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NEW CLASSES OF ANALYTIC FUNCTIONS

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Abstract: Inequalities conditions for certain new classes of univalent functions were determined. Also,the distortion inequalities for the the new class of univalent functions was established.

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1. Introduction and Definitions

Let $\mathcal{A}(w)$ denote the class of functions of the form

$$f(z) = (z-w) + \sum_{n=2}^{\infty} a_n (z-w)^n, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$ and normalized with $f(w) = 0$, $f'(w) - 1 = 0$ nd w is an arbitrary fixed point in U, see [3], [6], [7]. pagebreak

Also let $A^{+}(w)$ denote the class of functions of the form

$$f(z) = (z-w) + \sum_{n=2}^{\infty} a_n (z-w)^n, \quad (a_n \geq 0), \tag{1.2}$$

which are analytic in U and w is an arbitrary fixed point in U.

Here, we denote by $S^{\star}(w, A, B)$ and $S^{c}(w, A, B)$, $(-1 \leq B < A \leq 1)$ the subclasses of w-starlike and w-convex functions respectively, (see [3], [6], [7] for details). That is

$$S^{\star}(w, A, B) = \left\{ f(z) \in A(w) : \frac{(z-w)f'(z)}{\cdots} \prec \frac{1 + A(z-w)}{\cdots} \right\}$$

and is said to be in a corresponding class of uniformly starlike functions, denoted by $US^{*}(w)$ if

$$Re\left(\frac{(z-w)f'(z)}{f(z)}\right) > \left|\frac{(z-w)f'(z)}{f(z)} - 1\right|.$$

Furthermore, a function $f(z) \in \mathcal{A}(w)$ is said to be in the class of α-uniformly w-convex functions of order β, denoted $US^{c}(w,\alpha,\beta)$ (see [4], [7]) if

$$Re\left(1 + \frac{(z-w)f''(z)}{f'(z)}\right) > \alpha\left|\frac{(z-w)f''(z)}{f'(z)}\right| + \beta) \quad (\alpha \geq 0, 0 \leq \beta < 1).$$

and is also in the corresponding class denoted by $US^{*}(w,\alpha,\beta)$ if

$$Re\left(\frac{(z-w)f'(z)}{f(z)}\right) > \alpha\left|\frac{(z-w)f'(z)}{f(z)} - 1\right| + \beta \quad (\alpha \geq 0; 0 \leq \beta < 1).$$

It is clear and obvious that $f(z) \in US^{c}(w,\alpha,\beta)$ if and only if $(z-w)f'(z) \in US^{*}(w,\alpha,\beta)$.

For $f(z) \in \mathcal{A}(w)$ we define the following linear operator by

$$J_w^k(a,\gamma,\lambda,l)f(z) = \varphi_w(a,\gamma;z) * I_w^k(\lambda,l)f(z), \tag{1.6}$$

where

$$\varphi_w = (z-w) + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(\gamma)_{n-1}}(z-w)^n \quad (z \in U, a \in \Re, \gamma \in \Re \setminus \{0,-1,-2,\ldots\},$$

$(a)_n$ is the pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & n = 0, \\ a(a+1)(a+2)...(a+n-1), & n \in \mathbb{N}, \end{cases}$$

and

$$I_w^k(\lambda,l)f(z) = (z-w) + \sum_{n=2}^{\infty}\left(\frac{1+\lambda(n-1)+l}{1+l}\right)^k a_n(z-w)^n$$

where $\lambda \geq 0, l \geq 0$ and w is an arbitrary fixed point in U.

Therefore, the linear operator $J_w^k(a,\gamma,\lambda,l)$ can be written as

$$J_w^k(a,\gamma,\lambda,l)f(z) = (z-w) + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(\gamma)_{n-1}}\sigma^m(\lambda,l)a_n(z-w)^n, \tag{1.7}$$

where

$$\sigma^k(\lambda,l) = \left(\frac{1+\lambda(n-1)+l}{1+l}\right)^k$$

and w is an arbitrary fixed point in U.

Remark A. If $\lambda = 1, l = 0$ and $w = 0$ the operator reduces to the one studied in [5]. For $k = 0$, the operator in (1.7)reduces to the Carlson-Shaeffer, also for different values of $\lambda, l, w = 0$ it imposes the Aouf et al [3], Al-boudi operator [2], and the Salagean [9].

