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ON A HYPERSURFACE OF A HYPERBOLIC COSINE FINSLER METRIC

H. Wosoughi
Department of Mathematics
Islamic Azad University
Babol Branch, IRAN

Abstract: The purpose of the present paper is to investigate the various kinds of hypersurface of Finsler space with (α, β)- metric of type $L = \alpha \cosh(\frac{\beta}{\alpha})$. The conditions under which this hypersuface to be a hyperplane of first, second or third kinds have been obtained.

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Key Words: Finsler space, hypersurface, induced Cartan connection, (α, β)-metric, hyperplane of 1^{st}, 2^{nd} and 3^{rd} kinds

1. Introduction

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space,i.e.,a pair consisting of an n-dimensional differential manifold M^n equipped with a fundamental function L(x, y). The concept of the (α, β)-metric was introduced by M. Matsumoto [4] and has been studied by many authors (see [1], [2], [7],...), where $\alpha^2 = a_{ij}(x)\, y^i y^j$ is a Riemannian metric and $\beta = b_i(x)\, y^i$ is a 1-form on M^n.

A hypersurface M^{n-1} of the M^n may be represented parametrically by the equation $x^i = x^i(u^\alpha)$, $\alpha = 1, ..., n-1$, where u^α are Gaussian coordinates on M^{n-1}.

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Since the function $\underline{L} =: L(x(u), y(u,v))$ gives rise to a Finsler metric of M^{n-1}, we get an (n-1)-dimensional Finsler space $F^{n-1} = (M^{n-1}, \underline{L}(u,v))$. The hypersurface of Finler Space with some given special metrics has been studied by authors (see [8], [9], [10]).

with (α, β)-metric $L = \alpha \cosh(\frac{\beta}{\alpha})$ and the hypersurface of F^n with $b_i = \partial_i b$ being the

In the present paper, we consider an n-dimensional Finsler space $F^n = (M^n, L)$ gradient of a scalar function $b(x)$. We prove the conditions for this hypersurface to be hyperplane of 1^{st} kind, 2^{nd} kind and3^{rd} kind.

2. Preliminaries

Let $F^n = (M^n, L)$ be a special Finsler space with the metric

$$L(\alpha, \beta) = \alpha \cosh(\frac{\beta}{\alpha}) \tag{1}$$

The derivatives of the (1) with respect to α and β are given by

$$\begin{aligned} L_\alpha &= \cosh(\tfrac{\beta}{\alpha}) - \tfrac{\beta}{\alpha}\sinh(\tfrac{\beta}{\alpha}), \\ L_\beta &= \sinh(\tfrac{\beta}{\alpha}), \\ L_{\alpha\alpha} &= \tfrac{\beta^2}{\alpha^3}\cosh(\tfrac{\beta}{\alpha}), \\ L_{\beta\beta} &= \tfrac{1}{\alpha}\cosh(\tfrac{\beta}{\alpha}), \\ L_{\alpha\beta} &= -\tfrac{\beta}{\alpha^2}\cosh(\tfrac{\beta}{\alpha}), \end{aligned}$$

where $L_\alpha = \partial L/\partial\alpha$, $L_\beta = \partial L/\partial\beta$, $L_{\alpha\alpha} = \partial L_\alpha/\partial\alpha$, $L_{\beta\beta} = \partial L_\beta/\partial\beta$ and $L_{\alpha\beta} = \partial L_\alpha/\partial\beta$.

