GEOMETRICAL HYPERCOMPLEX COUPLING BETWEEN ELECTRIC AND GRAVITATIONAL FIELDS

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Abstract: The present work shows a coupling of electrical and gravitational fields through Cauchy-Riemann conditions for quaternions present in a previous paper [1]. It is also obtained an extended version of the Laplace-like equations for quaternions, now written in terms of both electric and gravitational fields.

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1. Initial Provisions

Throughout this work, are considered quaternionic functions which follow the pattern $f_i(t, x, y, z)$, with $i = 1, 2, 3, 4$, where $t$ is the time and the coordinates $x, y$ and $z$ are considered the spatial coordinates. Thus, the quaternion $q$ is written here as follows:

$$q = t + xi + yj + zk,$$

or

$$q = t + \mathbf{u}.$$  \hfill (1)

The next section based on a paper by Borges and Machado [2] shows a set

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Cauchy-Riemann like relations for quaternionic functions. These equations will be adapted to the particular case where \( x_1 \) will be replaced by the time and the other coordinates \( x_2, x_3 \) and \( x_4 \) will be identified here for \( x, y \) and \( z \), respectively.

### 2. Cauchy-Riemann Conditions for Quaternionic Functions

The conditions named here as Cauchy-Riemann like relations for quaternionic functions, are treated in detail in [1]. It follows the theorem:

**Theorem 1.** For any pair points \( a \) and \( b \) and any path joining them simply connect subdomain of the four-dimensional space, the integral \( \int_a^b f dq \) is independent form the given path if and only if there is a function \( F = F_1 + F_2i + F_3j + F_4k \) such that \( \int_a^b f dq = F(a)F(b) \), and satisfying the following relations:

\[
\begin{align*}
\frac{\partial F}{\partial t} &= \frac{\partial F_2}{\partial x} = \frac{\partial F_3}{\partial y} = \frac{\partial F_4}{\partial z}, \\
\frac{\partial F_2}{\partial t} &= -\frac{\partial F_1}{\partial x} = -\frac{\partial F_3}{\partial z} = \frac{\partial F_4}{\partial y}, \\
\frac{\partial F_3}{\partial t} &= -\frac{\partial F_1}{\partial y} = -\frac{\partial F_2}{\partial z} = \frac{\partial F_4}{\partial x}, \\
\frac{\partial F_4}{\partial t} &= \frac{\partial F_1}{\partial z} = -\frac{\partial F_2}{\partial y} = \frac{\partial F_3}{\partial x}.
\end{align*}
\]

*Proof.* The proof of this theorem can be analyzed in greater detail in [1].

### 3. The Laplace’s Equations

In this section it will be determined that from the relations showed in Theorem 1, naturally follows a new set of quaternionic Laplacelike equations. Therefore, the functions that make up the quaternionic function depend on \( t, x, y \) and \( z \) and are supposed of class \( C^2 \).

The first step to obtain the Laplace equations is the derivation of equations (5), (6), (7) and (8) over \( t, x, y \) and \( z \). That will be done as follows: Deriving
the conditions of equation (5), we have that:
\[
\begin{align*}
\frac{\partial^2 F_1}{\partial y \partial t} &= \frac{\partial^2 F_2}{\partial t \partial x} = \frac{\partial^2 F_3}{\partial t \partial y} = \frac{\partial^2 F_4}{\partial t \partial z} \\
\frac{\partial^2 F_1}{\partial t \partial x} &= \frac{\partial^2 F_2}{\partial y \partial x} = \frac{\partial^2 F_3}{\partial x \partial y} = \frac{\partial^2 F_4}{\partial x \partial z} \\
\frac{\partial^2 F_1}{\partial t \partial y} &= \frac{\partial^2 F_2}{\partial y \partial y} = \frac{\partial^2 F_3}{\partial y \partial y} = \frac{\partial^2 F_4}{\partial z \partial y} \\
\frac{\partial^2 F_1}{\partial t \partial z} &= \frac{\partial^2 F_2}{\partial z \partial x} = \frac{\partial^2 F_3}{\partial z \partial y} = \frac{\partial^2 F_4}{\partial z^2}.
\end{align*}
\] (7)