Let $L_w^{k,\rho}(a, \gamma, \alpha, \lambda, l, A, B)$ denote the subclass of $\mathcal{A}(w)$ consisting of functions $f(z)$ which satisfy the following inequality

$$\frac{J_w^k(a,\gamma,\lambda,l)f(z)}{J_w^\rho(a,\gamma,\lambda,l)f(z)} - \alpha\left|\frac{J_w^k(a,\gamma,\lambda,l)f(z)}{J_w^\rho(a,\gamma,\lambda,l)f(z)} - 1\right| \alpha\frac{1+A(z-w)}{1+B(z-w)}, \tag{1.8}$$

for $\lambda \geq 0, l \geq 0, -1 \leq B < A \leq 1, \alpha \geq 0$ and w is an arbitrary fixed point in U.

Also, let $y_{k,\rho}^{s,w}(a, \alpha, \gamma, \lambda, l, A, B)$ be the subclass of $\mathcal{A}(w)$ consisting of functions $f(z)$ which satisfy the following condition

$$f(z) \in y_{k,\rho}^{s,w}(a,\alpha,\gamma,\lambda,l,A,B) \Longleftrightarrow J_w^k(a,\gamma,\lambda,l)f(z) \in L_w^{k,\rho}(a,\alpha,\gamma,\lambda,l,A,B).$$

For $s = 0$, it is easy to see that $y_{k,\rho}^{0,w}(a, \gamma, \alpha, \lambda, l, A, B) = L_w^{k,\rho}(a, \gamma, \alpha, \lambda, l, A, B)$.

With various special choices of the parameters involved, various existing classes studied by various authors shall be derived. See [1], [3], [5], [6], [7], [8].

Let $P_w \subset P(w)$ denote the class of analytic functions of the form $P_w(z) = 1 + \sum_{n=1}^{\infty} B_n(z-w)^k$ satisfying $ReP_w(z) > 0$ and $P_w(w) = 1$.

2. Coefficient Inequalities for Class $L_w^{k,\rho}(a, \gamma, \alpha, \lambda, l, A, B)$ and $y_{k,\rho}^{s,w}(a, \alpha, \gamma, \lambda, l, A, B)$

Theorem 2.1. *If $f(z) \in \mathcal{A}(w)$ satisfies*

$$\sum_{n=2}^{\infty} \Phi(a,\gamma,\alpha,k,\rho,n,\lambda,l,A,B)(r+d)^{n-1}|a_n| \leq A - B, \tag{2.1}$$

where

$$\Phi(a,\gamma,\alpha,k,\rho,n,\lambda,l,A,B) = (1+2\alpha)\frac{(a)_{n-1}}{(\gamma)_{n-1}}\left|\sigma^k(\lambda,l) - \sigma^\rho(\lambda,l)\right| + \frac{(a)_{n-1}}{(\gamma)_{n-1}}\left|B\sigma^k(\lambda,l) - A\sigma^\rho(\lambda,l)\right|$$

and

$$\sigma^k(\lambda,l)=\left(\frac{1+\lambda(n-1)+l}{1+l}\right)^k \text{ and } \sigma^\rho(\lambda,l)=\left(\frac{1+\lambda(n-1)+l}{1+l}\right)^\rho,$$

for some $\alpha\geq 0,\lambda\geq 0,l\geq 0,-1\leq B<A\leq 1$ *then* $f(z)\in L_w^{k,\rho}(a,\gamma,\alpha,\lambda,l,A,B)$.

Proof. Suppose that (2.1) is true for all the parameters as earlier defined. For $f(z)\in\mathcal{A}(w)$, let us define the function $P_w(z)$ by

$$P_w(z)=\frac{J_w^k(a,\gamma,\lambda,l)f(z)}{J_w^\rho(a,\gamma,\lambda,l)f(z)}-\alpha\left|\frac{J_w^k(a,\gamma,\lambda,l)f(z)}{J_w^\rho(a,\gamma,\lambda,l)f(z)}-1\right|. \tag{2.2}$$

It suffices to show that

$$\left|\frac{P_w(z)-1}{A-BP_w(z)}\right|<1 \quad (z\in U),$$

where w is an arbitrary fixed point in U.

