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ON THE CURVATURE FORMS OF NIL SPACE

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Abstract: In this work, we study structure equations in Nil space. Also, the submanifold of Nil space is given and sectional, Ricci curvatures in this submanifold are obtained by using Nil metric.

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1. Introduction

A Riemann manifold is called homogeneous such that for every two point p and q in M, there exist an isometry mapping p into q. The study of Homogeneous geometries has played a main role in the development modern theory of three manifolds. The famous conjecture of Thurston on the classification of a compact $3-$manifolds in eight 'model geometries', that is, E^3, H^3, S^3, $S^2 \times R$, $H^2 \times R, SL(2,R)$, Nil and Sol in [6].

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Our aim is just to study the Nil geometry. The left invariant metric on Nil_3 defined by

$$ds^2 = dx^2 + dy^2 + (dz + \tau (ydx - xdy))^2 ,$$

here τ is non-zero real number. On the other hand biharmonic maps into Nil space is examined in [5] , higher order paralel surfaces in Nil space is studied in [7] and different aspect is investigated in [2], [5].

In this paper, first of all the structure equations in Nil space is given, secondly submanifold of Nil space is studied and these manifolds obtain that are not totally geodesic. Finally, sectional and Ricci curvatures of this submanifold are obtained.

2. Structure Equations of Nil Space

Let (R^3, g_{Nil}) denote Nil space such that the left-invariant metric can be written as

$$g_{Nil} = ds^2 = dx^2 + dy^2 + (dz + \tau (ydx - xdy))^2 ,$$

where τ is non-zero real number. This metric can be written as

$$ds^2 = \sum_1^3 w^i \otimes w^i,$$

where

$$w^1 = dx, \quad w^2 = dy, \quad w^3 = dz + \tau (ydx - xdy) ,$$

and the orthonormal basis of dual vector fields to the 1-form is

$$\begin{aligned} e_1 &= \frac{\partial}{\partial x} - \tau y \frac{\partial}{\partial z}, \\ e_2 &= \frac{\partial}{\partial y} + \tau x \frac{\partial}{\partial z}, \\ e_3 &= \frac{\partial}{\partial z}. \end{aligned} \tag{2.1}$$

The corresponding Lie brackets are

$$[e_1, e_2] = 2\tau e_3, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = 0. \tag{2.2}$$

Levi-civita connection of Nil_3 is given by

$$\begin{aligned} &\tilde{\nabla}_{e_1} e_1 = 0, && \tilde{\nabla}_{e_1} e_2 = \tau e_3, && \tilde{\nabla}_{e_1} e_3 = -\tau e_2, \\ &\tilde{\nabla}_{e_2} e_1 = -\tau e_3, && \tilde{\nabla}_{e_2} e_2 = 0, && \tilde{\nabla}_{e_2} e_3 = \tau e_1, \\ &\tilde{\nabla}_{e_3} e_1 = -\tau e_2, && \tilde{\nabla}_{e_3} e_2 = \tau e_1, && \tilde{\nabla}_{e_3} e_3 = 0, \end{aligned}$$

in [7]. The non-vanishing Christoffel symbols are obtained by

$$\Gamma_{12}^{3} = -\Gamma_{21}^{3} = \Gamma_{23}^{1} = \Gamma_{32}^{1} = -\Gamma_{13}^{2} = -\Gamma_{31}^{2} = \tau. \tag{2.3}$$

Let $\{e_1, e_2, e_3\}$ be a local orthonormal frame on Nil_3 and $\{w^1, w^2, w^3\}$ be the dual basis $\{e_1, e_2, e_3\}$, for $1 \leq i, j \leq 3$, the connection form w_j^i are calculated by

$$w_j^i = \sum_k g_{Nil} e\left(_i, \tilde{\nabla}_{e_k} e_j\right) w^k = \sum_k \Gamma_{kj}^i w^k,$$

in [3]. Then, we obtain the connection form w_j^i on Nil_3 as follows

$$\begin{array}{ccc} w_1^1 = 0, & w_1^2 = -\tau w^3, & w_1^3 = -\tau w^2, \\ w_2^1 = \tau w^3, & w_2^2 = 0, & w_2^3 = \tau w^1, \\ w_3^1 = \tau w^2, & w_3^2 = -\tau w^1 & w_3^3 = 0. \end{array} \tag{2.4}$$