In the special Finsler space $F^n = (M^n, L)$ the normalized element of support $l_i = \dot{\partial} L$ and the angular metric tensor h_{ij} are given by [6]:

$$\begin{aligned} l_i &= \alpha^{-1} L_\alpha Y_i + L_\beta b_i, \\ h_{ij} &= p a_{ij} + q_0 b_i b_j + q_1 (b_i Y_j + b_j Y_i) + q_2 Y_i Y_j \end{aligned}$$

where

$$\begin{aligned} Y_i &= a_{ij} y^j, \\ p &= L L_\alpha \alpha^{-1} = \tfrac{1}{\alpha}(\cosh(\tfrac{\beta}{\alpha}))(\alpha\cosh(\tfrac{\beta}{\alpha}) - \beta\sinh(\tfrac{\beta}{\alpha}), \\ q_0 &= L L_{\beta\beta} = \cosh^2(\tfrac{\beta}{\alpha}), \\ q_1 &= L L_{\alpha\beta}\alpha^{-1} = -\tfrac{\beta}{\alpha^2}\cosh^2(\tfrac{\beta}{\alpha}) \\ q_2 &= L\alpha^{-2}(L_{\alpha\alpha} - L_\alpha\alpha^{-1}) = \tfrac{1}{\alpha^4}(\cosh(\tfrac{\beta}{\alpha}))[(\beta^2 - \alpha^2)\cosh(\tfrac{\beta}{\alpha}) + \alpha\beta\sinh(\tfrac{\beta}{\alpha})] \end{aligned} \tag{2}$$

The fundamental tensor $g_{ij} = \frac{1}{2} \dot{\partial} \dot{\partial} L^2$ and it's reciprocal tensor is given by [6]

$$g_{ij} = pa_{ij} + p_0 b_i b_j + p_1(b_i Y_j + b_j Y_i) + p_2 Y_i Y_j,$$

where

$$\begin{aligned} p_0 &= q_0 + L_\beta^2 = \cosh 2(\beta/\alpha) \\ p_1 &= q_1 + L^{-1} p L_\beta = \tfrac{1}{\alpha^2}\{-\beta \cosh^2(\tfrac{\beta}{\alpha}) + \sinh(\tfrac{\beta}{\alpha})[\alpha \cosh(\tfrac{\beta}{\alpha}) - \beta \sinh(\tfrac{\beta}{\alpha})]\} \\ p_2 &= q_2 + p^2 L^{-2} = \tfrac{1}{\alpha^4}\{\cosh(\tfrac{\beta}{\alpha})[(\beta^2 - \alpha^2)\cosh(\tfrac{\beta}{\alpha}) + \alpha\beta \sinh(\tfrac{\beta}{\alpha})] + \\ &\quad [\alpha \cosh(\tfrac{\beta}{\alpha}) - \beta \sinh(\tfrac{\beta}{\alpha})]^2\} \end{aligned} \tag{3}$$

$$g^{ij} = p^{-1} a^{ij} + S_0 b^i b^j + S_1(b^i y^j + b^j y^i) + S_2 y^i y^j, \tag{4}$$

where

$$\begin{aligned} & b^i = a^{ij} b_j, \\ & S_0 = (pp_0 + (p_0 p_2 - p_1^2)\alpha^2)/\varsigma \\ & S_1 = (pp_1 + (p_0 p_2 - p_1^2)\beta)/\varsigma p \\ & S_2 = (pp_2 + (p_0 p_2 - p_1^2) b^2)/\varsigma p \\ & b^2 = a_{ij} b^i b^j, \\ & \varsigma = p(p + p_0 b^2 + p_1 \beta) + (p_0 p_2 - p_1^2)(\alpha^2 b^2 - \beta^2) \end{aligned} \tag{5}$$

The $hv-$torsion tensor $C_{ijk} = \frac{1}{2} \dot{\partial} g_{ij}$ is given by [6]

$$2pC_{ijk} = p_1(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + \gamma_1 m_i m_j m_k,$$

where

$$\gamma_1 = p \frac{\partial p_0}{\partial \beta} - 3 p_1 q_0, \quad m_i = b_i - \alpha^{-2} \beta Y_i. \tag{6}$$

Here m_i be a non-vanishing covariant vector orthogonal to the element of support y^i.

Let $\{^{\,i}_{jk}\}$ be the component of Christoffel symbols of the associated Riemannian space R^n and ∇_k be covariant differentiation with respect to x^k relative to this Christoffel symbols. We put

$$2E_{ij} = b_{ij} + b_{ij}, \quad 2F_{ij} = b_{ij} - b_{ji}, \tag{7}$$

where $b_{ij} = \nabla_j b_i$.