We note that

$$\left|\frac{P_w(z)-1}{A-BP_w(z)}\right|=\left|\frac{J_w^k(a,\gamma,\lambda,l)f(z)-\alpha e^{i\theta}\left|J_w^k(a,\gamma,\lambda,l)f(z)-J_w^\rho(a,\gamma,\lambda,l)f(z)\right|-J_w^\rho(a,\gamma,\lambda,l)f(z)}{AJ_w^\rho(a,\gamma,\lambda,l)f(z)-B\left(J_w^k(a,\gamma,\lambda,l)f(z)-\alpha e^{i\theta}\left|J_w^k(a,\gamma,\lambda,l)f(z)-J_w^\rho(a,\gamma,\lambda,l)f(z)\right|\right)}\right|$$

$$=\left|\frac{\left(J_w^k(a,\gamma,\lambda,l)f(z)-J_w^\rho(a,\gamma,\lambda,l)f(z)\right)-\alpha e^{i\theta}\left|J_w^k(a,\gamma,\lambda,l)f(z)-J_w^\rho(a,\gamma,\lambda,l)f(z)\right|}{(A-B)J_w^\rho(a,\gamma,\lambda,l)f(z)-B\left((J_w^k(a,\gamma,\lambda,l)f(z)-J_w^\rho(a,\gamma,\lambda,l)f(z))\right)-\alpha e^{i\theta}\left|J_w^k(a,\gamma,\lambda,l)f(z)-J_w^\rho(a,\gamma,\lambda,l)f(z)\right|}\right|$$

$$=\left|\frac{\sum_{n=2}^\infty\frac{(a)_{n-1}}{(\gamma)_{n-1}}\left(\sigma^k(\lambda,l)-\sigma^\rho(\lambda,l)\right)a_n(z-w)^{n-1}-\alpha e^{i\theta}\left|\sum_{n=2}^\infty\frac{(a)_{n-1}}{(\gamma)_{n-1}}\left(\sigma^k(\lambda,l)-\sigma^\rho(\lambda,l)\right)a_n(z-w)^{n-1}\right|}{(A-B)-\sum_{n=2}^\infty\frac{(a)_{n-1}}{(\gamma)_{n-1}}\left(B\sigma^k(\lambda,l)-A\sigma^\rho(\lambda,l)\right)a_n(z-w)^{n-1}-\alpha e^{i\theta}\left|\sum_{n=2}^\infty\frac{(a)_{n-1}}{(\gamma)_{n-1}}\left(\sigma^k(\lambda,l)-\sigma^\rho(\lambda,l)\right)a_n(z-w)^{n-1}\right|}\right|$$

$$\leq\frac{\sum_{n=2}^\infty\frac{(a)_{n-1}}{(\gamma)_{n-1}}\left|\sigma^k(\lambda,l)-\sigma^\rho(\lambda,l)\right||a_n||z-w|^{n-1}+\alpha|e|^{i\theta}\sum_{n=2}^\infty\frac{(a)_{n-1}}{(\gamma)_{n-1}}\left|\sigma^k(\lambda,l)-\sigma^\rho(\lambda,l)\right||a_n||z-w|^{n-1}}{(A-B)-\sum_{n=2}^\infty\frac{(a)_{n-1}}{(\gamma)_{n-1}}\left|B\sigma^k(\lambda,l)-A\sigma^\rho(\lambda,l)\right||a_n||z-w|^{n-1}-\alpha|e|^{i\theta}\sum_{n=2}^\infty\frac{(a)_{n-1}}{(\gamma)_{n-1}}\left|\sigma^k(\lambda,l)-\sigma^\rho(\lambda,l)\right||a_n||z-w|^{n-1}}$$

$$\leq\frac{\sum_{n=2}^\infty\frac{(a)_{n-1}}{(\gamma)_{n-1}}(r+d)^{n-1}\left|\sigma^k(\lambda,l)-\sigma^\rho(\lambda,l)\right||a_n|+\alpha\sum_{n=2}^\infty\frac{(a)_{n-1}}{(\gamma)_{n-1}}(r+d)^{n-1}\left|\sigma^k(\lambda,l)-\sigma^\rho(\lambda,l)\right||a_n|}{(A-B)-\sum_{n=2}^\infty\frac{(a)_{n-1}}{(\gamma)_{n-1}}(r+d)^{n-1}\left|\sigma^k(\lambda,l)-\sigma^\rho(\lambda,l)\right||a_n|-\alpha\sum_{n=2}^\infty\frac{(a)_{n-1}}{(\gamma)_{n-1}}(r+d)^{n-1}\left|\sigma^k(\lambda,l)-\sigma^\rho(\lambda,l)\right||a_n|}.$$

