contract.

Corollary 1.1. *Suppose that the conditions of Th.* (1.1) *hold and let* $g = 0$*, then, taking the stationary solution* $u_0 = 1$*, the resulting fundamental solution* $p(x, y, \tau)$ *satisfies* $\int_0^\infty p(x, y, \tau)dy = 1$*. It follows that if* ρ *is also positive, then it is a probability density function.*

The following proposition, see [3], Prop. 2.11, for details, shows the link between fundamental solutions and the expectation of the associated SDE solutions.

Proposition 1.1. *Let* $X = \{X_\tau : \tau \geq 0\}$ *be an Itô process solution of*

$$dX_\tau = f(X_\tau)d\tau + \sigma\sqrt{X_\tau}dW_\tau \,,$$

where W_τ *is a standard Brownian motion, and* f *is measurable. Suppose that there exist positive constants* a, K *such that* $|f(x)| \leq Ke^{ax}$*. Then there exists* $T > 0$ *such that* $u(x, \tau, \lambda) := \mathbb{E}_x[e^{-\lambda X_\tau}]$ *is the unique solution of*

$$u_\tau + \lambda^2 \frac{\sigma^2}{2} u_\lambda + \lambda \mathbb{E}_x[f(X_\tau)e^{-\lambda X_\tau}] = 0 \,,$$

subject to $u(x, 0, \lambda) = e^{-\lambda x}$ *for* $0 \leq \tau < T$, $\lambda > a$*.*

Moreover, see [4] for details, the following holds

Theorem 1.2. *Let us suppose* f *is a solution of the equation*

$$\sigma f + \frac{1}{2} f^2 + 2\sigma\mu x^2 = \frac{1}{2} A x^2 + Bx + C, \quad \text{with } A > 0 \,. \tag{4}$$

Then there exists a fundamental solution of

$$u_\tau = \sigma x u_{xx} + f(x)u_x - \mu x u, \quad x \geq 0 \,, \tag{5}$$

of the form

$$p(x, y, \tau) = \frac{\sqrt{Axy} e^{-(F(x)-F(y)/(2\sigma))}}{2\sigma \sinh(\sqrt{A}\tau/2)} \exp\left(-\frac{B\tau}{2\sigma} - \frac{\sqrt{A}(x+y)}{2\sigma \tanh(\sqrt{A}\tau/2)} \right) \tag{6}$$

$$\left(C_1(y) I_\nu \left(\frac{\sqrt{Axy}}{\sigma \sinh(\sqrt{A}\tau/2)} \right) + (C_2(y) I_{-\nu} \left(\frac{\sqrt{Axy}}{\sigma \sinh(\sqrt{A}\tau/2)} \right) \right)$$

where $\nu = \frac{\sqrt{\sigma^2 + 2C}}{\sigma}$ *and* $I_\nu(x) := (\frac{x}{2})^\nu \sum_{n \geq 0} \frac{(x/2)^{2n}}{n!\Gamma(\nu+n+1)}$ *, is a first-order modified Bessel function.*

2. CIR Process

Let us now focus on the CIR process and on its symmetric group.

We will study the stochastic process $X = \{X_\tau : \tau \geq 0\}$ solution of the CIR model, see [2] for details, described by the following SDE

$$\begin{cases} dX_\tau = k(\theta - X_\tau)d\tau + \sigma\sqrt{X_\tau}dW_\tau \\ X_0 = x \end{cases} . \tag{7}$$

We want to recover the fundamental solution associated to the SDE (7) exploiting the method described in Section (1). In particular the PDE associated to equation (7) is of the form described by equation (1), i.e. it reads as follows

$$u_\tau = \frac{\sigma^2}{2} x u_{xx} + k(\theta - x)u_x . \tag{8}$$

According to notation introduced in Th. (1.1), we have

$$\gamma = 1 \; ; \; g(x) = 0 \; ; \; f(x) = k(\theta - x) \; ; \; h(x) = f(x) = k(\theta - x) \; ,$$

and in order $h(x)$ to satisfy an equation of type (2) we obtain

$$-\frac{\sigma^2}{2} xk + \frac{\sigma^2}{2} xk - \sigma k\theta + \frac{(k\theta)^2}{2} + \frac{k^2 x^2}{2} - k^2\theta x = \frac{A}{2} x^2 + Bx + C \; ,$$

if and only if parameters A, B, C satisfy

$$A = k^2 \; ; \; B = -k^2\theta \; , \; C = \frac{k\theta}{2}(k\theta - \sigma^2) \; .$$

