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OSCILLATION OF CAPUTO LIKE DISCRETE FRACTIONAL EQUATIONS

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Abstract: This paper deals with some oscillation criteria of forced nonlinear fractional difference equations of the form

$$\Delta_*^\alpha x(t) - g(t+\alpha-1, x(t+\alpha-1)) + f_1(t+\alpha-1, x(t+\alpha-1)) = v(t),$$

$$\Delta_*^\alpha x(t) - g(t+\alpha-1, x(t+\alpha-1)) + f_1(t+\alpha-1, x(t+\alpha-1)) + f_2(t+\alpha-1, x(t+\alpha-1)) = v(t)$$

where Δ_*^α is a Caputo like discrete fractional difference operator, $t \in N_{1-\alpha}$, $0 < \alpha \leq 1$, $x(0) = x_0$. $g, f_i : [0, +\infty) \times R \to R$, $i = 1, 2$ and $v : [0, +\infty) \to R$ are continuous with respect t and x and $N_{1-\alpha} = \{1-\alpha, 2-\alpha, \ldots\}$.

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1. Introduction

The study of oscillation of fractional differential equations is discussed in [2] - [8]. Motivated by [2], we present the oscillatory behavior of forced nonlinear fractional difference equations of the form

$$\Delta_*^\alpha x(t) - g(t+\alpha-1, x(t+\alpha-1)) + f_1(t+\alpha-1, x(t+\alpha-1)) = v(t), \quad (1)$$

$$\begin{aligned} \Delta_*^\alpha x(t) - g(t+\alpha-1, x(t+\alpha-1)) + f_1(t+\alpha-1, x(t+\alpha-1)) \\ +f_2(t+\alpha-1, x(t+\alpha-1)) = v(t), \end{aligned} \quad (2)$$

where Δ_*^α is a Caputo like discrete fractional difference operator, $t \in N_{1-\alpha}$, $0 < \alpha \leq 1$, $x(0) = x_0$. $g, f_i : [0, +\infty) \times R \to R$, $i = 1, 2$ and $v : [0, +\infty) \to R$ are continuous with respect to t and x and $N_{1-\alpha} = \{1-\alpha, 2-\alpha, \ldots\}$.

Fractional calculus is more than 300 years old. In 1695, ĹHospital raised the question as to the meaning of $d^n y/dx^n$ if n = 1/2, that is, what if n is fractional?. This is an apparent paradox from which, one day, useful consequences will be drawn, Leibniz replied. We observe that fractional differential equations play an important role in many research areas, such as physics, electro chemistry, population dynamics, biotechnology, viscoelasticity, diffusion equations, electromagnetism and economics, see [16]-[21] and the references cited therein. However, very little progress has been made in developing the theory of the analogous fractional difference equations [9]-[15].

The solution of equation (1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory.

2. Definitions and Basic Lemmas

In this section, we introduce some preliminary results of discrete fractional calculus, so that this paper is self-contained.

Definition 2.1. (see [9, 11]) Let $\nu > 0$. The ν-th fractional sum f is defined by

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} f(s),$$

where f is defined for $s \equiv a \bmod (1)$ and $\Delta^{-\nu} f$ is defined for $t \equiv (a+\nu) \bmod (1)$, and $t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t-\nu+1)}$. The fractional sum $\Delta^{-\nu} f$ maps functions defined on N_a to functions defined on N_{a+v}.

Definition 2.2. (see [13]) Let $\mu > 0$ and $m-1 < \mu < m$, where m denotes a positive integer, $m = \lceil \mu \rceil$. Set $\nu = m - \mu$. The μ-th fractional Caputo like difference is defined as

$$\Delta_*^{\mu} f(t) = \Delta^{-\nu}(\Delta^m f(t)) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} (\Delta^m f)(s), \forall t \in N_{a+\nu}.$$

Here Δ^m is the m^{th} order forward difference operator

$$(\Delta^m f)(s) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} f(s+k).$$

Theorem 2.3. *(see [13]) For $\mu > 0$, μ non integer, $m = \lceil \mu \rceil$, $\nu = m - \mu$, it holds*

$$f(t) = \sum_{k=0}^{m-1} \frac{(t-a)^{(k)}}{k!} \Delta^k f(a) + \frac{1}{\Gamma(\mu)} \sum_{s=a+\nu}^{t-\mu} (t-s-1)^{(\mu-1)} \Delta_*^{\mu} f(s),$$

where f is defined on N_a with $a \in Z^+$. $Z^+ = \{0, 1, 2, \ldots\}$.

