

**A HOCHSTADT-LIEBERMAN THEOREM FOR
INTEGRO-DIFFERENTIAL OPERATOR**

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Abstract: In this study, we solve a half inverse problem for integro-differential operator that consists of Sturm-Liouville differential part and integral part of Volterra type on a finite interval by using Hochstadt-Lieberman's method. We consider that q is a squared integrable function on $[0, \pi]$ and the kernel of the integral perturbation is integrable on its domain. Especially, we show that the potential function q is known as uniquely when the kernel of the integral perturbation is given.

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1. Introduction

A classical regular Sturm-Liouville problem is related to a differential equation of the form

$$-y'' + q(x)y = \lambda y, \quad (1.1)$$

with the following seperated boundary conditions

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$$\left. \begin{array}{l} y'(0) - hy(0) = 0 \\ y'(\pi) + Hy(\pi) = 0 \end{array} \right\}, \quad (1.2)$$

where $q \in L_2(0, \pi)$ and h, H are finite numbers. Inverse problems of spectral analysis consist in recovering operators from given some spectral datas. The first problem in this directon was solved by Ambarzumyan in 1929 [1]. This type of inverse problems were further solved by many authors with different methods [2], [3], [4], [5], [6], [7], [8]. Suppose that the spectrum $\{\lambda_n\}_{n \in \mathbb{N}}$ of Sturm-Liouville operator, the number h or H and the potential function q on a half interval $(0, \frac{\pi}{2})$ are known; one can recover the Sturm-Liouville operator based on this information. These problems are known as half-inverse problems in literature [9], [10], [11], [12], [13]. It was firstly solved by Hochstadt and Lieberman. They showed that if q is known on $[\frac{\pi}{2}, \pi]$, then only one spectrum is sufficient to determine q on $[0, \frac{\pi}{2}]$. Later, Hald [14] proved that if the potential function is known over half the interval and if one boundary condition is given, then the potential and the other boundary conditions are uniquely determined by using eigenvalues. This result generalizes a theorem which is given by Hochstadt and Lieberman. In [6] and [15], Gesztesy, Simons and Malamud gave some new uniqueness results in inverse spectral analysis with partial information on the potential function for scalar and matrix Sturm-Liouville equations, respectively. In 2001, Sakhnovich [12] studied existence of solution to the half-inverse problem; that is, he presented sufficient conditions on a function \tilde{q} on $(0, \frac{1}{2})$ and a sequence $\{\lambda_n^2\}$ of pairwise distinct real numbers tending to $+\infty$ in order that there existed a Sturm-Liouville operator on $(0, 1)$ with given spectrum $\{\lambda_n^2\}$ and potential function q coinciding with \tilde{q} on $(0, \frac{1}{2})$. In [10], authors gave the necessary and sufficient conditions for solvability of the half-inverse spectral problem for the class of Sturm-Liouville operators with singular potentials from the space $W_2^{-1}(0, 1)$.

Let $\{\lambda_n\}_{n \geq 1}$ be the eigenvalues of a boundary value problem $L = L(q, M)$ of the form

$$Ly(x) = -y''(x) + q(x)y(x) + \int_0^x M(x-t)y(t)dt = \lambda y(x), \lambda = \rho^2, x \in [0, \pi] \quad (1.3)$$

$$y(0) = y(\pi) = 0, \quad (1.4)$$

where q and M are real valued functions, $q \in L_2(0, \pi)$ and $M(x, t)$ is integrable on [4], [16]

$$D_0 = \{(x, t) \in R^2 : 0 \leq t \leq x \leq \pi\}.$$

Now, we define

$$\Delta(\lambda) = y(\pi, \lambda), \tag{1.5}$$

which is called the characteristic function of the boundary value problem (1.3), (1.4) where λ is spectral parameter and $y(x, \lambda_n)$ are eigenfunctions of the problem (1.3), (1.4). This function has a countable set of zeros and the eigenvalues $\{\lambda_n\}_{n \geq 1}$ of the boundary value problem (1.3), (1.4) coincide with the zeros of the function $\Delta(\lambda)$ [4].