We can finally apply (1.1) with $F(x) = k\theta \ln(x) - kx$ and $u_0(x) = 1$ to obtain the desired solution, namely

$$U_{\frac{\sigma^2(2-\gamma)^2\lambda}{2\sqrt{A}}}(x,\tau) = \frac{2k^{\frac{k\theta}{\sigma^2}} e^{\frac{2k^2\theta\tau}{\sigma^2}}}{(\frac{\sigma^2}{2}\lambda(e^{k\tau} - 1) + ke^{k\tau})^{\frac{2k\theta}{\sigma^2}}} \exp\left(\frac{-\lambda kx}{(\frac{\sigma^2}{2}\lambda(e^{k\tau} - 1) + ke^{k\tau})^{\frac{2k\theta}{\sigma^2}}}\right) ,$$

in fact we have that

$$U_\epsilon(x,\tau) = \left| \frac{\cosh(\frac{k\tau}{2}) + (1 + 2\epsilon)\sinh(\frac{k\tau}{2})}{\cosh(\frac{k\tau}{2}) - (1 - 2\epsilon)\sinh(\frac{k\tau}{2})} \right|^{\frac{-k\theta}{\sigma^2}} \times \exp\left(-\frac{1}{\sigma^2}(k\theta \ln(x) - kx) + \frac{2k^2\theta\tau}{\sigma^2}\right)$$

$$\times \exp\left(\frac{-k\epsilon x(\cosh(k\tau)+\epsilon\sinh(k\tau))}{\frac{\sigma^2}{2}(1+2\epsilon^2(\cosh(k\tau)-1)+2\epsilon\sinh(k\tau))}\right.$$

$$\left.+\ln\left(\frac{x}{1+2\epsilon^2(\cosh(k\tau)-1)+2\epsilon\sinh(k\tau)}\right)^{\frac{k\theta}{\sigma^2}}\right)$$

$$\times \exp\left(-\frac{kx}{\sigma^2}\left(\frac{1}{1+2\epsilon^2(\cosh(k\tau)-1)+2\epsilon\sinh(k\tau)}\right)\right)$$

$$= \underbrace{\begin{array}{c}\left|\frac{\cosh(\frac{k\tau}{2})+(1+2\epsilon)\sinh(\frac{k\tau}{2})}{\cosh(\frac{k\tau}{2})-(1-2\epsilon)\sinh(\frac{k\tau}{2})}\right|^{\frac{-k\theta}{\sigma^2}} e^{\frac{k^2\theta\tau}{\sigma^2}} \\ \times \exp\left[\ln\left(\frac{x(1+\epsilon^2(\cosh(k\tau)-1)+2\epsilon\sinh(k\tau))}{x}\right)^{-\frac{k\theta}{\sigma^2}}\right]\end{array}}_{\equiv Part1}$$

$$\times \underbrace{\exp\left(\frac{kx}{\sigma^2}\frac{1-2\epsilon(\cosh(k\tau)-\epsilon\sinh(k\tau)}{1+2\epsilon^2(\cosh(k\tau)-1)+2\epsilon\sinh(k\tau)}\right)}_{\equiv Part2}. \tag{9}$$

Let us focus our attention separately on $Part1$ and $Part2$ of the equation (9). For what concerns $Part1$ we have

$$Part1 = \left|\frac{\cosh(\frac{k\tau}{2})+(1+2\epsilon)\sinh(\frac{k\tau}{2})}{\cosh(\frac{k\tau}{2})-(1-2\epsilon)\sinh(\frac{k\tau}{2})}\right|^{\frac{-k\theta}{\sigma^2}} e^{\frac{k^2\theta\tau}{\sigma^2}}$$

$$\times [1+2\epsilon^2(\cosh(k\tau)-1)+2\epsilon\sinh(k\tau)]^{\frac{-k\theta}{\sigma^2}}$$

$$= \left|\frac{e^{\frac{k\tau}{2}}+e^{\frac{-k\tau}{2}}+(1+2\epsilon)\left(e^{\frac{k\tau}{2}}-e^{\frac{-k\tau}{2}}\right)}{e^{\frac{k\tau}{2}}+e^{\frac{-k\tau}{2}}-(1-2\epsilon)\left(e^{\frac{k\tau}{2}}-e^{\frac{-k\tau}{2}}\right)}\right|^{\frac{-k\theta}{\sigma^2}} e^{\frac{k^2\theta\tau}{\sigma^2}}$$