In particular, when $0 < \mu < 1$ and $a = 0$, we have

$$f(t) = f(0) + \frac{1}{\Gamma(\mu)} \sum_{s=1-\mu}^{t-\mu} (t-s-1)^{(\mu-1)} \Delta_*^{\mu} f(s).$$

Lemma 2.4. *A function $x(t) : N \to R$ is a solution of the IVP (1) if and only if $x(t)$ is a solution of the following fractional Taylor's difference formula:*

$$\begin{aligned} x(t) =& x_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} \big[v(s) + g(s+\alpha-1, x(s+\alpha-1)) \\ & - f_1(s+\alpha-1, x(s+\alpha-1)) \big], \ 0 < \alpha \leq 1, \\ x(0) =& x_0. \end{aligned} \tag{3}$$

Proof. Suppose that $x(t)$ for $t \in N$ is a solution of (1), that is $\Delta_*^{\alpha} x(t) = v(t) + g(t+\alpha-1, x(t+\alpha-1)) - f_1(t+\alpha-1, x(t+\alpha-1))$ for $t \in N_{1-\alpha}$, then we can obtain (3) according to Theorem (2.3).

Conversely, we assume that $x(t)$ is a solution of (3), then

$$\begin{aligned} x(t) =& x_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} [v(s) + g(s+\alpha-1, x(s+\alpha-1)) \\ & - f_1(s+\alpha-1, x(s+\alpha-1))]. \end{aligned}$$

On the other hand, Theorem (2.3) yields that

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} \Delta_*^\alpha x(s).$$

Comparing with the above two equations, we obtained

$$\frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} \big[\Delta_*^\alpha x(s) - \big(v(s) + g(s+\alpha-1, x(s+\alpha-1)) \\ -f_1(s+\alpha-1, x(s+\alpha-1))\big)\big] = 0.$$

Let $t = 1, 2, \ldots,$ respectively, we have that $\Delta_*^\alpha x(t) = v(t) + g(t+\alpha-1, x(t+\alpha-1)) - f_1(t+\alpha-1, x(t+\alpha-1))$ for $t \in N_{1-\alpha}$, which implies that $x(t)$ is a solution of (1). □

In order to discuss our results in Section 3, now we state the following Lemma.

Lemma 2.5. *(see [1]) For X and Y are nonnegative, we have*

$$\gamma X^\lambda + (\lambda-\gamma)Y^\lambda - \lambda X^\gamma Y^{\lambda-\gamma} \geq 0, \quad \text{for all } \lambda \geq \gamma > 0. \tag{4}$$

3. Main Results

In this section, we consider the following conditions:

$$x f_i(t,x) > 0 \quad (i = 1,2), \quad x \neq 0, \quad t \geq t_0 \tag{5}$$

and

$$\begin{aligned} |g(t,x)| \leq |p(t)|\,|x|^\gamma, |f_1(t,x)| \geq |q_1(t)|\,|x|^\lambda \\ \text{and} \quad |f_2(t,x)| \geq |q_2(t)|\,|x|^\mu, \quad x \neq 0, \quad t \geq t_0, \end{aligned} \tag{6}$$

where $p, q_1, q_2 \in C([0,+\infty), R^+)$ and $\beta, \gamma > 0$ are real numbers.
Now we prove our first theorem when $g = 0$.

Theorem 3.1. *Suppose*

$$\liminf_{t\to\infty} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} v(s) = -\infty, \tag{7}$$

and

$$\limsup_{t\to\infty} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} v(s) = \infty, \tag{8}$$

then every solution of equation (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1) with $g = 0$. Suppose that $T > t_0$ is large enough that $x(t) > 0$, for $t \geq T$. Using (3), we get

$$\begin{aligned}\Gamma(\alpha)x(t) =&\Gamma(\alpha)x_0 + \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} \big[v(s) \\ &+g(s+\alpha-1, x(s+\alpha-1)) - f_1(s+\alpha-1, x(s+\alpha-1))\big].\end{aligned} \tag{9}$$