The Riemannian curvature tensor R of Nil_3 space is defined by using the following convention

$$\tilde{R}(X, Y) Z = \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_X \tilde{\nabla}_Y Z + \tilde{\nabla}_{[X,Y]} Z, \tag{2.5}$$

or

$$\begin{aligned} \tilde{R}(X, Y) Z \;=\; & -3\tau^2 \left(g_{Nil}(X, Z) Y - g_{Nil}(Y, Z) X\right) \\ & +4\tau^2 \left(g_{Nil}(X, e_3) g_{Nil}(Z, e_3) Y\right. \\ & -g_{Nil}(Y, e_3) g_{Nil}(Z, e_3) X + g_{Nil}(Y, e_3) g_{Nil}(X, Z) e_3 \\ & \left.-g_{Nil}(X, e_3) g_{Nil}(Y, Z) e_3\right), \end{aligned} \tag{2.6}$$

where $X, Y, Z \in T_p(Nil_3)$. Then, for $1 \leq i, j, k \leq 3$, we can write the non-vanishing Riemannian curvature tensor R_{ijk} as follow

$$\begin{array}{ll} \tilde{R}_{121} = -\tilde{R}_{211} = -3\tau^2 e_1, & \tilde{R}_{122} = -\tilde{R}_{212} = 3\tau^2 e_1, \\ \tilde{R}_{131} = -\tilde{R}_{311} = \tau^2 e_3, & \tilde{R}_{133} = -\tilde{R}_{313} = -\tau^2 e_1, \\ \tilde{R}_{232} = -\tilde{R}_{321} = \tau^2 e_3, & \tilde{R}_{233} = -\tilde{R}_{323} = -\tau^2 e_2. \end{array} \tag{2.7}$$

Also, for $1 \leq i, j, k, l \leq 3$,

$$\tilde{R}_{klj}^i = g_{Nil}\left(\tilde{R}(e_k, e_l) e_j, e_i\right). \tag{2.8}$$

Considering equation (2.7), the non-vanishing Riemann curvature tensor R_{klj}^i on Nil_3 obtained in the following way

$$\begin{array}{ll} \tilde{R}_{121}^2 = -\tilde{R}_{211}^2 = -3\tau^2, & \tilde{R}_{122}^1 = -\tilde{R}_{212}^1 = 3\tau^2, \\ \tilde{R}_{131}^3 = -\tilde{R}_{311}^3 = \tau^2, & \tilde{R}_{133}^1 = -\tilde{R}_{313}^1 = -\tau^2, \\ \tilde{R}_{232}^3 = -\tilde{R}_{321}^3 = \tau^2, & \tilde{R}_{233}^2 = -\tilde{R}_{323}^2 = -\tau^2. \end{array} \tag{2.9}$$

The structure equations, for $1 \leq i, j, k, l, p \leq 3$,

$$dw^i = -\sum_p w_p^i \wedge w^p, \tag{2.10}$$

$$dw_j^i = -\sum_p w_p^i \wedge w_j^p - \frac{1}{2}\sum_k \sum_l \tilde{R}_{klj}^i w^k \wedge w^l, \quad . \tag{2.11}$$

in [3], [8]. Using above equations, we can show the following;

$$dw^1 = 0, \quad dw^2 = 0, \quad dw^3 = -2\tau w^1 \wedge w^2. \tag{2.12}$$

$$\begin{array}{ccc} dw_1^1 = 0, & dw_1^2 = 2\tau^2 w^1 \wedge w^2, & dw_1^3 = 0, \\ dw_2^1 = -2\tau^2 w^1 \wedge w^2, & dw_2^2 = 0, & dw_2^3 = 0, \\ dw_3^1 = 0, & dw_3^2 = 0, & dw_3^3 = 0. \end{array} \tag{2.13}$$

3. Sectional and Ricci Curvatures in Submanifold of Nil_3 Space

Let $\{e_{1,}e_2, e_3\}$ be a local orthonormal frame on Nil_3 and $\{w^1, w^2, w^3\}$ be the dual basis $\{e_{1,}e_2, e_3\}$, for $1 \leq i, j \leq 3$.The Riemannian curvature tensor R of Nil_3 is given by equation in (2.5).