The last expression is bounded above by 1 if

$$\sum_{n=2}^\infty\frac{(a)_{n-1}}{(\gamma)_{n-1}}(r+d)^{n-1}\left|\sigma^k(\lambda,l)-\sigma^\rho(\lambda,l)\right||a_n|$$
$$+\alpha\sum_{n=2}^\infty\frac{(a)_{n-1}}{(\gamma)_{n-1}}(r+d)^{n-1}\left|\sigma^k(\lambda,l)-\sigma^\rho(\lambda,l)\right||a_n|$$
$$\leq(A-B)-\sum_{n=2}^\infty\frac{(a)_{n-1}}{(\gamma)_{n-1}}(r+d)^{n-1}\left|\sigma^k(\lambda,l)-\sigma^\rho(\lambda,l)\right||a_n|$$

$$-\alpha\sum_{n=2}^{\infty}\frac{(a)_{n-1}}{(\gamma)_{n-1}}(r+d)^{n-1}\left|\sigma^{k}(\lambda,l)-\sigma^{\rho}(\lambda,l)\right|,$$

which is equivalent to the condition (2.1) and the proof is complete.

Corollary A. *If* $f(z)\in\mathcal{A}(w)$ *satisfies*

$$\sum_{n=2}^{\infty}\Phi(a,\gamma,\alpha,1,0,n,\lambda,l,A,B)(r+d)^{n-1}\left|a_n\right|\le A-B, \tag{2.3}$$

where

$$\Phi(a,\gamma,\alpha,1,0,n,\lambda,l,A,B) = (1+2\alpha)\frac{(a)_{n-1}}{(\gamma)_{n-1}}\left|\frac{\lambda(n-l)}{1+l}\right| + \frac{(a)_{n-1}}{(\gamma)_{n-1}}\left|B\left(\frac{1+\lambda(n-1)+l}{1+l}\right)-A\right|,$$

for some $\alpha\ge 0, -1\le B<A\le 1, \lambda>0, l\ge 0$ *then* $US^{\square}(w,\alpha,A,B)$.

Corollary B. *If* $f(z)\in\mathcal{A}(w)$ *satisfies*

$$\sum_{n=2}^{\infty}\Phi(a,\gamma,\alpha,1,0,n,1,0,A,B)(r+d)^{n-1}\left|a_n\right|\le A-B, \tag{2.4}$$

where

$$\Phi(a,\gamma,\alpha,1,0,n,1,0,A,B) = (1+2\alpha)\frac{(a)_{n-1}}{(\gamma)_{n-1}}\left|n-1\right| + \frac{(a)_{n-1}}{(\gamma)_{n-1}}\left|Bn-A\right|,$$

for some $\alpha\ge 0, -1\le B<A\le 1,$ *then* $f(z)\in US^{\square}(w,\alpha,A,B)$.

Corollary C. *If* $f(z)\in\mathcal{A}(w)$ *satisfies*

$$\sum_{n=2}^{\infty}\Phi(a,\gamma,\alpha,2,1,n,\lambda,l,A,B)(r+d)^{n-1}\left|a_n\right|\le A-B, \tag{2.5}$$

where

$$\begin{aligned}\Phi(a,\gamma,\alpha,2,1,n,\lambda,l,A,B) &= (1+2\alpha)\frac{(a)_{n-1}}{(\gamma)_{n-1}}\left(\frac{1+\lambda(n-l)+l}{1+l}\right)\left|\frac{\lambda(n-l)}{1+l}\right|\\ &\quad + \frac{(a)_{n-1}}{(\gamma)_{n-1}}\left(\frac{1+\lambda(n-1)+l}{1+l}\right)\left|B\left(\frac{1+\lambda(n-1)+l}{1+l}\right)-A\right|,\end{aligned}$$

for some $\alpha \geq 0, -1 \leq B < A \leq 1$, *then* $f(z) \in US^c(w, \alpha, A, B)$.