Let $F(t) = v(t) + g(t+\alpha-1, x(t+\alpha-1)) - f_1(t+\alpha-1, x(t+\alpha-1))$, then we obtain

$$\Gamma(\alpha)x(t) \leq \Gamma(\alpha)x_0 + \sum_{s=1-\alpha}^{T-1-\alpha} (t-s-1)^{(\alpha-1)} |F(s)| + \sum_{s=T-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} v(s), \quad t \geq T,$$

and hence

$$\Gamma(\alpha)x(t) \leq c(T) + \sum_{s=T-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} v(s), \quad t \geq T, \tag{10}$$

where

$$c(T) = \Gamma(\alpha)x_0 + \sum_{s=1-\alpha}^{T-1-\alpha} \left(\frac{1}{T-s-1}\right)^{(1-\alpha)} |F(s)|$$

and $\lim_{t\to\infty} c(t) = M < \infty, \quad t \geq T$ is convergent. Taking the limit inferior of both sides of inequality as $t \to \infty$, we get a contradiction to condition (7). This completes the proof of the theorem. □

Next we have the following results.

Theorem 3.2. *Suppose that*

$$\liminf_{t\to\infty} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} \left[v(s) + g_1(s)\right] = -\infty, \tag{11}$$

and

$$\limsup_{t\to\infty} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} \left[v(s)+g_1(s)\right] = \infty, \tag{12}$$

where

$$g_1(s) = (\lambda-\gamma)\lambda^{\lambda/(\gamma-\lambda)}\gamma^{\gamma/(\lambda-\gamma)}p^{\lambda/(\lambda-\gamma)}q_1^{\gamma/(\gamma-\lambda)}. \tag{13}$$

Then every solution of equation (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1), say, $x(t) > 0$ for $t \geq T > t_0$. Using (3), we have

$$\begin{aligned}\Gamma(\alpha)x(t) = \Gamma(\alpha)x_0 + &\sum_{s=1-\alpha}^{T-1-\alpha} (t-s-1)^{(\alpha-1)}\big[v(s)+g(s+\alpha-1,x(s+\alpha-1)) \\ &-f_1(s+\alpha-1,x(s+\alpha-1))\big] + \sum_{s=T-\alpha}^{t-\alpha}(t-s-1)^{(\alpha-1)}\big[v(s) \\ &+g(s+\alpha-1,x(s+\alpha-1)) - f_1(s+\alpha-1,x(s+\alpha-1))\big] \\ &\leq \Gamma(\alpha)x_0 + \sum_{s=1-\alpha}^{T-1-\alpha}(t-s-1)^{(\alpha-1)}|F(s)| + \sum_{s=T-\alpha}^{t-\alpha}(t-s-1)^{(\alpha-1)}\big[v(s) \\ &+g(s+\alpha-1,x(s+\alpha-1)) - f_1(s+\alpha-1,x(s+\alpha-1))\big] \\ &\leq c(T) + \sum_{s=T-\alpha}^{t-\alpha}(t-s-1)^{(\alpha-1)}\big[v(s)+g(s+\alpha-1,x(s+\alpha-1)) \\ &\qquad -f_1(s+\alpha-1,x(s+\alpha-1))\big],\end{aligned}$$

where

$$c(T) = \Gamma(\alpha)x_0 + \sum_{s=1-\alpha}^{T-1-\alpha}\left(\frac{1}{T-s-1}\right)^{(1-\alpha)} |F(s)|, \quad \text{for} \quad t \geq T$$

$$\Gamma(\alpha)x(t) \leq c(T) + \sum_{s=T-\alpha}^{t-\alpha}(t-s-1)^{(\alpha-1)}\left[v(s)+p(s)x^{\gamma} - q_1(s)x^{\lambda}\right] \tag{14}$$

We apply Lemma (2.5) with

$$X = \gamma^{-1/\lambda}q_1^{1/\lambda}x, \quad Y = \lambda^{1/\gamma-\lambda}\gamma^{\gamma/\lambda(\lambda-\gamma)}p^{1/(\lambda-\gamma)}q_1^{\gamma/\lambda(\gamma-\lambda)}$$

The last two terms in (14) can be written as

$$\begin{aligned} px^{\gamma} - q_1 x^{\lambda} =& \lambda \left(\gamma^{-\gamma/\lambda} q_1^{\gamma/\lambda} x^{\gamma}\right) \left(\lambda^{-1}\gamma^{\gamma/\lambda} p q_1^{-\gamma/\lambda}\right) - \gamma \left(\gamma^{-1} q_1 x^{\lambda}\right) \\ =& \lambda\, X^{\gamma} Y^{\lambda-\gamma} - \gamma X^{\lambda} \leq (\lambda-\gamma) Y^{\lambda} = g_1(t) \end{aligned}$$

and we obtain

$$\Gamma(\alpha)x(t) \leq c(T) + \sum_{s=T-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} [v(s) + g_1(s)], \quad t \geq T.$$