Let $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^i_{jk})$ be the Cartan connection of F^n. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \{^{\,i}_{jk}\}$ of the special Finsler space $F^n$is given by [3]

$$\begin{aligned} D^i_{jk} = {} & B^i E_{jk} + F^i_k B_j + F^i_j B_k + B^i_j b_{0k} + B^i_k b_{0j} \\ & - b_{0m} g^{im} B_{jk} - C^i_{jm} A^m_k - C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} \\ & + \lambda^s (C^i_{jm} C^m_{sk} + C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms}), \end{aligned} \tag{8}$$

where

$$\begin{aligned}
&B_k = p_0 b_k + p_1 Y_k, \quad B^i = g^{ij} B_j, \quad F_i^k = g^{kj} F_{ji}\\
&B_{ij} = \{p_1(a_{ij} - \alpha^{-2} Y_i Y_j) + \tfrac{\partial p_0}{\partial \beta} m_i m_j\}/2\\
&B_i^k = g^{kj} B_{ji}\\
&A_k^m = b_k^m E_{00} + B^m E_{k0} + B_k F_0^m + B_0 F_k^m\\
&\lambda^m = B^m E_{00} + 2B_0 F_0^m, \quad B_0 = B_i y^i
\end{aligned} \tag{9}$$

where '0' denote contraction with y^i except for the quantities p_0, q_0 and S_0.

3. Induced Cartan Connection

Let F^{n-1} be a hypersurface of F^n given by the equations $x^i = x^i(u^\alpha)$. The element of support y^i of F^n is to be taken tangential to F^{n-1}, that is

$$y^i = B_\alpha^i(u) v^\alpha. \tag{10}$$

The metric tensor $g_{\alpha\beta}$ and $v-$torsion tenor $C_{\alpha\beta\gamma}$ of F^{n-1} are given by

$$g_{\alpha\beta} = g_{ij} B_\alpha^i B_\beta^j, \quad C_{\alpha\beta\gamma} = C_{ijk} B_\alpha^i B_\beta^j B_\gamma^k$$

At each point u^α of F^{n-1}, a unit normal vector $N^i(u,v)$ is defined by

$$g_{ij}(x(u,v), y(u,v)) B_\alpha^i N^j = 0, \quad g_{ij}(x(u,v), y(u,v)) N^i N^j = 1.$$

As for the angular metric tensor h_{ij}, we have

$$h_{\alpha\beta} = h_{ij} B_\alpha^i B_\beta^j, h_{ij} B_\alpha^i N^j = 0, h_{ij} N^i N^j = 1 \tag{11}$$

If (B_i^α, N_i) denote the inverse of (B_α^i, N^i), then we have

$$\begin{aligned}
&B_i^\alpha = g^{\alpha\beta} g_{ij} B_\beta^j, \quad B_\alpha^i B_i^\beta = \delta_\alpha^\beta,\\
&B_i^\alpha N^i = 0, \quad B_\alpha^i N_i = 0, \quad N_i = g_{ij} N^i,\\
&B_i^k = g^{kj} B_{ji},\\
&B_\alpha^i B_j^\alpha + N^i N_j = \delta_j^i
\end{aligned}$$

The induced connection $IC\Gamma = (\Gamma^{*\alpha}_{\beta\gamma}, G^\alpha_\beta, C^\alpha_{\beta\gamma})$ of

$$\begin{aligned}
&\Gamma^{*\alpha}_{\beta\gamma} = B_i^\alpha (B_{\beta\gamma}^i + \Gamma^{*i}_{jk} B_\beta^j B_\gamma^k) + M_\beta^\alpha H_\gamma,\\
&G_\beta^\alpha = B_i^\alpha (B_{0\beta}^i + \Gamma^{*i}_{0j} B_\beta^j),\\
&G_{\beta\gamma}^\alpha = B_i^\alpha C_{jk}^i B_\beta^j B_\gamma^k,
\end{aligned}$$