$$\times [1+\epsilon^2(e^{k\tau}+e^{-k\tau}-2)+\epsilon(e^{k\tau}-e^{-k\tau})]^{\frac{-k\theta}{\sigma^2}}$$

$$= \left|\frac{e^{\frac{k\tau}{2}}2\epsilon+e^{\frac{-k\tau}{2}}(2-2\epsilon)}{e^{\frac{k\tau}{2}}(2+2\epsilon)-e^{\frac{-k\tau}{2}}2\epsilon}\right|^{\frac{k\theta}{\sigma^2}}$$

$$\times \left(\frac{1}{e^{k\tau}(\epsilon^2+\epsilon)+e^{-k\tau}(\epsilon^2-\epsilon)-2\epsilon^2+1}\right)^{\frac{k\theta}{\sigma^2}} e^{\frac{k^2\theta\tau}{\sigma^2}}$$

$$= \left(\frac{e^{k\tau}}{(\epsilon-e^{k\tau}(1+\epsilon))^2}\right)^{\frac{k\theta}{\sigma^2}} e^{\frac{k^2\theta\tau}{\sigma^2}},$$

hence, taking $\epsilon = \frac{\sigma^2\lambda}{2k}$ we get $Part1 = \left(\frac{2ke^{k\tau}}{\sigma^2\lambda(e^{k\tau}-1)+2ke^{k\tau}}\right)^{\frac{2k\theta}{\sigma^2}}$.

Concerning the second part of equation (9), we have

$$\begin{aligned}
Part2 &= \exp\left(\frac{kx}{\sigma^2}\left(\frac{1+2\epsilon^2(\cosh(k\tau)-1)+2\epsilon\sinh(k\tau)-1-2\epsilon(\cosh(k\tau)-2\epsilon^2\sinh(k\tau))}{1+2\epsilon^2(\cosh(k\tau)-1)+2\epsilon\sinh(k\tau)}\right)\right)\\
&= \exp\left(\frac{kx}{\sigma^2}\left(\frac{2\epsilon^2\,(\cosh(k\tau)-1)+2\epsilon\sinh(k\tau)-2\epsilon\cosh(k\tau)-2\epsilon^2\sinh(k\tau)}{1+2\epsilon^2(\cosh(k\tau)-1)+2\epsilon\sinh(k\tau)}\right)\right)\\
&= \exp\left(\frac{kx}{\sigma^2}\left(\frac{2\epsilon^2\left(\frac{e^{k\tau}+e^{-k\tau}}{2}-1\right)+2\epsilon\left(\frac{e^{k\tau}-e^{-k\tau}}{2}\right)-2\epsilon\left(\frac{e^{k\tau}+e^{-k\tau}}{2}\right)-2\epsilon^2\left(\frac{e^{k\tau}-e^{-k\tau}}{2}\right)}{1+2\epsilon^2(\cosh(k\tau)-1)+2\epsilon\sinh(k\tau)}\right)\right)\\
&= \exp\left(\frac{kx}{\sigma^2}\left(\frac{e^{k\tau}(\epsilon^2-\epsilon^2+\epsilon-\epsilon)+e^{-k\tau}(2\epsilon^2-2\epsilon)-2\epsilon^2}{1+2\epsilon^2\left(\frac{e^{k\tau}+e^{-k\tau}}{2}-1\right)+2\epsilon\left(\frac{e^{k\tau}-e^{-k\tau}}{2}\right)}\right)\right)\\
&= \exp\left(\frac{kx}{\sigma^2}\left(\frac{e^{-k\tau}(2\epsilon^2-2\epsilon)-2\epsilon^2}{e^{k\tau}(\epsilon^2+\epsilon)+e^{-k\tau}(\epsilon^2-\epsilon)-2\epsilon^2+1}\right)\right) = \exp\left(\frac{-2kx\epsilon}{-\sigma^2\epsilon+e^{k\tau}\sigma^2(1+\epsilon)}\right),
\end{aligned}$$

thus, taking $\epsilon = \frac{\sigma^2\lambda}{2k}$, we obtain $Part2 = \exp\left(\frac{-2kx\lambda}{\sigma^2\lambda(e^{k\tau}-1)+2ke^{k\tau}}\right)$.