Theorem 2.2. *If* $f(z) \in \mathcal{A}(w)$ *satisfies*

$$\sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(\gamma)_{n-1}} \left(\frac{1+\lambda(n-1)+l}{1+l}\right)^s \Phi(a,\gamma,\alpha,k,\rho,n,\lambda,l,A,B)(r+d)^{n-1}|a_n| \leq A-B, \quad (2.6)$$

where $\Phi(a,\gamma,\alpha,k,\rho,n,\lambda,l,A,B)$ *is as defined in Theorem 2.1, then*

$$f \in y^s_{k,\rho}(a,\gamma,\alpha,k,\rho,n,\lambda,l,A,B).$$

Theorem 2.3. *If* $f \in L^{k,\rho}_w(a,\gamma,\alpha,\lambda,l,A,B)$, *then for* $|z-w| = r+d < 1$,

$$\frac{1-(A-B)(r+d)-AB(r+d)^2}{1-B^2(r+d)^2} \leq Re\left[\frac{J^k_w(a,\gamma,\lambda,l)f(z)}{J^\rho_w(a,\gamma,\lambda,l)f(z)} - \alpha\left|\frac{J^k_w(a,\gamma,\lambda,l)f(z)}{J^\rho_w(a,\gamma,\lambda,l)f(z)} - 1\right|\right] \leq \frac{1+(A-B)(r+d)-AB(r+d)^2}{1-B^2(r+d)^2}, \quad B \neq 0, \quad (2.7)$$

$$1-A(r+d) \leq Re\left[\frac{J^k_w(a,\gamma,\lambda,l)f(z)}{J^\rho_w(a,\gamma,\lambda,l)f(z)} - \alpha\left|\frac{J^k_w(a,\gamma,\lambda,l)f(z)}{J^\rho_w(a,\gamma,\lambda,l)f(z)} - 1\right|\right] \leq 1+A(r+d), \quad B = 0. \quad (2.8)$$

Proof. Let $P_w(z)$ be defined as

$$P(z) < \frac{1+A(z-w)}{1+B(z-w)}, \quad |z-w| = (r+d) < 1,$$

then

$$\left|P_w(z) - \frac{1-AB(r+d)^2}{1-B^r(r+d)^2}\right| < \frac{(A-B)(r+d)}{1-B^r(r+d)^2}, \quad B \neq 0,$$
$$|P_w(z)-1| < A(r+d), \quad B = 0.$$

By the definition of $L^{k,\rho}_w(a,\gamma,\alpha,\lambda,l,A,B)$, the inequalities (2.7) and (2.8) can be written in the form

$$\left|\frac{J_w^k(a,\gamma,\lambda,l)f(z)}{J_w^\rho(a,\gamma,\lambda,l)f(z)} - \alpha\left|\frac{J_w^k(a,\gamma,\lambda,l)f(z)}{J_w^\rho(a,\gamma,\lambda,l)f(z)} - 1\right| - \frac{1-AB(r+d)^2}{1-B^r(r+d)^2}\right| < \frac{(A-B)(r+d)}{1-B^2(r+d)^2}, \quad B \neq 0,$$

$$\left|\frac{J_w^k(a,\gamma,\lambda,l)f(z)}{J_w^\rho(a,\gamma,\lambda,l)f(z)} - \alpha\left|\frac{J_w^k(a,\gamma,\lambda,l)f(z)}{J_w^\rho(a,\gamma,\lambda,l)f(z)} - 1\right| - 1\right| < A(r+d), \quad B = 0.$$

Hence, we obtain (2.7) and (2.8) of Theorem 2.3.

Theorem 2.4. *If* $f(z) \in y_{k,\rho}^{s,w}(a,\gamma,\lambda,l,A,B)$ *then for* $|z-w| = r+d < 1$

$$\frac{1-(A-B)(r+d)-AB(r+d)^2}{1-B^2(r+d)^2} \leq Re\left[\frac{J_w^k(a,\gamma,\lambda,l)J_w^s(a,\gamma,\lambda,l)(f(z)}{J_w^\rho(a,\gamma,\lambda,l)J_w^s(a,\gamma,\lambda,l)f(z)} - \alpha\left|\frac{J_w^k(a,\gamma,\lambda,l)J_w^s(a,\gamma,\lambda,l)f(z)}{J_w^\rho(a,\gamma,\lambda,l)J_w^s(a,\gamma,\lambda,l)f(z)} - 1\right|\right] \leq \frac{1+(A-B)(r+d)-AB(r+d)^2}{1-B^2(r+d)^2}, \quad B \neq 0,$$

$$1-A(r+d) \leq Re\left[\frac{J_w^k(a,\gamma,\lambda,l)J_w^s(a,\gamma,\lambda,l)f(z)}{J_w^\rho(a,\gamma,\lambda,l)J_w^s(a,\gamma,\lambda,l)f(z)} - \alpha\left|\frac{J_w^k(a,\gamma,\lambda,l)J_w^s(a,\gamma,\lambda,l)f(z)}{J_w^\rho(a,\gamma,\lambda,l)J_w^s(a,\gamma,\lambda,l)f(z)} - 1\right|\right] \leq 1+A(r+d), \quad B = 0.$$