Lemma 1.1. (see [4], [16]) *The functions $\varphi(x, \lambda_n)$ and $\psi(x, \lambda_n)$ are the eigenfunctions of the problem (1.3), (1.4) and there is a sequence $\{c_n\}_{n=0}^\infty$ such that*

$$\psi(x, \lambda_n) = c_n \varphi(x, \lambda_n), \quad c_n \neq 0. \tag{1.6}$$

Lemma 1.2. (see [16]) *The following relation holds*

$$\Delta(\lambda) = -\rho \sin(\rho\pi) + \omega \cos(\rho\pi) + r(\rho), \tag{1.7}$$

where

$$\omega = \frac{1}{2} \int_0^\pi q(x) dx \tag{1.8}$$

and

$$r(\rho) = \frac{1}{2} \int_0^\pi \cos[\rho(\pi - 2t)] q(t) dt + \int_0^\pi \cos(\rho t) dt \int_t^\pi M(\xi, t) \cos[\rho(\pi - \xi)] d\xi + O\left(\frac{e^{|\tau|\pi}}{\rho}\right). \tag{1.9}$$

for $\tau = \text{Im } \rho$.

Lemma 1. 3. (see [4], [16]) *The eigenvalues $\{\lambda_n\}$ of the problem (1.3), (1.4) satisfy to the following relation*

$$\rho_n = \sqrt{\lambda_n} = n + \frac{\omega}{\pi n} + \frac{\kappa_n}{n} \tag{1.10}$$

where

$$\omega = \frac{1}{2} \int_0^\pi q(x) dx, \quad \{\kappa_n\} \in l_2, \quad n = \overline{0, \infty}.$$

uniformly in $x \in [0, \pi]$.

Lemma 1.4. (see [4]) *The function $\Delta(\lambda)$ is uniquely determined by its zeros. Moreover,*

$$\Delta(\lambda) = \pi \prod_{n=1}^{\infty} \left(\frac{\lambda_n - \lambda}{n^2} \right).$$

The inverse problem consists in the reconstruction of the potential function q from the spectra by the integro-differential expression for the problem (1.3), (1.4) where the function $M(x, t)$ is assumed to be given. But, the classical methods are not always applicable in inverse problems for integro-differential operators. Whence, there are comparatively little references on inverse problem for integro-differential operators. Nevertheless, some important results for this operator have been obtained by several authors [4], [16], [17], [18], [19], [20], [21], [22], [23].

In this study, the perturbation of the Sturm-Liouville operator by a Volterra type integral operator is considered. We give a half inverse problem for integro-differential operator. And, by using Hochstadt-Lieberman method, we show that if the potential function q and $M(x, t)$ are known on $[\frac{\pi}{2}, \pi]$ and $[0, \pi]$, respectively, then only one spectrum is also sufficient to determine the potential function q as uniquely on $[0, \pi]$.

2. Uniqueness Result

In this section, we give a uniqueness theorem. Let consider another perturbation of Sturm-Liouville operator \tilde{L} satisfying

$$\tilde{L}\tilde{y} = -\tilde{y}'' + \tilde{q}(x)\tilde{y} + \int_0^x M(x-t)\tilde{y}(t)dt, \quad (2.1)$$

with the boundary conditions

$$\tilde{y}(0) = \tilde{y}(\pi) = 0, \quad (2.2)$$

where \tilde{q} satisfies suitable regularity conditions on $[0, \pi]$.

Theorem 2.1. *Let $\{\lambda_n\}_{n \geq 1}$ be the spectrum of the problem (1.3), (1.4) and $\{\tilde{\lambda}_n\}_{n \geq 1}$ be the spectrum of the problem (2.1), (2.2). If $\lambda_n = \tilde{\lambda}_n, n = 1, 2, \dots$ and $\tilde{q}(x) = q(x)$ on $[\frac{\pi}{2}, \pi]$ then $\tilde{q}(x) = q(x)$ almost everywhere on $[0, \pi]$.*

