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SOME PROPERTIES OF $N(k)$-QUASI EINSTEIN MANIFOLDS

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Abstract: In this paper, an example of an $N(k)$-quasi Einstein manifold with closed associated 1-form is given. Also we show that if a quasi Einstein manifold $(M_1^{n_1}, g_1)$ and an Einstein manifold $(M_2^{n_2}, g_2)$ satisfy a certain condition, then the Riemannian product manifold $(M^n, g) = (M_1^{n_1} \times M_2^{n_2}, g_1 + g_2)$ is a quasi Einstein manifold. In particular, in $N(k)$-quasi Einstein case, we show that there exists a quasi Einstein product manifold $(M^n, g) = (M_1^{n_1} \times M_2^{n_2}, g_1 + g_2)$ but not an $N(k)$-quasi Einstein manifold, which consists of an $N(k)$-quasi Einstein manifold $(M_1^{n_1}, g_1)$ and an Einstein manifold $(M_2^{n_2}, g_2)$ satisfying the certain condition. Finally we study an $N(k)$-quasi Einstein manifold satisfying the condition $R(U, X) \cdot G = 0$.

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1. Introduction

A non-flat Riemannian manifold (M^n, g) of dimension $n \geq 3$ is said to be a quasi Einstein manifold if the Ricci tensor r of (M^n, g) is not identically zero

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and satisfies the condition

$$r(X,Y) = ag(X,Y) + bu(X)u(Y) \tag{1.1}$$

for some smooth functions a and $b \neq 0$, where u is a non-zero 1-form such that

$$g(X,U) = u(X), g(U,U) = u(U) = 1$$

for the associated unit vector field U. The 1-form u is called the associated 1-form and the unit vector field U is called the generator of the quasi Einstein manifold (M^n, g). The notion of quasi Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi umbilical hypersurfaces. Investigations in [2] and others have revealed that a conformally flat quasi Einstein manifold has the geometric structure of quasi constant curvature. Also it has been found that a manifold of quasi constant constant is a natural subclass of quasi Einstein manifold [4].

Let R denote the Riemannian curvature tensor of (M^n, g). The k-nullity distribution $N(k)$ of (M^n, g) is defined for all $X, Y \in T_pM^n$ by

$$N(k) : p \mapsto N_p(k) = \{Z \in T_pM^n : R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y)\},$$

where k is a smooth function. If the generator U of a quasi Einstein manifold (M^n, g) belongs to the k-nullity distribution $N(k)$, then the quasi Einstein manifold (M^n, g) is called as an $N(k)$-quasi Einstein manifold. In [12], it was shown that a conformally flat quasi Einstein manifold is an $N(k)$-quasi Einstein manifold and in particular, a quasi Einstein manifold of dimension 3 is an $N(k)$-quasi Einstein manifold.

In [5], [6] and [7], Pokhariyal and Mishra introduced a type of curvature-like tensor called G-curvature tensor and studied its relativistic significance. According to them, a G-curvature tensor G on a Riemannian manifold (M^n, g) is defined by

$$\begin{aligned} G(X,Y,Z,V) = R(X,Y,Z,V) - \frac{1}{2(n-1)}(&g(X,Z)r(Y,V) - g(X,V)r(Y,Z) \\ &-g(Y,Z)r(X,V) + g(Y,V)r(X,Z)). \end{aligned} \tag{1.2}$$

In this connection, they introduced a W^*-curvature tensor defined on the line of Weyl projective curvature tensor. The G-curvature tensor G has been defined by breaking W^* into skew-symmetric parts [7].