There are three cases for the submanifold of Nil_3 :

*Case 1.*Let M_1 be differentiable 2-manifold of Nil_3 space. Suppose that e_3 is normal at M_1, that is, $e_3 \in \chi(M_1)^{\perp}$. Then

$$\begin{aligned} \tilde{e}_1 &= a_{11}e_1 + a_{12}e_2, \\ \tilde{e}_2 &= a_{21}e_1 + a_{22}e_2, \end{aligned}$$

where $\tilde{e}_1, \tilde{e}_2$ are basis vector of M_1 and for $1 \leq i, j \leq 2$, a_{ij} are real valued differentiable functions such that $A = [a_{ij}]$ and $\det A \neq 0$.

Corollary 3.1. *M_1 can not be totally geodesic submanifold of Nil_3.*

Proof. Let $\tilde{\nabla}'$ and $\tilde{\nabla}$ be connection M_1 and Nil_3, respectively, and $\widetilde{II}$ be second fundemental form tensor. Then, Gauss equation can be written as follow

$$\tilde{\nabla}_{\tilde{e}_i}\tilde{e}_j = \tilde{\nabla}'_{\tilde{e}_i}\tilde{e}_j + \widetilde{II}(\tilde{e}_i, \tilde{e}_j), \quad 1 \leq i, j \leq 2, \tag{3.1}$$

and so obtained

$$\widetilde{II}(\tilde{e}_i, e_j) = nor\tilde{\nabla}_{\tilde{e}_i}\tilde{e}_j.$$

Hence

$$\widetilde{II}\left(\tilde{e}_1,\tilde{e}_1\right)=nor\tilde{\nabla}_{\tilde{e}_1}\tilde{e}_1=0. \tag{3.2}$$

$$\widetilde{II}\left(\tilde{e}_1,\tilde{e}_2\right)=nor\tilde{\nabla}_{\tilde{e}_1}\tilde{e}_2=\det A\tau e_3. \tag{3.3}$$

$$\widetilde{II}\left(\tilde{e}_2,\tilde{e}_1\right)=nor\tilde{\nabla}_{\tilde{e}_2}\tilde{e}_1=-\det A\tau e_3. \tag{3.4}$$

$$\widetilde{II}\left(\tilde{e}_2,\tilde{e}_2\right)=nor\tilde{\nabla}_{\tilde{e}_2}\tilde{e}_2=0. \tag{3.5}$$

If M_1 is totally geodesic, $\widetilde{II}$ is zero and $\tau\neq 0$. $\widetilde{II}$ equals to zero in $(3.2),(3.3),(3.4)$ and (3.5), we get

$$a_{21}a_{12}-a_{22}a_{11}=0,$$

this means that $\det A=0$. This is a contradiction, that is, $M_1\subset Nil_3$ can not be totally geodesic.

Corollary 3.2. *The sectional curvature $\tilde{K}_1$ of $M_1\subset Nil_3$ is given by*

$$\tilde{K}_1=\frac{a_{21}\left(B_1-D_1+F_1\right)+a_{22}\left(B_2-D_2+F_2\right)}{\left(\det A\right)^2}, \tag{3.6}$$

where
$B_1=a_{21}e_1[A_1]+a_{22}e_2[A_1]$, $B_2=a_{21}e_1[A_2]+a_{22}e_2[A_2]$, $B_3=\left(a_{21}A_2-a_{22}A_1\right)\tau$
$(A_1=a_{11}e_1[a_{11}]+a_{12}e_2[a_{11}]$, $A_2=a_{11}e_1[a_{12}]+a_{12}e_2[a_{12}]$, $A_3=0)$,
$D_1=a_{11}e_1[C_1]+a_{12}e_2[C_1]+a_{12}C_3\tau, D_2=a_{11}e_1[C_2]+a_{12}e_2[C_2]-a_{11}C_3\tau$,
$D_3=a_{11}e_1[C_3]+a_{12}e_2[C_3]-a_{12}C_1\tau+a_{11}C_2\tau$,
$(C_1=a_{21}e_1[a_{11}]+a_{22}e_2[a_{11}]$, $C_2=a_{21}e_1[a_{12}]+a_{22}e_2[a_{12}]$, $C_3=-\tau\det A.)$,
$F_1=2\tau\det A\left(e_3[a_{11}]+\tau a_{12}\right)$, $F_2=2\tau\det A\left(e_3[a_{12}]-\tau a_{11}\right)$.