Combining $Part1$ and $Part2$, we finally have

$$\begin{aligned}
U_{\frac{\sigma^2(2-\gamma)^2\lambda}{2\sqrt{A}}}(x,\tau) &= \int_0^\infty e^{-\lambda y}p(x,y,\tau)dy\\
&= \frac{k^{\frac{2k\theta}{\sigma^2}}e^{\frac{2k^2\theta\tau}{\sigma^2}}}{(\frac{\sigma^2}{2}\lambda(e^{k\tau}-1)+ke^{k\tau})^{\frac{2k\theta}{\sigma^2}}}\exp\left(\frac{-2\lambda kx}{\sigma^2\lambda(e^{k\tau}-1)+2ke^{k\tau}}\right),
\end{aligned}$$

which is nothing but the Laplace transform of $p(x,y,\tau)$, moreover it can be inverted, see [3, 4] for details, to have that the fundamental solution of (8) reads as follows

$$\begin{aligned}
&p_{CIR}(x,y,\tau)\\
&\quad= \frac{2ke^{k(\frac{2k\theta}{\sigma^2}+1)\tau}}{\sigma^2(e^{k\tau}-1)}\left(\frac{y}{x}\right)^{\frac{\nu}{2}}\exp\left(\frac{-2k(x+e^{k\tau}y)}{\sigma^2(e^{k\tau}-1)}\right)I_\nu\left(\frac{2k\sqrt{xy}}{\sigma^2\sinh(\frac{k\tau}{2})}\right), \qquad (10)
\end{aligned}$$

where $\nu := \frac{2k\theta}{\sigma^2} - 1$.

Then by proposition (1.1) we have that the fundamental solution represented in (10) is in fact the transition density function of the CIR process.

Moreover, exploiting Th. (1.2), we can compute the ZCB fair price, see, e.g., [1] for a description of such type of bond. To this end let us recall that given a differential operator L of elliptic type L such that

$$u_\tau - Lu = 0 \qquad (11)$$

with initial datum $u(0,x) = f(x)$, where $(x,\tau) \in \Omega \times [0,T]$, and Ω is a measurable subset of $\mathbb{R}$ representing the domain of definition of f which is assumed to be sufficiently smooth on it, a fundamental solution for (11) is a kernel $p(x,y,\tau)$ such that the following conditions hold: (i) for fixed $y \in \Omega$, $p(x,y,\tau)$ is a solution of the PDE (11) on $\Omega \times (0,T]$; (ii) $u(x,\tau) = \int_\Omega f(x)p(x,y,\tau)dy$ is a solution of the Cauchy problem for a given initial datum f. We anyway refer to [5] for a complete treatment of fundamental solutions for partial differential equations of parabolic type.

Then it follows that if the ZCB associated interested rate is driven by the CIR process defined by equation (7), then its fair price is given by theorem (1.2) setting

$$\mu = 1\,;\; A = k^2 + 2\sigma^2\,;\; B = -k^2\theta\,,\; C = \frac{\theta k}{2}(\theta k - \sigma^2)\,;$$

$$F(x) = \theta k \ln(x) - kx\,;\; C_1 = 1\,;\; C_2 = 0\,.$$

Then denoting with P_t the fair price of our ZCB with underlying interest rate process given by (7) we can recover a classical solution from the fundamental one exploiting point (ii) above. In particular for every $t \in [0,T)$ the price at time t of the ZCB reads as follows

$$\begin{aligned} P_t := u(x,t) = & \int_0^\infty \frac{\gamma x^{(\sigma^2-2\theta k)/2\sigma^2} y^{(\sigma^2+2\theta k)/2\sigma^2} e^{\frac{k}{\sigma^2}(x-y)}}{\sigma^2 \sinh(\gamma(T-t)/2)} \\ & \times \exp\left(\frac{k^2\theta(T-t)}{\sigma^2} - \frac{\gamma(x+y)}{\sigma^2\tanh(\gamma(T-t)/2)}\right) \\ & \times I_{\frac{\sigma^2-2k\theta}{\sigma^2}}\left(\frac{2\gamma\sqrt{xy}}{\sigma^2\sinh(\gamma(T-t)/2)}\right) dy \\ & \times \exp\left[\frac{-2x\left(e^{\gamma(T-t)}-1\right)}{(\gamma+k)(e^{\gamma(T-t)}-1)+2\gamma}\right] \left[\frac{2\gamma\exp\left[(\gamma+k)(T-t)/2\right]}{(\gamma+k)(e^{\gamma(T-t)}-1)+2\gamma}\right]^{\frac{2k\theta}{\sigma^2}}, \end{aligned} \tag{12}$$

where $\gamma := \sqrt{k^2+2\sigma^2}$.

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