Proof. Let consider following initial value problems

$$-y'' + q(x)y + \int_0^x M(x-t)y(t)dt = \lambda y, \tag{2.3}$$

$$y(0) = 0, \quad y'(0) = 1, \tag{2.4}$$

and

$$-\tilde{y}'' + \tilde{q}(x)\tilde{y} + \int_0^x M(x-t)\tilde{y}(t)dt = \lambda\tilde{y}, \tag{2.5}$$

$$\tilde{y}(0) = 0, \quad \tilde{y}'(0) = 1. \tag{2.6}$$

$\psi(x, \rho)$ and $\tilde{\psi}(x, \rho)$ are the solutions of (2.3), (2.4) and (2.5), (2.6), respectively, for $\rho = \sqrt{\lambda}$. $\psi(x, \rho)$ and $\tilde{\psi}(x, \rho)$ have the following representations [4]

$$\psi(x, \rho) = \frac{\sin(\rho x)}{\rho} + \int_0^x S(x, t) \frac{\sin(\rho t)}{\rho} dt, \tag{2.7}$$

$$\tilde{\psi}(x, \rho) = \frac{\sin(\rho x)}{\rho} + \int_0^x \tilde{S}(x, t) \frac{\sin(\rho t)}{\rho} dt. \tag{2.8}$$

respectively where $S(x, t)$ and $\tilde{S}(x, t)$ are continuous functions. Using (2.7) and (2.8) for the solutions $\psi(x, \rho)$ and $\tilde{\psi}(x, \rho)$, we get

$$\begin{aligned} \psi\tilde{\psi} &= \frac{\sin^2(\rho x)}{\rho^2} + \int_0^x \left[S(x, t) + \tilde{S}(x, t) \right] \frac{\sin(\rho x)}{\rho} \frac{\sin(\rho t)}{\rho} dt \\ &\quad + \int_0^x S(x, t) \frac{\sin(\rho t)}{\rho} dt \int_0^t \tilde{S}(x, s) \frac{\sin(\rho s)}{\rho} ds. \end{aligned}$$

By using some trigonometric addition formulas and extending the range of $S(x, t)$ and $\tilde{S}(x, t)$ with respect to t , we can obtain that

$$\psi\tilde{\psi} = \frac{1}{2\rho^2} \left\{ 1 - \cos(2\rho x) + \int_0^x \hat{S}(x, \tau) \cos(2\rho\tau) d\tau \right\}, \tag{2.9}$$

where

$$\hat{S}(x, \tau) = 2 \left[S(x, x - 2\tau) + \tilde{S}(x, x - 2\tau) + \int_{-x+2\tau}^x S(x, s) \tilde{S}(x, s - 2\tau) ds + \int_{-x}^{x-2\tau} S(x, s) \tilde{S}(x, s + 2\tau) ds \right].$$

Now, we consider

$$\Delta(\rho) = \psi(\pi, \rho). \tag{2.9}$$

Asymptotic results for $\psi(x, \rho)$ imply that this function is an entire function of order $\frac{1}{2}$ according to λ . Then $\Delta(\rho)$ is also an entire function of order $\frac{1}{2}$ according to λ . If we multiply (2.3) by $\tilde{\psi}$ and (2.5) by ψ and subtract after integration from 0 to π , we get

$$\left(\tilde{\psi} \psi' - \tilde{\psi}' \psi \right) \Big|_0^\pi + \int_0^\pi \int_0^x M(x-t) \left[\tilde{\psi}(t, \rho) \psi(x, \rho) - \psi(t, \rho) \tilde{\psi}(x, \rho) \right] dt dx + \int_0^\pi (\tilde{q} - q) \tilde{\psi} \psi dx = 0.$$

and

$$\left(\tilde{\psi}(\pi, \lambda) \psi'(\pi, \lambda) - \tilde{\psi}'(\pi, \lambda) \psi(\pi, \lambda) \right) + \int_0^\pi \int_0^x M(x-t) \left[\tilde{\psi}(t, \rho) \psi(x, \rho) - \psi(t, \rho) \tilde{\psi}(x, \rho) \right] dt dx = \int_0^{\frac{\pi}{2}} (q - \tilde{q}) \tilde{\psi} \psi dx. \tag{2.10}$$

By virtue of (2.7) and its analogue for \tilde{L} , we get

$$\tilde{\psi}(t, \rho) \psi(x, \rho) - \psi(t, \rho) \tilde{\psi}(x, \rho) = O\left(\frac{1}{\rho^2}\right).$$