where

$$M_{\beta\gamma} = N_i C^i_{jk} B^j_\beta B^k_\gamma, \quad M^\alpha_\beta = g^{\alpha\gamma} M_{\beta\gamma}, H_\beta = N_i(B^i_{0\beta} + \Gamma^{*i}_{0j} B^j_\beta) \tag{12}$$

and $B^i_{\beta\gamma} = \partial B^i_\beta \big/ \partial u^\gamma$, $B^i_{0\beta} = B^i_{\alpha\beta} v^\alpha$. The quantities $M_{\beta\gamma}$ and H_β are called the second fundamental $v-$tensor and normal curvature vector respectively [5]. The second fundamental $h-$tensor $H_{\beta\gamma}$ is defined as [5]

$$H_{\beta\gamma} = N_i(B^i_{\beta\gamma} + \Gamma^{*i}_{jk} B^j_\beta B^k_\gamma) + M_\beta H_\gamma, \tag{13}$$

where

$$M_\beta = N_i C^i_{jk} B^j_\beta N^k. \tag{14}$$

The relative h and $v-$covariant derivatives of projection factor B^i_α with respect to $IC\Gamma$ are given by

$$B^i_{\alpha|\beta} = H_{\alpha\beta} N^i, B^i_\alpha|_\beta = M_{\alpha\beta} N^i. \tag{15}$$

The equation (13) shows that $H_{\beta\gamma}$ is generally not symmetric and

$$H_{\beta\gamma} - H_{\gamma\beta} = M_\beta H_\gamma - M_\gamma H_\beta. \tag{16}$$

The above equation yield

$$H_{0\gamma} = H_\gamma, H_{\gamma 0} = H_\gamma + M_\gamma H_0. \tag{17}$$

We use following lemmas which are due to Matsumoto [5]:

Lemma 1. *The normal curvature $H_0 = H_\beta v^\beta$ vanishes if and only if the normal curvature vector H_β vanishes.*

Lemma 2. *A hypersurface F^{n-1} is a hyperplane of 1st kind if and only if $H_\alpha = 0$.*

Lemma 3. *A hypersurface F^{n-1} is a hyperplane of the 2nd kind with respect to the connection $C\Gamma$ if and only if $H_\alpha = 0$.and $H_{\alpha\beta} = 0$.*

Lemma 4. *A hyperplane of the 3rd kind is characterized by $H_{\alpha\beta} = 0$ and $M_{\alpha\beta} = 0$.*

4. Hypersurface $F^{n-1}(c)$ of the Special Finsler Space

Let us consider special Finsler metric $L=\alpha\cosh(\frac{\beta}{\alpha})$ with a gradient $b_i(x)=\partial_i b$ for a scalar function $b(x)$ and a hypersurface $F^{n-1}(c)$ given by the equation $b(x)$=c(constant) (see [8], [10]). From parametric equation $x^i = x^i(u^\alpha)$ of $F^{n-1}(c)$, we get $\partial_\alpha b(x(u)) = 0 = b_i B^i_\alpha$, so that $b_i(x)$ are regarded as covariant components of a normal vector field of $F^{n-1}(c)$. Therefore, along the $F^{n-1}(c)$. Therefore, along the $F^{n-1}(c)$ we have

$$b_i B^i_\alpha = 0, \quad b_i y^i = 0 \tag{18}$$

The induced metric $L(u,v)$ of $F^{n-1}(c)$ is given by

$$L(u,v) = a_{ij} v^\alpha v^\beta, \quad a_{\alpha\beta} = a_{ij} B^i_\alpha B^j_\beta, \tag{19}$$

which is the Riemannian metric. At a point of $F^{n-1}(c)$, from (2), (3) and (5), we have

$$\begin{aligned} &p=1, \quad q_0=1, \quad q_2=-\alpha^{-2}, \quad p_0=1, \quad p_1=0, \quad p_2=0,\\ &\varsigma=1+b^2, \quad S_0=\tfrac{1}{1+b^2}, \quad S_1=0, \quad S_2=0. \end{aligned} \tag{20}$$