Taking limit inferior of both sides of inequality as $t \to \infty$, we get a contradiction to condition (11). This completes the proof of the theorem. □

Theorem 3.3. *Suppose*

$$\liminf_{t\to\infty} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} \left[v(s) + g_2(s)\right] = -\infty, \tag{15}$$

and

$$\limsup_{t\to\infty} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} \left[v(s) + g_2(s)\right] = \infty, \tag{16}$$

where

$$\begin{aligned} g_2 =& (\lambda-\gamma)\lambda^{\lambda/(\gamma-\lambda)} \gamma^{\gamma/(\lambda-\gamma)} \delta^{\lambda/(\lambda-\gamma)} p^{\lambda/(\lambda-\gamma)} q_1^{\gamma/(\gamma-\lambda)} \\ &+ (\mu-\gamma)\mu^{\mu/(\gamma-\mu)}\ \gamma^{\gamma/(\mu-\gamma)} (1+\delta)^{\mu/(\mu-\gamma)} p^{\mu/(\mu-\gamma)} q_2^{\gamma/(\gamma-\mu)}. \end{aligned} \tag{17}$$

Then every solution of equation (2) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (2), say, $x(t) > 0$ for $t \geq T > t_0$. Using (3), we have

$$\begin{aligned} \Gamma(\alpha)x(t) =& \Gamma(\alpha)x_0 + \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} \big[v(s) + g(s+\alpha-1, x(s+\alpha-1)) \\ & - f_1(s+\alpha-1, x(s+\alpha-1)) - f_2(s+\alpha-1, x(s+\alpha-1))\big] \end{aligned}$$

$$\Gamma(\alpha)x(t) \leq \Gamma(\alpha)x_0 + \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} \left[v(s) + p(s)x^{\gamma} - q_1(s)x^{\lambda} - q_2(s)x^{\mu}\right]$$

$$\begin{aligned} \Gamma(\alpha)x(t) = \Gamma(\alpha)x_0 + \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} \big[v(s) \\ + \delta p(s)x^{\gamma} - q_1(s)x^{\lambda} + (1-\delta)p(s)x^{\gamma} - q_2(s)x^{\mu}\big] \end{aligned}$$

$$\leq \Gamma(\alpha)x_0 + \sum_{s=1-\alpha}^{T-1-\alpha} (t-s-1)^{(\alpha-1)} \times \Big[v(s) + \delta p(s)x^\gamma - q_1(s)x^\lambda + (1+\delta)p(s)x^\gamma - q_2(s)x^\mu\Big]$$

$$+ \sum_{s=T-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} \Big[v(s) + \delta p(s)x^\gamma - q_1(s)x^\lambda + (1+\delta)p(s)x^\gamma - q_2(s)x^\mu\Big].$$

Define $X_1, X_2 \geq 0$ and $Y_1, Y_2 \geq 0$ by

$$X_1 = \gamma^{-1/\lambda} q_1^{1/\lambda} x,$$
$$Y_1 = \lambda^{1/(\gamma-\lambda)} \gamma^{\gamma/(\lambda(\lambda-\gamma))} \delta^{1/(\lambda-\gamma)} p^{1/(\lambda-\gamma)} q_1^{\gamma/(\lambda(\gamma-\lambda))}$$

and

$$X_2 = \gamma^{-1/\mu} q_2^{1/\mu} x,$$
$$Y_2 = \mu^{1/(\gamma-\mu)} \gamma^{\gamma/(\mu(\mu-\gamma))} (1+\delta)^{1/(\mu-\gamma)} p^{1/(\mu-\gamma)} q_2^{\gamma/(\mu(\gamma-\mu))}$$

which implies by Lemma 2.6 that

$$\begin{aligned}\delta p x^\gamma - q_1 x^\lambda &= \lambda \left(\gamma^{-1/\lambda} q_1^{1/\lambda} x\right)^\gamma \\ &\times \left(\lambda^{1/(\gamma-\lambda)} \gamma^{\gamma/(\lambda(\lambda-\gamma))} \delta^{1/(\lambda-\gamma)} p^{1/(\lambda-\gamma)} q_1^{\gamma/(\lambda(\gamma-\lambda))}\right)^{\lambda-\gamma} - \gamma \left(\gamma^{-1/\lambda} q_1^{1/\lambda} x\right)^\lambda \\ &= \lambda X_1^\gamma Y_1^{\lambda-\gamma} - \gamma X_1^\lambda \leq (\lambda-\gamma) Y_1^\lambda\end{aligned}$$