In this paper, we give an example of an $N(k)$-quasi Einstein manifold with closed associated 1-form. In case of Riemannian product manifolds, we show that if a quasi Einstein manifold $(M_1^{n_1}, g_1)$ and an Einstein manifold $(M_2^{n_2}, g_2)$ satisfy the condition $n_2 a_1 = s_2$, then the Riemannian product manifold (M^n, g) of $(M_1^{n_1}, g_1)$ and $(M_2^{n_2}, g_2)$ is a quasi Einstein manifold. Therefore we can raise a natural question whether or not the Riemannian product manifold $(M^n, g) = (M_1^{n_1} \times M_2^{n_2}, g_1 + g_2)$ is an $N(k)$-quasi Einstein manifold if the quasi Einstein manifold $(M_1^{n_1}, g_1)$ from the above mentioned statement is in fact an $N(k)$-quasi Einstein manifold. This paper gives a negative answer of this question. Finally an $N(k)$-quasi Einstein manifold satisfying the condition $R(U, X) \cdot G = 0$ has been studied.

2. Preliminaries

Let (M^n, g) be a Riemannian manifold. The Riemannian curvature tensor R, Ricci tensor r and scalar curvature s are defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{2.1}$$

And

$$R(X, Y, Z, W) = g(R(X, Y)Z, W). \tag{2.2}$$

$$r(Y, Z) = \sum_{i=1,\dots,n} R(e_i, Y, Z, e_i), \tag{2.3}$$

$$s = \sum_{i=1,\dots,n} r(e_i, e_i), \tag{2.4}$$

where $\{e_i\}_{i=1,..,n}$ is an orthonormal frame. The Riemannian manifold (M^n, g) is called an Einstein manifold if the Ricci tensor r is proportional to the metric tensor g , i.e., $r = \frac{s}{n} g$. The Riemannian curvature tensor R has the following well known $SO(n)$-decomposition [1];

$$R = \frac{s}{2n(n-1)} g \bullet g + \frac{1}{n-2} r_o \bullet g + W, \tag{2.5}$$

where r_o is the traceless Ricci tensor and W is the Weyl curvature tensor. Here the symbol $\bullet$ is the Nomizu-Kulkarni product of symmetric (0,2)-tensors generating a curvature type tensor:

$$h \bullet k(X, Y, Z, V) = h(X, Z)k(Y, V) + h(Y, V)k(X, Z) - h(X, V)k(Y, Z) - h(Y, Z)k(X, V).$$

Note that $r_o = 0$ if and only if (M^n, g) is Einstein, and $W = 0$ if and only if (M^n, g) is conformally flat. The Weyl curvature tensor depends only on the conformal class of (M^n, g). Moreover, it satisfies the curvature symmetries and so we can treat it as a conformal curvature tensor. Note that the Weyl curvature tensor is traceless.

For a $(0, k)$-tensor field A on M^n, we define a $(0, k+2)$-tensor field $R \cdot A$ (see [10] and [11]) by

$$\begin{aligned}(R \cdot A)(X_1, \ldots, X_k; X, Y) &= (R(X, Y) \cdot A)(X_1, \ldots, X_k) \\ &= -A(R(X, Y)X_1, X_2, \ldots, X_k) - \ldots - A(X_1, \ldots, X_{k-1}, R(X, Y)X_k). \quad (2.6)\end{aligned}$$

If a Riemannian manifold (M^n, g) satisfies the condition $R \cdot R = 0$ (resp. $R \cdot r = 0$), then (M^n, g) is said to be semisymmetry (resp. Ricci-semisymmetry) (see [10] and [11]). It is well known that the class of semisymmetric manifolds includes the set of locally symmetric manifolds ($\nabla R = 0$) as a proper subset and that the class of Ricci -semisymmetric manifolds includes the set of Ricci-symmetric manifolds ($\nabla r = 0$) as a proper subset (see [10] and [11]).

In a quasi Einstein manifold (M^n, g) if the generator U belongs to some k-nullity distribution $N(k)$, then we get

$$R(X, Y)U = k(u(Y)X - u(X)Y), \quad (2.7)$$

which is equivalent to

$$R(X, U)Y = k(u(Y)X - g(X, Y)U) = -R(U, X)Y. \quad (2.8)$$

Moreover in [8] and [12], for an $N(k)$-quasi Einstein manifold (M^n, g)

$$k = \frac{a + b}{n - 1}. \quad (2.9)$$

3. Product Manifolds and $N(k)$-Quasi Einstein Manifolds

In this section, we investigate some relations between product manifolds and $N(k)$-quasi Einstein manifolds. First of all, we give an example of an $N(k)$-quasi Einstein manifold with closed associated 1-form.