Corollary E. *If* $f(z) \in US^{*}(w,\alpha,A,B)$, *then for* $|z-w| = r+d < 1$

$$\frac{1-(A-B)(r+d)-AB(r+d)^2}{1-B^2(r+d)^2} \leq Re\left[\frac{(z-w)f'(z)}{f(z)} - \alpha\left|\frac{(z-w)f'(z)}{f(z)} - 1\right|\right] \leq \frac{1+(A-B)(r+d)-AB(r+d)^2}{1-B^2(r+d)^2}, \quad B \neq 0,$$

$$1-A(r+d) \leq Re\left[\frac{(z-w)f'(z)}{f(z)} - \alpha\left|\frac{(z-w)f'(z)}{f(z)} - 1\right|\right] \leq 1+A(r+d), \quad B \neq 0.$$

Corollary F. *If* $f(z) \in US^c(w,\alpha,A,B)$, *then for* $|z-w| = r+d < 1$

$$\frac{1-(A-B)(r+d)-AB(r+d)^2}{1-B^2(r+d)^2}$$
$$\leq Re\left[1+\frac{(z-w)f''(z)}{f'(z)}-\alpha\left|\frac{(z-w)f''(z)}{f'(z)}\right|\right]$$
$$\leq \frac{1+(A-B)r-AB(r+d)^2}{1-B^2(r+d)^2}, \quad B\neq 0$$

$$1-A(r+d)\leq Re\left[1+\frac{(z-w)f''(z)}{f'(z)}-\alpha\left|\frac{(z-w)f''(z)}{f'(z)}-1\right|\right]\leq 1+A(r+d), \quad B=0.$$

3. Distortion Inequalities

Lemma B. *If $f(z)\in L_w^{k,\rho}(a,\gamma,\alpha,\lambda,l,A,B)$, then we have*

$$\sum_{n=p+1}^{\infty} a_n \leq \frac{(A-B)-\sum_{n=2}^{p}\Phi(a,\gamma,\alpha,k,\rho,n,\lambda,l,A,B)(r+d)^{n-1}a_n}{\Phi(a,\gamma,\alpha,k,\rho,p+1,\lambda,l,A,B)}, \quad (3.1)$$

where $\Phi(a,\gamma,\alpha,k,\rho,n,\lambda,l,A,B)$ is as earlier defined in Theorem 2.1.

Proof. In view of Theorem 2.1, we can write

$$\sum_{n=p+1}^{\infty}\Phi(a,\gamma,\alpha,n,k,\rho,\lambda,l,A,B)(r+d)^{n-1}a_n$$
$$\leq (A-B)-\sum_{n=2}^{p}\Phi(a,\gamma,\alpha,n,\lambda,l,A,B)(r+d)^{n-1}a_n.$$

Clearly, $\Phi(a,\gamma,\alpha,n,k,\rho,\lambda,l,A,B)(r+d)^{n-1}$ is an increasing function for n.

Then from (2.1), we have

$$\Phi(a,\gamma,\alpha,p+1,k,\rho,\lambda,l,A,B)\sum_{n=p+1}^{\infty}(r+d)^{n-1}a_n$$
$$\leq (A-B)-\sum_{n=2}^{p}\Phi(a,\gamma,\alpha,k,\rho,n,\lambda,l,A,B)(r+d)^{n-1}a_n.$$

Thus, we obtain

$$\sum_{n=p+1}^{\infty} a_n \leq \frac{(A-B)-\sum_{n=2}^{p}\Phi(a,\gamma,\alpha,k,\rho,n,\lambda,l,A,B)(r+d)^{n-1}a_n}{\Phi(a,\gamma,\alpha,k,\rho,p+1,\lambda,l,A,B)}$$
$$=A_n. \quad (3.2)$$

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