and

$$\begin{aligned}(1+\delta)p x^\gamma - q_2 x^\mu =&\mu \left(\gamma^{-1/\mu} q_2^{1/\mu} x\right)^\gamma \times \big(\mu^{1/(\gamma-\mu)} \gamma^{\gamma/(\mu(\mu-\gamma))} (1+\delta)^{1/(\mu-\gamma)} \\ & p^{1/(\mu-\gamma)} q_2^{\gamma/(\mu(\gamma-\mu))}\big)^{\mu-\gamma} - \gamma \left(\gamma^{-1/\mu} q_2^{1/\mu} x\right)^\mu \\ &= \mu X_2^\gamma Y_2^{\mu-\gamma} - \gamma X_2^\mu \leq (\mu-\gamma) Y_2^\mu.\end{aligned}$$

It follows that

$$\Gamma(\alpha)x(t) \leq c(T) + \sum_{s=T-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} [v(s) + (\lambda-\gamma)Y_1^\lambda + (\mu-\gamma)Y_2^\mu] \tag{18}$$

where

$$c(T) = \Gamma(\alpha)x_0 + \sum_{s=1-\alpha}^{T-1-\alpha} \left(\frac{1}{T-s-1}\right)^{(1-\alpha)} \big[v(s) + \delta p(s)x^\gamma - q_1(s)x^\lambda + (1+\delta)p(s)x^\gamma - q_2(s)x^\mu\big], \text{ for } t \geq T$$

$$\Gamma(\alpha)x(t) \leq c(T) + \sum_{s=T-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)}[v(s) + g_2(s)]$$

Taking limit inferior of both sides of inequality as $t \to \infty$, we get a contradiction to condition (15). This completes the proof of the theorem. □

We now give an example to show that the condition (11) cannot be dropped.

Example 3.4. Consider the Caputo like fractional difference equation

$$\Delta_*^\alpha x(t) - e^{t+\alpha-1}x(t+\alpha-1) + e^{t+\alpha-1}x^3(t+\alpha-1) = \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-(1-\alpha)} (t-s-1)^{(-\alpha)} + (t+\alpha-1)e^{t+\alpha-1}((t+\alpha-1)^2 - 1), \quad (19)$$

where $0 < \alpha \leq 1$. Here $\alpha = 0.5$, $\gamma = 1$, $\lambda = 3$, $p(t) = q_1(t) = e^{t+\alpha-1}$ and $v(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-(1-\alpha)} (t-s-1)^{(-\alpha)} + e^{t+\alpha-1}((t+\alpha-1)^2 - 1)$. But the condition (11) is not satisfied in view of $v(t) \geq 0$ and $\liminf_{t\to\infty} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)}[v(s) + g_1(s)] \geq 0$.

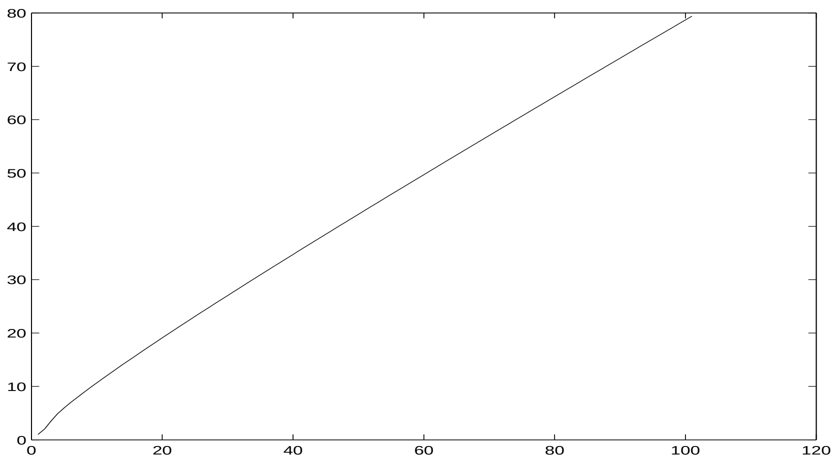

Figure 1: Dynamical behavior of equation (19)

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