Example 3.1. Let (R^n, g) $(n \geq 3)$ be a Riemannian manifold endowed with the metric g given by

$$g = e^{2x_1}(dx_1^2 + dx_2^2 + \cdots + dx_n^2).$$

It is easy to see that the Riemannian manifold (M^n, g) is conformally flat. From (2.5), it follows that on (R^n, g)

$$r_{11} = 2 - n$$

and

$$r_{ij} = 0$$

otherwise, where $r_{ij} = r(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$.

Therefore we get

$$r = 0g + (2-n)e^{-2x_1} u \otimes u,$$

where $u = e^{x_1} dx_1$. And hence (R^n, g) is a quasi Einstein manifold with $a = 0$ and $b = (2-n)e^{-2x_1} \neq 0$. In fact, (R^n, g) is an $N(k)$-quasi Einstein manifold because (R^n, g) is conformally flat. Furthermore, it is easy to see that

$$du = 0.$$

Summing up the above arguments, we obtain the following: R^n allows a Riemannian metric g such that (R^n, g) is an $N(k)$-quasi Einstein manifold whose associated 1-form u is closed.

Concerning Riemannian product manifolds, we have

Theorem 3.2. *Let $(M_1^{n_1}, g_1)$ and $(M_2^{n_2}, g_2)$ be a quasi Einstein manifold and an Einstein manifold, respectively. Then the Riemannian product manifold $(M^n, g) = (M_1^{n_1} \times M_2^{n_2}, g_1 + g_2)$ is a quasi Einstein manifold if the following condition*

$$n_2 a_1 = s_2 \tag{3.1}$$

is satisfied. (Here a_1 is the associated scalar of a quasi Einstein manifold $(M_1^{n_1}, g_1)$ and s_2 is the scalar curvature of an Einstein manifold $(M_2^{n_2}, g_2)$)

Proof. Since $(M_1^{n_1}, g_1)$ and $(M_2^{n_2}, g_2)$ are a quasi Einstein manifold and an Einstein manifold, respectively, we get

$$r_1 = a_1 g_1 + b_1 u \otimes u$$

and

$$r_2 = \frac{s_2}{n_2} g_2.$$

By virtue of the condition $n_2 a_1 = s_2$, the Riemannian product manifold (M^n, g) $= (M_1^{n_1} \times M_2^{n_2}, g_1 + g_2)$ has its Ricci tensor r as

$$r = r_1 + r_2 = a_1(g_1 + g_2) + b_1 u \otimes u$$

$$= a_1 g + b_1 u \otimes u,$$

which implies that (M^n, g) is a quasi Einstein manifold. This completes the proof of theorem 3.1. □

In case of $N(k)$-quasi Einstein manifolds, we are able to show that the $N(k)$-quasi Einstein version of Theorem 3.1 does not hold true:

Example 3.3. Let (R^3, g_1) be a conformally flat manifold endowed with the metric g_1 given by

$$g_1 = e^{2x_1}(dx_1^2 + dx_2^2 + dx_3^2).$$

From the previous example, it follows that (R^3, g_1) is an $N(k)$-quasi Einstein manifold. More precisely, its Ricci tensor r_1 satisfies the relation

$$r_1 = 0g_1 + (-e^{-2x_1})u \otimes u,$$

where $a_1 = 0, b_1 = -e^{-2x_1} \neq 0$ and $u = e^{x_1}dx_1$. On the other hand, let (R^{n-3}, g_2) be a standard flat manifold with the metric g_2 given by