Deriving the conditions of equation (6), we obtain:
\[
\begin{align*}
\frac{\partial^2 F_2}{\partial^2 t} &= -\frac{\partial^2 F_1}{\partial t \partial x} = -\frac{\partial^2 F_3}{\partial t \partial z} = \frac{\partial^2 F_4}{\partial t \partial y} \\
\frac{\partial^2 F_2}{\partial t \partial x} &= -\frac{\partial^2 F_1}{\partial x^2} = \frac{\partial^2 F_3}{\partial x \partial z} = \frac{\partial^2 F_4}{\partial x \partial y} \\
\frac{\partial^2 F_2}{\partial t \partial y} &= -\frac{\partial^2 F_1}{\partial y^2} = -\frac{\partial^2 F_3}{\partial y \partial z} = \frac{\partial^2 F_4}{\partial y \partial y} \\
\frac{\partial^2 F_2}{\partial t \partial z} &= -\frac{\partial^2 F_1}{\partial z^2} = \frac{\partial^2 F_3}{\partial z \partial x} = \frac{\partial^2 F_4}{\partial z \partial y}.
\end{align*}
\] (8)

Deriving the conditions of equation (7), we obtain:
\[
\begin{align*}
\frac{\partial^2 F_3}{\partial^2 t} &= \frac{\partial^2 F_1}{\partial t \partial y} = \frac{\partial^2 F_2}{\partial t \partial z} = -\frac{\partial^2 F_4}{\partial t \partial x} \\
\frac{\partial^2 F_3}{\partial t \partial x} &= \frac{\partial^2 F_1}{\partial x \partial y} = -\frac{\partial^2 F_2}{\partial x \partial z} = \frac{\partial^2 F_4}{\partial x^2} \\
\frac{\partial^2 F_3}{\partial t \partial y} &= \frac{\partial^2 F_1}{\partial y^2} = \frac{\partial^2 F_2}{\partial y \partial z} = -\frac{\partial^2 F_4}{\partial y \partial x} \\
\frac{\partial^2 F_3}{\partial t \partial z} &= \frac{\partial^2 F_1}{\partial z^2} = -\frac{\partial^2 F_2}{\partial z \partial x} = -\frac{\partial^2 F_4}{\partial z \partial y}.
\end{align*}
\] (9)

And finally, in deriving the conditions of equation (8), it follows that:
\[
\begin{align*}
\frac{\partial^2 F_4}{\partial^2 t} &= \frac{\partial^2 F_1}{\partial t \partial z} = -\frac{\partial^2 F_2}{\partial t \partial y} = -\frac{\partial^2 F_3}{\partial t \partial x} \\
\frac{\partial^2 F_4}{\partial t \partial x} &= \frac{\partial^2 F_1}{\partial x \partial z} = -\frac{\partial^2 F_2}{\partial x \partial y} = \frac{\partial^2 F_3}{\partial x^2} \\
\frac{\partial^2 F_4}{\partial t \partial y} &= \frac{\partial^2 F_1}{\partial y^2} = \frac{\partial^2 F_2}{\partial y \partial z} = \frac{\partial^2 F_3}{\partial y \partial x} \\
\frac{\partial^2 F_4}{\partial t \partial z} &= \frac{\partial^2 F_1}{\partial z^2} = -\frac{\partial^2 F_2}{\partial z \partial x} = \frac{\partial^2 F_3}{\partial z \partial y}.
\end{align*}
\] (10)
Correlating groups of partial derivatives in (9), (10), (11) and (12), then immediately follows the Laplace-like Equations:

\[ \frac{\partial^2 F_1}{\partial t^2} + \frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2} = 0 \]  \hspace{1cm} (11)

\[ \frac{\partial^2 F_2}{\partial t^2} + \frac{\partial^2 F_2}{\partial x^2} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_2}{\partial z^2} = 0 \]  \hspace{1cm} (12)

\[ \frac{\partial^2 F_3}{\partial t^2} + \frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_3}{\partial z^2} = 0 \]  \hspace{1cm} (13)

and

\[ \frac{\partial^2 F_4}{\partial t^2} + \frac{\partial^2 F_4}{\partial x^2} + \frac{\partial^2 F_4}{\partial y^2} + \frac{\partial^2 F_4}{\partial z^2} = 0 \]  \hspace{1cm} (14)