Proof. We know that

$$\tilde{K}_1\left(\tilde{e}_1,\tilde{e}_2\right)=\frac{g_{Nil}\left(\tilde{R}\left(\tilde{e}_1,\tilde{e}_2\right)\tilde{e}_1,\tilde{e}_2\right)}{g_{Nil}\left(\tilde{e}_1,\tilde{e}_1\right)g_{Nil}\left(\tilde{e}_2,\tilde{e}_2\right)-g_{Nil}\left(\tilde{e}_1,\tilde{e}_2\right)^2},$$

hence, by a straightforward computation, we have

$$\tilde{K}_1=\frac{a_{21}\left(B_1-D_1+F_1\right)+a_{22}\left(B_2-D_2+F_2\right)}{\left(\det A\right)^2}.$$

Corollary 3.3. *For $1\leq i,j,k\leq 3$, if $e_k[a_{ij}]$ is zero, then $\tilde{K}_1=-3\tau^2$, where $e_k[a_{ij}]$ is the directional derivatives of the functions a_{ij} in terms of the vector e_k.*

Proof. Since $e_k[a_{ij}] = 0$, from Corollary 3.2, we get

$$\begin{aligned} B_1 &= 0, B_2 = 0, \\ D_1 &= -\tau^2 a_{12} \det A, D_2 = \tau^2 a_{11} \det A, \\ F_1 &= 2\tau^2 a_{12} \det A, F_2 = -2\tau^2 a_{11} \det A. \end{aligned} \tag{3.7}$$

we substitue (3.7) into (3.6), we have $\tilde{K}_1 = -3\tau^2$.

By Gauss Bonnet theorem, we give the folllowing result.

Corollary 3.4. *If M_1 is compact, then the Euler characteristic of M_1 is negative.*

Corollarry 3.5. *The Ricci curvature of M_1 is always negative.*

Proof. Let $p \in M_1$, $\tilde{e}_1, \tilde{e}_2 \in T_p(M_1)$ and $e_3 \in T_p(M_1)^{\perp}$,

$$Ric_1(e_3) = \frac{1}{2}\sum_i \left\langle \tilde{R}(e_3, \tilde{e}_i)\, e_3, \tilde{e}_i \right\rangle = -\frac{\tau^2}{2}\sum_{i,j} a_{ij}^2, \quad 1 \le i,j \le 2, \tag{3.8}$$

Furthermore computation leads to

$$Ric_1(e_3) = -\frac{\tau^2}{2}\sum_{i,j} a_{ij}^2, \quad 1 \le i,j \le 2.$$

Case 2. Let M_2 be differentiable 2-manifold of Nil_3 space. Suppose that e_2 is normal at M_2, that is, $e_2 \in \chi(M_2)^{\perp}$. Then

$$\begin{aligned} \tilde{e}_1 &= b_{11}e_1 + b_{12}e_3, \\ \tilde{e}_2 &= b_{21}e_1 + b_{22}e_3, \end{aligned}$$

where $\tilde{e}_1, \tilde{e}_2$ are basis vector of M_2 and for $1 \le i,j \le 2$, b_{ij} are real valued differentiable functions such that $B = [b_{ij}]$ and $\det B \neq 0$.