$$g_2 = dx_4^2 + dx_5^2 + \cdots + dx_n^2.$$

It is obvious that its Ricci tensor r_2 satisfies the relation

$$r_2 = 0g_2,$$

which implies that (R^{n-3}, g_2) is an Einstein manifold with $s_2 = 0$. Therefore the Riemannian product manifold $(R^n, g) = (R^3 \times R^{n-3}, g_1 + g_2)$ is a quasi Einstein manifold since the condition (3.12) is satisfied. Note that the generator of the quasi Einstein product manifold $(R^n, g) = (R^3 \times R^{n-3}, g_1 + g_2)$ is given by

$$U = e^{-x_1}\frac{\partial}{\partial x_1}.$$

However the quasi Einstein product manifold $(R^n, g) = (R^3 \times R^{n-3}, g_1 + g_2)$ is not an $N(k)$-quasi Einstein manifold. In fact, if we assume that the Riemannian product manifold $(R^n, g) = (R^3 \times R^{n-3}, g_1 + g_2)$ is an $N(k)$-quasi Einstein manifold, then taking account of (2.4),(2.9) and (2.11), we have

$$R(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_4})$$
$$= \frac{-e^{-2x_1}}{n-1}[g(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_4})g(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_4}) - g(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1})g(\frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_4})]$$

$$= \frac{1}{n-1}.$$

On the other hand, from the definition of Riemannian product manifold, it follows that

$$R(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_4})$$
$$= R_1(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_4}) + R_2(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_4}) = 0,$$

where R_1 and R_2 are the Riemannian curvature tensors of (R^3, g_1) and (R^{n-3}, g_2), respectively. This is a contradiction. Summing up the above arguments, we obtain the following: There exists a quasi Einstein product manifold $(R^n, g) = (R^3 \times R^{n-3}, g_1 + g_2)$ but not an $N(k)$-quasi Einstein manifold, which consists of an $N(k)$-quasi Einstein manifold (R^3, g_1) and an Einstein manifold (R^{n-3}, g_2) satisfying the condition (3.12).

4. $N(k)$-Quasi Einstein Manifold Satisfying $R(U, X) \cdot G = 0$

In this section, we consider an $N(k)$-quasi Einstein manifold satisfying the condition $R(U, X) \cdot G = 0$. First of all, we have

Lemma 4.1. *In an $N(k)$-quasi Einstein manifold (M^n, g), the G-curvature tensor satisfies the relation*

$$G(X, Y, Z, U) = \frac{4a + 3b}{2(n-1)}(u(X)g(Y, Z) - u(Y)g(X, Z)) \tag{4.1}$$

for all vector fields X, Y, Z on M^n.

Proof. Taking account of (1.1),(1.2),(2.9) and (2.11), we have

$$G(X, Y, Z, U) = R(X, Y, Z, U) - \frac{1}{2(n-1)}[g(X, Z)r(Y, U) -$$
$$-g(X, U)r(Y, Z) - g(Y, Z)r(X, U) + g(Y, U)r(X, Z)]$$
$$= \frac{a+b}{n-1}[u(X)g(Y, Z) - u(Y)g(X, Z)] - \frac{1}{2(n-1)}[g(X, Z)(ag(Y, U) + bu(Y)u(U))$$
$$-g(X, U)(ag(Y, Z) + bu(Y)u(Z)) - g(Y, Z)(ag(X, U) + bu(X)u(U)) +$$
$$+g(Y, U)(ag(X, Z) + bu(X)u(Z))]$$

$$= \frac{4a+3b}{2(n-1)}(u(X)g(Y,Z) - u(Y)g(X,Z)).$$

This completes the proof of Lemma 4.1. □

Now we give the main results of this section.