Taking now into account the functions \( F_3(t, x, y, z) \) and \( F_4(t, x, y, z) \) at (13) and (14), and making the limit in these equations when \( t \) tends to zero, we have that:

\[ \lim_{t \to 0} \left[ \frac{\partial^2 F_3(t, x, y, z)}{\partial t^2} + \frac{\partial^2 F_3(t, x, y, z)}{\partial x^2} + \frac{\partial^2 F_3(t, x, y, z)}{\partial y^2} + \frac{\partial^2 F_3(t, x, y, z)}{\partial z^2} \right] = 0, \]  \hspace{1cm} (15)

and

\[ \lim_{t \to 0} \left[ \frac{\partial^2 F_4(t, x, y, z)}{\partial t^2} + \frac{\partial^2 F_4(t, x, y, z)}{\partial x^2} + \frac{\partial^2 F_4(t, x, y, z)}{\partial y^2} + \frac{\partial^2 F_4(t, x, y, z)}{\partial z^2} \right] = 0. \]  \hspace{1cm} (16)

As already mentioned earlier, the functions \( F_3 \) and \( F_4 \) are of class \( C^2 \) and thus making the limit as \( t \) tends to zero, these functions will depend only of \( x, y \) and \( z \), and will be denoted by \( \varphi(x, y, z) \) and \( \Phi(x, y, z) \), respectively. Moreover, in the second set of partial derivatives respect to \( t \) in the limit as \( t \) tends to zero are allowed constants, and now will be made the following identifications:

\[ \lim_{t \to 0} \frac{\partial^2 F_3(t, x, y, z)}{\partial t^2} = \frac{\rho_f}{\varepsilon} \]  \hspace{1cm} (17)

and

\[ \lim_{t \to 0} \frac{\partial^2 F_4(t, x, y, z)}{\partial t^2} = 4\pi G \rho, \]  \hspace{1cm} (18)
where $\rho_f$ is free charge density, $\varepsilon$ is permittivity of the medium. Furthermore, $\rho$ is density and $G$ is gravitational constant. Soon, with the identifications and the limits indicated above, we have the following equations:

$$\frac{\partial^2 \varphi(x, y, z)}{\partial x^2} + \frac{\partial^2 \varphi(x, y, z)}{\partial y^2} + \frac{\partial^2 \varphi(x, y, z)}{\partial z^2} = \frac{-\rho_f}{\varepsilon}$$

and

$$\frac{\partial^2 \Phi(x, y, z)}{\partial x^2} + \frac{\partial^2 \Phi(x, y, z)}{\partial y^2} + \frac{\partial^2 \Phi(x, y, z)}{\partial z^2} = -4\pi G \rho$$

There is the possibility of determining the solutions of the equations above, but considering that they are related by the Cauchy-Riemann conditions, after the treatment is again considered the limit when $t$ tends to zero. Therefore, a system of partial differential equations, which arise from the Riemann Cauchy like conditions, is presented only for the functions $F_3$ and $F_4$. It follows that:

$$\frac{\partial^2 F_3}{\partial t \partial y} = \frac{\partial^2 F_4}{\partial t \partial z}, \quad \frac{\partial^2 F_3}{\partial x \partial y} = \frac{\partial^2 F_4}{\partial x \partial z},$$

$$\frac{\partial^2 F_3}{\partial y^2} = \frac{\partial^2 F_4}{\partial z \partial y}, \quad \frac{\partial^2 F_3}{\partial z \partial y} = \frac{\partial^2 F_4}{\partial z^2},$$

$$-\frac{\partial^2 F_3}{\partial t \partial z} = \frac{\partial^2 F_4}{\partial t \partial y}, \quad -\frac{\partial^2 F_3}{\partial x \partial z} = \frac{\partial^2 F_4}{\partial y \partial x},$$

$$-\frac{\partial^2 F_3}{\partial y \partial z} = \frac{\partial^2 F_4}{\partial y^2}, \quad -\frac{\partial^2 F_3}{\partial z \partial y} = \frac{\partial^2 F_4}{\partial z^2}$$

$$\frac{\partial^2 F_3}{\partial t^2} = \frac{\partial^2 F_4}{\partial t \partial x}, \quad \frac{\partial^2 F_3}{\partial t \partial x} = \frac{\partial^2 F_4}{\partial x^2},$$