Corollary 3.6. *M_2 can not be totally geodesic submanifold of Nil_3.*

Proof. Let $\tilde{\nabla}'$ and $\tilde{\nabla}$ be connection M_2 and Nil_3, respectively, and $\widetilde{II}$ be second fundemental form tensor. Then, from (3.1),

$$\widetilde{II}(\tilde{e}_1, \tilde{e}_1) = nor\tilde{\nabla}_{\tilde{e}_1}\tilde{e}_1 = -(2b_{11}b_{12})\,\tau e_2 \tag{3.9}$$

$$\widetilde{II}(\tilde{e}_1, \tilde{e}_2) = nor\tilde{\nabla}_{\tilde{e}_1}\tilde{e}_2 = (b_{11}b_{22} + b_{12}b_{21})\,\tau e_2. \tag{3.10}$$

$$\widetilde{II}\left(\tilde{e}_2,\tilde{e}_1\right) = nor\tilde{\nabla}_{\tilde{e}_2}\tilde{e}_1 = \left(b_{21}b_{12}+b_{22}b_{11}\right)\tau e_2. \tag{3.11}$$

$$\widetilde{II}\left(\tilde{e}_2,\tilde{e}_2\right) = nor\tilde{\nabla}_{\tilde{e}_2}\tilde{e}_2 = -\left(2b_{21}b_{22}\right)\tau e_2. \tag{3.12}$$

If M_2 is totally geodesic, $\widetilde{II}$ is zero and $\tau \neq 0$. $\widetilde{II}$ equals to zero in $(3.9), (3.10), (3.11)$ and (3.12), we get

$$\begin{aligned} b_{21}b_{12}+b_{22}b_{11} &= 0, \\ b_{11}b_{12} &= 0, \\ b_{21}b_{22} &= 0, \end{aligned}$$

which means $\det B = 0$. This is a contradiction, so $M_2 \subset Nil_3$ can not be totally geodesic.

Corollary 3.7. *The sectional curvature $\tilde{K}_2$ of $M_2 \subset Nil_3$ is given by*

$$\tilde{K}_2 = \frac{b_{21}\left(H_1-J_1\right)+b_{22}\left(H_3-J_3\right)}{\left(\det B\right)^2}, \tag{3.13}$$

where $H_1 = b_{21}e_1[G_1]+b_{22}e_3[G_1]+b_{22}G_2\tau$,
$H_2 = b_{21}e_1[G_2]+b_{22}e_3[G_2]-b_{21}G_3\tau-b_{22}G_1\tau$,
$H_3 = b_{21}e_1[G_3]+b_{22}e_3[G_3]+b_{21}G_2\tau$,
$\left(G_1 = b_{11}e_1[b_{11}]+b_{12}e_3[b_{11}], G_2 = -2\tau b_{11}b_{12}, G_3 = b_{11}e_1[b_{12}]+b_{12}e_3[b_{12}]\right)$,
$J_1 = b_{11}e_1[I_1]+b_{12}e_3[I_1]+b_{12}I_2\tau$,
$J_2 = b_{11}e_1[I_2]+b_{12}e_3[I_2]-b_{12}I_1\tau-b_{11}I_3\tau$,
$J_3 = b_{11}e_1[I_3]+b_{12}e_3[I_3]+b_{11}I_2\tau$,
$\left(I_1 = b_{21}e_1[b_{11}]+b_{22}e_3[b_{11}], I_2 = -\left(b_{21}b_{12}+b_{22}b_{11}\right)\tau, I_3 = b_{21}e_1[b_{12}]+a_{22}e_3[b_{12}]\right)$.

Proof. We know that

$$\tilde{K}_2\left(\tilde{e}_1,\tilde{e}_2\right) = \frac{g_{Nil}\left(\tilde{R}\left(\tilde{e}_1,\tilde{e}_2\right)\tilde{e}_1,\tilde{e}_2\right)}{g_{Nil}\left(\tilde{e}_1,\tilde{e}_1\right)g_{Nil}\left(\tilde{e}_2,\tilde{e}_2\right)-g_{Nil}\left(\tilde{e}_1,\tilde{e}_2\right)^2},$$

and so this proof can be easily shown by using above equation.

Corollary 3.8. *For $1 \leq i,j,k \leq 3$, if $e_k[a_{ij}]$ is zero, then $\tilde{K}_2 = \tau^2$, where $e_k[b_{ij}]$ is the directional derivatives of the functions b_{ij} in terms of the vector e_k.*

Proof. Since $e_k[b_{ij}]=0$, from Corollary 3.7, we get

$$\begin{aligned} H_1 &= -2\tau^2 b_{11}b_{12}b_{22}, H_3 = -2\tau^2 b_{21}b_{11}b_{12}, \\ J_1 &= -\tau^2 b_{12}\left(b_{21}b_{12}+b_{22}b_{11}\right), J_3 = -\tau^2 b_{11}\left(b_{21}b_{12}+b_{22}b_{11}\right). \end{aligned} \tag{3.14}$$

we substitue (3.14) in (3.13), we have $\tilde{K}_2=\tau^2$.