Theorem 4.2. *Let (M^n, g) be an $N(k)$-quasi Einstein manifold. If (M^n, g) satisfies the condition $R(U,X)\cdot G = 0$, then $a+b=0$.*

Proof. Assume that (M^n, g) is an $N(k)$-quasi Einstein manifold and satisfies the condition $R(U,X)\cdot G = 0$. Then from (2.8) we can write

$$0 = G(R(U,X)Y,Z,V,W) + G(Y,R(U,X)Z,V,W) + G(Y,Z,R(U,X)V,W) + \\ + G(Y,Z,V,R(U,X)W). \tag{4.2}$$

From (2.10) and (4.14), it follows that

$$0 = \frac{a+b}{n-1}[G(g(X,Y)U - g(U,Y)X, Z,V,W) + G(Y, g(X,Z)U - g(U,Z)X, V,W) + \\ + G(Y,Z,g(X,V)U - g(U,V)X, W) + G(Y,Z,V,g(X,W)U - g(U,W)X)].$$

From the above identity, it follows that

$$0 = \frac{a+b}{n-1}[g(X,Y)G(U,Z,V,W) - g(U,Y)G(X,Z,V,W) + \\ + g(X,Z)G(Y,U,V,W) - g(U,Z)G(Y,X,V,W) + g(X,V)G(Y,Z,U,W) - \\ - g(U,V)G(Y,Z,X,W) + g(X,W)G(Y,Z,V,U) - g(U,W)G(Y,Z,V,X)].$$

Substituting $W = U$ into the above identity, we get either

$$a + b = 0 \tag{4.3}$$

or

$$0 = g(X,Y)G(U,Z,V,U) - g(U,Y)G(X,Z,V,U) + \\ + g(X,Z)G(Y,U,V,U) - g(U,Z)G(Y,X,V,U) + g(X,V)G(Y,Z,U,U) - \\ - g(U,V)G(Y,Z,X,U) + g(X,U)G(Y,Z,V,U) - g(U,U)G(Y,Z,V,X). \tag{4.4}$$

Now assume that $a+b \neq 0$. Taking account of (4.13) and (4.16), we have

$$G(Y,Z,V,X) = \frac{4a+3b}{2(n-1)}(g(X,Y)g(Z,V) - g(X,Z)g(Y,V)).$$

By virtue of (1.2), we have

$$\begin{aligned} R(Y,Z,V,X) = \frac{4a+3b}{2(n-1)}&(g(X,Y)g(Z,V) - g(X,Z)g(Y,V)) \\ &+ \frac{1}{2(n-1)}(g(Y,V)r(Z,X) \\ &- g(Y,X)r(Z,V) - g(Z,V)r(Y,X) + g(Z,X)r(Y,V)). \quad (4.5) \end{aligned}$$

Contracting (4.17) over X and Y, we get

$$r(Z,V) = \frac{(4a+3b)(n-1)-s}{3n-4}g(Z,V),$$

which contradicts our assumption that (M^n, g) is an $N(k)$-quasi Einstein manifold. Hence we can conclude that (M^n, g) satisfies the condition $a + b = 0$. This completes the proof of Theorem 4.2. □

As a consequence we obtain

Corollary 4.3. *Let (M^n, g) be an $N(k)$-quasi Einstein manifold. If (M^n, g) satisfies the condition $R(U,X) \cdot G = 0$, then $R(X,Y,Z,U) = 0$.*

Proof. By virtue of Theorem 4.2 we have

$$a + b = 0,$$

which implies from (2.4),(2.9) and (2.11) that

$$R(X,Y,Z,U) = 0.$$

This completes the proof. □

Let C denote the symmetric endomorphism of T_pM^n corresponding to the Ricci tensor r, that is, $g(CX,Y) = r(X,Y)$, where $X,Y \in T_pM^n$. The transformations $R(X,Y)$ and C are called the curvature transformation and the Ricci transformation, respectively. It is known that in a quasi Einstein manifold the curvature and the Ricci transformations commute if and only if the relation $R(X,Y,Z,U) = 0$ holds [3]. According to this fact we get the following:

Corollary 4.4. *Let (M^n, g) be an $N(k)$-quasi Einstein manifold. If (M^n, g) satisfies the condition $R(U,X) \cdot G = 0$, then the curvature and the Ricci transformations commute.*

In the compact case, if we assume that the generator U is a Killing vector field, then we are able to show that U ia a parallel vector field. More precisely we have

Theorem 4.5. *Let (M^n, g) be a compact $N(k)$-quasi Einstein manifold with Killing vector field U. If (M^n, g) satisfies the condition $R(U, X) \cdot G = 0$, then U is a parallel vector field.*

Proof. It is known that for a vector field X on a compact Riemannian manifold M^n the following relation holds [9]:

$$\int_{M^n} (r(X, X) - |\nabla X|^2 - (div\ X)^2) dM \leq 0.$$

and equality holds if and only if X is a Killing vector field [9]. Moreover if X is a Killing vector field then $div\ X = 0$. Therefore the above inequality takes the following form:

$$\int_{M^n} (r(X, X) - |\nabla X|^2) dM = 0.$$

In case of a Killing vector field U, the above identity yields by virtue of (2.4),(2.5),(2.9) and (2.11) that

$$\int_{M^n} ((a + b) - |\nabla U|^2) dM = 0.$$

From Theorem 4.2 and the last identity, it follows that

$$\nabla U = 0.$$

This completes the proof of Theorem 4.5. □

According to Theorem 4.5 we obtain the following:

Corollary 4.6. *Let (M^n, g) be a compact $N(k)$-quasi Einstein manifold with Killing vector field U. If (M^n, g) satisfies the condition $R(U, X) \cdot G = 0$, then (M^n, g) is a product manifold.*

Proof. Since the generator U is a Killing vector field we have

$$g(\nabla_X U, Y) + g(\nabla_Y U, X)$$

for every $X, Y \in T_p M^n$. Now let $U^\perp$ denote the $(n-1)$-dimensional distribution in (M^n, g) orthogonal to U and if X and Y belong to $U^\perp$ then

$$g(X, U) = g(Y, U) = 0.$$

From $(\nabla_X g)(Y, U) = 0$ and Theorem 4.5, it follows that

$$0 = g(\nabla_X Y, U) + g(Y, \nabla_X U) = g(\nabla_X Y, U).$$

Similarly we have

$$g(\nabla_Y X, U) = 0.$$

Therefore since the Riemannian connection ∇ is of vanishing torsion, that is,

$$[X, Y] = \nabla_X Y - \nabla_Y X,$$

we obtain that

$$g([X, Y], U) = g(\nabla_X Y - \nabla_Y X, U) = 0,$$

which means $[X, Y] \in U^{\perp}$ for every $X, Y \in U^{\perp}$. From the Frobenius Theorem, it follows that $U^{\perp}$ is integrable. Hence (M^n, g) is a product manifold. This completes the proof. □

Due to Theorem 4.5 we also have

Corollary 4.7. *Let (M^n, g) be a compact $N(k)$-quasi Einstein manifold with Killing vector field U. If (M^n, g) satisfies the condition $R(U, X) \cdot G = 0$, then (M^n, g) is a Ricci-semisymmetric manifold.*

Proof. In virtue of $a + b = 0$, we have

$$r = ag + bu \otimes u = a(g - u \otimes u). \tag{4.6}$$

From (4.18), $\nabla U = 0$ and $a = -b \neq 0$, it follows that

$$\nabla r = da(g - u \otimes u) = \frac{da}{a} r = (d \log a) r. \tag{4.7}$$

Taking account of (4.19), we have

$$(\nabla_X \nabla_Y r)(V, W) = (X(d \log a)(Y) + (d \log a)(X)(d \log a)(Y)) r(V, W),$$

which yields from (2.3)

$$(R(X, Y) \cdot r)(V, W) = 2d(d \log a)(X, Y) r(V, W) = 0.$$

Hence we conclude that (M^n, g) is a Ricci-semisymmetric manifold. This completes the proof. □

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