Then, we can rewrite (2.10) as

$$\begin{aligned} &\left(\tilde{\psi}(\pi, \lambda)\psi'(\pi, \lambda) - \tilde{\psi}'(\pi, \lambda)\psi(\pi, \lambda)\right) + \int_0^\pi \int_0^x M(x-t)O\left(\frac{1}{\rho^2}\right) dt dx \\ &= \int_0^{\frac{\pi}{2}} (q - \tilde{q})\tilde{\psi}\psi dx. \end{aligned}$$

Let define

$$\int_0^{\frac{\pi}{2}} (q - \tilde{q})\tilde{\psi}(x, \rho)\psi(x, \rho) dx = H(\rho). \tag{2.11}$$

Using the properties of ψ and $\tilde{\psi}$, we obtain that $H(\rho)$ is an entire function. For $\rho = \rho_n$, we see that the first and the second terms go to zero and hence $H(\rho_n) = 0$ as $\rho \rightarrow \infty$. In addition, using (2.7) and (2.11), we see that

$$|H(\rho)| \leq M. \tag{2.12}$$

for all complex ρ and some constant M . Now, we consider quotient

$$\Psi(\rho) = \frac{H(\rho)}{\Delta(\rho)}. \tag{2.13}$$

It is evident that $\Psi(\rho)$ is an entire function and from (2.9) and (2.12)

$$|\Psi(\rho)| = O\left(\frac{1}{\rho}\right),$$

for large $|\rho|$. By the Liouville's theorem

$$\Psi(\rho) = 0 \text{ or } H(\rho) = 0. \tag{2.14}$$

for all ρ . Substituting (2.9) in (2.11), we yield

$$\frac{1}{2\rho^2} \int_0^{\frac{\pi}{2}} (q - \tilde{q}) \left[1 - \cos(2\rho x) + \int_0^x \hat{S}(x, \tau) \cos(2\rho\tau) d\tau \right] dx = 0,$$

for all ρ . We can change the order of integration where $0 < \tau < x$ and $0 < x < \frac{\pi}{2}$. Hence last equation can be rewritten as

$$\int_0^{\frac{\pi}{2}} (q - \tilde{q}) dx + \int_0^{\frac{\pi}{2}} \cos(2\rho\tau) \left\{ [q(\tau) - \tilde{q}(\tau)] + \int_{\tau}^{\frac{\pi}{2}} [q(x) - \tilde{q}(x)] \hat{S}(x, \tau) dx \right\} d\tau = 0. \quad (2.15)$$

Letting $\rho \rightarrow \infty$ for real ρ and considering Riemann-Lebesgue lemma, we obtain that

$$\int_0^{\frac{\pi}{2}} [q(x) - \tilde{q}(x)] dx = 0,$$

and

$$\int_0^{\frac{\pi}{2}} \cos(2\rho\tau) \left\{ [q(\tau) - \tilde{q}(\tau)] + \int_{\tau}^{\frac{\pi}{2}} [q(x) - \tilde{q}(x)] \hat{S}(x, \tau) dx \right\} d\tau = 0.$$

From the completeness of the functions $\cos(2\rho\tau)$, the last integral must be vanish. Then, we get

$$q(\tau) - \tilde{q}(\tau) + \int_{\tau}^{\frac{\pi}{2}} [q(x) - \tilde{q}(x)] \hat{S}(x, \tau) dx = 0,$$

for $0 < \tau < \frac{\pi}{2}$. The last equation is a Volterra type integral equation and has only trivial solution. Then, we obtain that

$$q(x) - \tilde{q}(x) = 0,$$

almost everywhere on $[0, \pi]$. This completes the proof.

Remark. Let consider following initial value problems. If we assume that $\varphi(x, \lambda)$ and $\tilde{\varphi}(x, \lambda)$ are the solutions of

$$-y'' + q(x)y + \int_0^x M(x-t)y(t)dt = \lambda y \quad (2.16)$$

$$y(0) = 1, y'(0) = 0 \quad (2.17)$$

and

$$-\tilde{y}'' + \tilde{q}(x)\tilde{y} + \int_0^x M(x-t)\tilde{y}(t)dt = \lambda\tilde{y} \quad (2.18)$$

$$\tilde{y}(0) = 1, \tilde{y}'(0) = 0 \quad (2.19)$$

respectively, we can prove a uniqueness theorem for the problems (2.16), (2.17) and (2.18), (2.19) similarly.

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