By Gauss Bonnet theorem, we give the folllowing result.

Corollary 3.9. *If M_2 is compact, then the Euler characteristic of M_2 is positive.*

Corollarry 3.10. *The Ricci curvature of M_2 is given by*

$$Ric_2\left(e_2\right)=\frac{\tau^2}{2}\left(3b_{11}^2-b_{12}^2+3b_{21}^2-b_{22}^2\right). \tag{3.15}$$

Proof. Let $p\in M_2$, $\tilde{e}_1,\tilde{e}_2\in T_p\left(M_2\right)$ and $e_2\in T_p\left(M_2\right)^{\perp}$,

$$Ric_2\left(e_2\right)=\frac{1}{2}\sum_i\left\langle \tilde{R}\left(e_2,\tilde{e}_i\right)e_2,\tilde{e}_i\right\rangle,$$

Furthermore computation leads to

$$Ric_2\left(e_2\right)=\frac{\tau^2}{2}\left(3b_{11}^2-b_{12}^2+3b_{21}^2-b_{22}^2\right).$$

Case 3. Let M_3 be differentiable 2-manifold of Nil_3 space. Suppose that e_1 is normal at M_3, that is, $e_1\in\chi\left(M_3\right)^{\perp}$. Then

$$\begin{aligned} \tilde{e}_1 &= c_{11}e_2+c_{12}e_3, \\ \tilde{e}_2 &= c_{21}e_3+c_{22}e_3, \end{aligned}$$

where $\tilde{e}_1,\tilde{e}_2$ are basis vector of M_3 and for $1\leq i,j\leq 2$, c_{ij} are real valued differentiable functions such that $C=[c_{ij}]$ and $\det C\neq 0$.

Corollary 3.11. *M_3 can not be totally geodesic submanifold of Nil_3.*

Proof. Let $\tilde{\nabla}'$ and $\tilde{\nabla}$ be connection M_3 and Nil_3, respectively, and $\widetilde{II}$ be second fundemental form tensor. Then, from (3.1),

$$\widetilde{II}\left(\tilde{e}_1,\tilde{e}_1\right)=nor\tilde{\nabla}_{\tilde{e}_1}\tilde{e}_1=2b_{11}b_{12}\tau e_1. \tag{3.16}$$

$$\widetilde{II}\left(\tilde{e}_1, \tilde{e}_2\right) = nor\tilde{\bigtriangledown}_{\tilde{e}_1}\tilde{e}_2 = \left(c_{11}c_{22} + c_{12}c_{21}\right)\tau e_1. \tag{3.17}$$

$$\widetilde{II}\left(\tilde{e}_2, \tilde{e}_1\right) = nor\tilde{\bigtriangledown}_{\tilde{e}_2}\tilde{e}_1 = \left(c_{21}c_{12} + c_{22}c_{11}\right)\tau e_1. \tag{3.18}$$

$$\widetilde{II}\left(\tilde{e}_2, \tilde{e}_2\right) = nor\tilde{\bigtriangledown}_{\tilde{e}_2}\tilde{e}_2 = 2b_{21}b_{22}\tau e_1. \tag{3.19}$$

If M_3 is totally geodesic, $\widetilde{II}$ is zero and $\tau \neq 0$. $\widetilde{II}$ equals to zero in $(3.7), (3.8), (3.9)$ and (3.10), we get

$$\begin{aligned} c_{21}c_{12} + c_{22}c_{11} &= 0, \\ c_{11}c_{12} &= 0, \\ c_{21}c_{22} &= 0, \end{aligned}$$

which means $\det C = 0$. This is a contradiction, that is, $M_3 \subset Nil_3$ cannot be totally geodesic.