Therefore, from (4) we get

$$g^{ij} = a^{ij} - \frac{1}{1+b^2} b^i b^j. \tag{21}$$

Thus along $F^{n-1}(c)$, (21) and (18) lead to $g^{ij} b_i b_j = \frac{b^2}{1+b^2}$. Therefore, we get

$$b_i(x(u)) = \sqrt{\frac{b^2}{1+b^2}} N_i, b^2 = a^{ij} b_i b_j, \tag{22}$$

where b is the length of the vector b^i. Again from (21) and (22) we get

$$b^i = a^{ij} b_j = \sqrt{b^2(1+b^2)} N^i + b^2 \alpha^{-1} y^i. \tag{23}$$

Thus we have

Theorem 4.1. *In the special Finsler hypersurface $F^{n-1}(c)$,the induced metric a is Riemannian metric given by (19) and the scalar function $b(x)$ is given by (22) and (23).*

Theorem 4.2. *The second fundamental $v-$tensor of special Finsler hypersurface $F^{n-1}(c)$ vanishes and the second fundamental $h-$tensor $H_{\alpha\beta}$ is symmetric.*

Proof. The angular metric tensor and metric tensor of F^n are given by

$$h_{ij} = a_{ij} + b_i b_j - \frac{Y_i Y_j}{\alpha^2}, g_{ij} = a_{ij} + b_i b_j. \tag{24}$$

From (18), (24) and (11) it follows that if $h^{(a)}_{\alpha\beta}$ denote the angular metric tensor of the Riemannian $a_{ij}(x)$, then along $F^{n-1}(c), h_{\alpha\beta} = h^{(a)}_{\alpha\beta}$. From (3), we get

$$\frac{\partial p_0}{\partial \beta} = (\frac{2}{\alpha}) \sinh 2(\frac{\beta}{\alpha}).$$

Thus along $F^{n-1}(c)$, $\frac{\partial p_0}{\partial \beta} = 0$ and therefore gives $\gamma_1 = 0$, $m_i = b_i$. Therefore the $hv-$torsion tensor becomes

$$C_{ijk} = 0, \tag{25}$$

in a special Finsler hypersurface $F^{n-1}(c)$. Therefore, (11), (21), (14), (18) and (25) give

$$M_{\alpha\beta} = 0, M_\alpha = 0. \tag{26}$$

Now (16) implies that $H_{\alpha\beta}$ is symmetric.

Next, we give conditions under which $F^{n-1}(c)$ is a hyperplane of first, second and third kind:

Theorem 4.3. *The hypersurface $F^{n-1}(c)$ of special Finsler space is hyperplane of first kind if and only if $2b_{ij} = b_i c_j + b_j c_i$ holds.*

Proof. From (18), we obtain $b_{i|j} B^i_\alpha + b_i B^i_{\alpha|\beta} = 0$. Thus, from (15) and using $b_{i|j} = b_{i|j} B^j_\beta + b_i|_j N^i H_\beta$, we get

$$b_{i|j} B^i_\alpha B^j_\beta + b_i|_j B^i_\alpha N^j H_\beta + b_i H_{\alpha\beta} N^i = 0. \tag{27}$$

Since $b\ _i|_j = -b_h C^h_{ij}$, we get

$$b\ _i|_j\ B^i_\alpha N^j = 0.$$

Thus (27) gives

$$\sqrt{\frac{b^2}{1+b^2}} H_{\alpha\beta} + b_{i|j} B^i_\alpha B^j_\beta = 0. \tag{28}$$