$$\frac{\partial^2 F_3}{\partial y \partial t} = \frac{\partial^2 F_4}{\partial y \partial x}, \quad \frac{\partial^2 F_3}{\partial t \partial z} = \frac{\partial^2 F_4}{\partial z \partial x},$$

$$\frac{\partial^2 F_4}{\partial t^2} = -\frac{\partial^2 F_3}{\partial t \partial x}, \quad \frac{\partial^2 F_4}{\partial t \partial x} = -\frac{\partial^2 F_3}{\partial x^2},$$

$$\frac{\partial^2 F_4}{\partial y \partial t} = -\frac{\partial^2 F_3}{\partial y \partial x}, \quad \frac{\partial^2 F_4}{\partial t \partial z} = -\frac{\partial^2 F_3}{\partial z \partial x}.$$
\[
\lim_{t \to 0} \frac{\partial^2 F_4}{\partial t^2} = 4\pi G\rho, \quad \lim_{t \to 0} \frac{\partial^2 F_3}{\partial t^2} = \frac{\rho f}{\varepsilon},
\]
\[
\frac{\partial F_3}{\partial y} = \frac{\partial F_4}{\partial z}, \quad -\frac{\partial F_3}{\partial z} = \frac{\partial F_4}{\partial y},
\]
\[
\frac{\partial F_3}{\partial t} = \frac{\partial F_4}{\partial x}, \quad \frac{\partial F_4}{\partial t} = -\frac{\partial F_3}{\partial x}.
\] (21)

The above system has the following solution (solution that verifies the Laplace like Equation for \(F_3\) and \(F_4\), where \(C_1\) and \(C_2\) are constants. Hence it follows that:

\[
F_3(t, x, y, z) = -\frac{1}{2} \left( \frac{\rho f}{\varepsilon} \right) x^2 - (4\pi \rho G t + f_1(z - yi) + f_2(z + yi)) x
\]
\[
+ \frac{1}{2} \left( \frac{\rho f}{\varepsilon} \right) t^2 + (if_1(z - yi) - if_2(z + yi) + C_1)t + if_3(z - yi)
\]
\[
- if_4(z + yi) + C_2,
\] (22)

and

\[
F_4(t, x, y, z) = \frac{1}{2} (4\pi \rho G) t^2 - \left( \left( \frac{\rho f}{\varepsilon} \right) x + f_1(z - iy) + f_2(z + iy) \right) t
\]
\[
- \frac{1}{2} (4\pi \rho G) x^2 + (if_1(z - iy) - if_2(z + iy) + C_1)x + f_3(z - y)
\]
\[
+ f_4(z + iy).
\] (23)

Making \(F_3 - iF_4\) the threshold \(t\) tends to zero, ie, \(F_3(x, y, z) - iF_4(x, y, z)\) we have that:

\[
F_3(x, y, z) - iF_4(x, y, z) = -\frac{1}{2} \left( \frac{\rho f}{\varepsilon} \right) x^2 + \frac{1}{2} (4\pi G\rho) x^2 i - 2f_2(z + iy)x
\]
\[
+ C_2 - C_1 xi - 2if_4(z + iy).
\] (24)

On the other hand, performing the partial derivatives of the above functions and taken to the limit when \(t \to 0\), and making the appropriate identifications, we have that:

\[
i \frac{\partial F_3}{\partial y}(x, y, z) = -Df_1(z - iy)x + Df_2(z + iy)x + iDf_3(z - iy) + iDf_4(z + iy),
\] (25)

\[
\frac{\partial F_4}{\partial y}(x, y, z) = Df_1(z - iy)x + Df_2(z + iy)x - iDf_3(z - iy) + iDf_4(z + iy),
\] (26)
which together give us:

\[
\frac{\partial F_4}{\partial y}(x, y, z) + i \frac{\partial F_3}{\partial y}(x, y, z) = 2Df_2(z + iy)x + 2iDf_4(z + iy)x. \tag{27}
\]