Corollary 3.12. *The sectional curvature $\tilde{K}_3$ of $M_3 \subset Nil_3$ is*

$$\tilde{K}_3 = \frac{c_{21}\left(L_2 - N_2\right) + c_{22}\left(L_3 - N_3\right)}{\left(\det C\right)^2}, \tag{3.20}$$

where $L_1 = c_{21}e_2[K_1] + c_{22}e_3[K_1] + c_{21}K_3\tau + c_{22}K_2\tau$, $L_2 = c_{21}e_2[K_2] + c_{22}e_3[K_2] - c_{22}M_1\tau$, $L_3 = c_{21}e_2[K_3] + c_{22}e_3[K_3] - c_{21}K_1\tau$, $(K_1 = 2\tau c_{11}c_{12}$, $K_2 = c_{11}e_2[c_{11}] + c_{12}e_3[c_{11}]$, $K_3 = c_{11}e_2[c_{12}] + c_{12}e_3[c_{12}])$, $N_1 = c_{11}e_2[M_1] + c_{12}e_3[M_1] + c_{11}M_3\tau + c_{12}M_2\tau$, $N_2 = c_{11}e_2[M_2] + c_{12}e_3[M_2] - c_{12}M_1\tau$, $N_3 = c_{11}e_2[M_3] + c_{12}e_3[M_3] - c_{11}M_1\tau$, $(M_1 = \left(c_{21}c_{12} + c_{22}c_{11}\right)\tau$, $M_2 = c_{21}e_2[c_{11}] + c_{22}e_3[c_{11}]$, $M_3 = c_{21}e_2[c_{12}] + c_{22}e_3[c_{12}])$.

Proof. We know that

$$\tilde{K}_3\left(\tilde{e}_1, \tilde{e}_2\right) = \frac{g_{Nil}\left(\tilde{R}\left(\tilde{e}_1, \tilde{e}_2\right)\tilde{e}_1, \tilde{e}_2\right)}{g_{Nil}\left(\tilde{e}_1, \tilde{e}_1\right)g_{Nil}\left(\tilde{e}_2, \tilde{e}_2\right) - g_{Nil}\left(\tilde{e}_1, \tilde{e}_2\right)^2},$$

so this proof can be easily shown by using basic calculation.

Corollary 3.13. *For $1 \leq i, j, k \leq 3$, if $e_k[c_{ij}]$ is zero, then $\tilde{K}_3 = \tau^2$, .where $e_k[c_{ij}]$ is the directional derivatives of the functions c_{ij} in terms of the vector e_k.*

Proof. Since $e_k[c_{ij}] = 0$, from Corollary 3.12, we obtain

$$L_2 = -2\tau^2 c_{11}c_{12}c_{22}, L_3 = -2\tau^2 c_{21}c_{11}c_{12}, \tag{3.21}$$

$$N_2 = -\tau^2 c_{12}\left(c_{21}c_{12}+c_{22}c_{11}\right), N_3 = -\tau^2 c_{11}\left(c_{21}c_{12}+c_{22}c_{11}\right).$$

we substitue (3.21) in (3.20), we have $\tilde{K}_3 = \tau^2$.

By Gauss Bonnet theorem, we have the folllowing result.

Corollary 3.14. *If M_3 is compact, then the Euler characteristic of M_3 is positive.*

Corollarry 3.15. *The Ricci curvature of M_3 is given by*

$$Ric_3\left(e_1\right) = \frac{\tau^2}{2}\left(3c_{11}^2 - c_{12}^2 + 3c_{21}^2 - c_{22}^2\right). \tag{3.22}$$

Proof. Let $p \in M_3$, $\tilde{e}_1, \tilde{e}_2 \in T_p\left(M_3\right)$ and $e_1 \in T_p\left(M_3\right)^{\perp}$,

$$Ric_3\left(e_1\right) = \frac{1}{2}\sum_i \left\langle \tilde{R}\left(e_1, \tilde{e}_i\right)e_1, \tilde{e}_i\right\rangle,$$

Furthermore computation leads to

$$Ric_3\left(e_1\right) = \frac{\tau^2}{2}\left(3c_{11}^2 - c_{12}^2 + 3c_{21}^2 - c_{22}^2\right).$$

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