It is noted that $b_{i|j}$ is symmetric. Furthermore, contracting (28) with v^β and then with v^α and using (10), (17) and (26) we get

$$\sqrt{\frac{b^2}{1+b^2}} H_\alpha + b_{i|j} B^i_\alpha y^j = 0, \tag{29}$$

$$\sqrt{\frac{b^2}{1+b^2}}H_0 + b_{i|j}y^i y^j = 0. \tag{30}$$

In view of lemmas (1) and (2), the hypersurface $F^{n-1}(c)$ is hyperplane of the first kind if and only if $H_0 = 0$. Thus from (30) it follows that $F^{n-1}(c)$ is a hyperplane of the first kind if and only if $b_{i|j}y^i y^j = 0$. Here $b_{i|j}$ being the covariant derivative with respect to $C\Gamma$ of F^n depends on y^i.

Since b_i is a gradient vector, from (7) we have $E_{ij} = b_{ij}, F_{ij} = 0$ and $F^i_j = 0$. Thus (8) reduces to

$$\begin{aligned} D^i_{jk} = B^i b_{jk} + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m} g^{im} B_{jk} \\ - C^i_{jm} A^m_k - C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} \\ + \lambda^n (C^i_{jm} C^m_{sk} + C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms}). \end{aligned} \tag{31}$$

In view of (20) and (21), the relations in (9) become to

$$\begin{aligned} & B_i = b_i, \qquad B^i = \tfrac{b^i}{1+b^2} \\ & B_{ij} = 0 \\ & B^i_j = g^{il} B_{lj} = 0 \\ & A^m_k = B^m b_{k0}, \qquad \lambda^m = B^m b_{00}. \end{aligned} \tag{32}$$

By virtue of (32) we have $B^i_0 = 0$, $B_{i0} = 0$ which leads $A^m_0 = B^m b_{00}$.Therefore we have

$$D^i_{j0} = B^i b_{j0}, \tag{33}$$

$$D^i_{00} = B^i b_{00} = (\frac{b^i}{1+b^2}) b_{00}. \tag{34}$$

Thus from the relation (18), we get

$$b_i D^i_{j0} = \frac{b^2}{1+b^2} b_{j0}, \tag{35}$$

$$b_i D^i_{00} = \frac{b^2}{1+b^2} b_{00}. \tag{36}$$

From (25) it follows that

$$b^m b_i C^i_{jm} B^j_\alpha = b^2 M_\alpha = 0.$$

Therefore, the relation $b_{i|j} = b_{ij} - b_r D^r_{ij}$ and equations (35), (36) give

$$b_{i|j} y^i y^j = b_{00} - b_r D^r_{00} = (\frac{1}{1+b^2}) b_{00}.$$

Consequently, (29) and (30) may be written as

$$\sqrt{b^2}H_\alpha + (\frac{1}{\sqrt{1+b^2}})b_{i0}B^i_\alpha = 0 \tag{37}$$

$$\sqrt{b^2}H_0 + (\frac{1}{\sqrt{1+b^2}})b_{00} = 0. \tag{38}$$

Thus the condition $H_0 = 0$ is equivalent to $b_{00} = 0$, where b_{ij} does not depend on y^i. Since y^i is to satisfy (18), the condition is written as $b_{ij} = (b_i y^i)(c_j y^j)$ for some $c_j(x)$, so that we have

$$2b_{ij} = b_i c_j + b_j c_i, \tag{39}$$

which completes the proof of the theorem.

Theorem 4.4. *If the special hypersurface $F^{n-1}(c)$ of Finsler space is a hyperplane of the first kind then it is a hyperplane of the second kind, too.*

Proof. using (25), (31) and (32), we get $b_k D^k_{ij} = \frac{b^2}{1+b^2} b_{ij}$. Substituting (39) in (28) and applying (18), we obtain

$$H_{\alpha\beta} = 0 \tag{40}$$

Therefore, by the help of Lemmas (1), (2) and (3) and Theorem (4.3) we get the required result.

Theorem 4.5. The hypersurface $F^{n-1}(c)$of Finsler space is a hyperplane of third kind if and only if it is a hyperplane of the first kind.

Proof. This follows from (17), (40) and Theorem 4.2.

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