Similarly,

\[
i \frac{\partial F_3}{\partial z}(x, y, z) = -iDf_1(z - iy)x - iDf_2(z + iy)x - Df_3(z - iy) + Df_4(z + iy), \tag{28}
\]

\[
\frac{\partial F_4}{\partial z}(x, y, z) = iDf_1(z - iy)x - iDf_2(z + iy)x + Df_3(z - iy) + iDf_4(z + iy), \tag{29}
\]

which together generate the following equality:

\[
\frac{\partial F_4}{\partial z}(x, y, z) + i \frac{\partial F_3}{\partial z}(x, y, z) = -2iDf_2(z + iy)x + 2Df_4(z + iy). \tag{30}
\]

Finally, by taking the sum:

\[
\frac{\partial F_4}{\partial y}(x, y, z) + i \frac{\partial F_3}{\partial y}(x, y, z) - \frac{\partial F_3}{\partial z}(x, y, z) + i \frac{\partial F_4}{\partial z}(x, y, z) = 4Df_1(z + iy)x + 4iDf_4(z + iy), \tag{31}
\]

or

\[
\left(\frac{\partial F_4}{\partial y}(x, y, z) - \frac{\partial F_3}{\partial z}(x, y, z)\right) + i \left(\frac{\partial F_3}{\partial y}(x, y, z) + \frac{\partial F_4}{\partial z}(x, y, z)\right) = 4Df_1(z + iy)x + 4iDf_4(z + iy). \tag{32}
\]

Integrating the terms of \(f_1(z + iy)\) and \(f_4(z + iy)\) we have:

\[
4Df_1(z + iy)x = \left(\frac{\partial F_4}{\partial y}(x, y, z) - \frac{\partial F_3}{\partial z}(x, y, z)\right); \tag{33}
\]

that is equal to

\[
f_1(z + iy)x = -\frac{1}{2}(F_3(x, y, z) - F_3(x, y, z_0)) + \frac{i}{2}(F_4(x, y, z) - F_4(x, y_0, z)). \tag{34}
\]

Similarly, we have:

\[
f_4(z + iy) = \frac{1}{2}(F_4(x, y, z) - F_4(x, y, z_0)) + \frac{i}{2}(F_3(x, y, z) - F_3(x, y_0, z)). \tag{35}
\]

Substituting the above results integrated \(y_0\) to \(y\) and \(z_0\) by \(z\) and substituting in equation (24) it follows that:
\((-1)[F_3(x, y, z) - F_3(x, y, z_0) - F_3(x, y_0, z)]\)

\[= \left( -\frac{1}{2}\left(\frac{\rho_f}{\varepsilon}\right)x^2 + C_2 \right) + i\left(\frac{1}{2}(4\pi G \rho)x^2 + C_1 x\right), \quad (36)\]

or

\[[F_3(x, y, z) - F_3(x, y, z_0) - F_3(x, y_0, z)]^2 + [F_4(x, y, z) - F_4(x, y, z_0) - F_4(x, y_0, z)]^2 = \left(\frac{1}{2}\left(\frac{\rho_f}{\varepsilon}\right)x^2 - C_2\right)^2 + \left(\frac{1}{2}(4\pi G \rho)x^2 + C_1 x\right)^2. \quad (37)\]

The results of this work can be summarized in the following theorem:

**Theorem 2.** Let \(f(q)\) quaternionic function that satisfies the Cauchy-Riemann conditions. If \(f(q)\) is of class \(C^2\), then it is possible to determine a relationship between gravitational and electrical potential as listed below:

\[[F_3(x, y, z) - F_3(x, y, z_0) - F_3(x, y_0, z)]^2 + [F_4(x, y, z) - F_4(x, y, z_0) - F_4(x, y_0, z)]^2 = \left(\frac{1}{2}\left(\frac{\rho_f}{\varepsilon}\right)x^2 - C_2\right)^2 + \left(\frac{1}{2}(4\pi G \rho)x^2 + C_1 x\right)^2. \quad (38)\]

4. Conclusion

The present results in the previous sections, showed the feasibility of obtaining the equations of Laplace through the Cauchy-Riemann like conditions for quaternions. This fact will allow the relationship between equations that can explain such physical phenomena e. g. the possibility of a geometrical coupling regarding gravitational and electric fields. You can also use the above equations as a way of stating the theorem for harmonic functions that satisfy the Cauchy conditions.

**References**
