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METHOD TO SOLVE FUZZY GAME MATRIX

Sanjiv Kumar[1 §], Ritika Chopra[2], Ratnesh R. Saxena[3]
[1,2]Department of Mathematics
University of Delhi
Delhi, 110007, INDIA
[3]Department of Mathematics
Deen Dayal Upadhyay College
Delhi University
Delhi, INDIA

Abstract: The paper considers a two person zero sum game with imprecise values in the payoff matrix. All the imprecise values are assumed to be triangular fuzzy numbers. The proposed method is an interactive method that integrates the concept of fuzzy ranking and the minimax principle to get an imprecise game value. The decision maker can amicably change the values of the parameter until a satisfactory result is obtained.

Key Words: two person zero sum game, fuzzy number, triangular fuzzy number, fuzzy ranking, minimax principle

1. Introduction

Game theory has a remarkable importance in the field of decision theory due to its great applicability. Many real problems can be modelled as games in the field of economic sciences and optimization. One of the basic problems in the game theory is the two player zero sum game [1]. A two player zero sum game is a game with only two players in which one player win what the other player losses. Let A be a matrix whose entries represents the win of player I. Let us suppose that the player I adopts strategies 'i' and player II adopts strategies 'j'

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§Correspondence author

then player I win be amount a_{ij} corresponding to the (i, j)th entries of matrix A. The matrix A is thus called the game matrix or the payoff matrix.

When real situation are concerned, the entries in the payoff matrix are rather imprecise. In such a case the theoretical support provided by the fuzzy subsets, proposed by Zadeh [2], can become a very useful tool to model the problem. After the pioneer work of Aubin [3, 4] in fuzzy game study an increasing literature has appeared on this topic in which several types of games have been studied.

In this paper, we consider the case of a two person game in which although the player have perfectly defined their sets of strategies, they have, however, some lack of precision on the knowledge of the associated payoffs. We have considered the payoffs as triangular fuzzy numbers [5].

1.1. Crisp Game Value of the Matrix

A game can be expressed as:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -4 & -3 \\ -2 & 5 & -1 \end{bmatrix} \tag{1.1}$$

In this game, players X and Y both have 3 strategies. For player X, the minimum value in each row represents the least gain (payoff) to him if he chooses this particular strategy. He will therefore select the strategy that maximizes his minimum gain.

Similarly, for player Y, the maximum value in each column represents the maximum loss to him if he chooses this particular strategy. He will thus select the strategy that minimizes his maximum losses.

1.2. Saddle Point

If the maxmin value equals the minimax value, then the game is said to have a *saddle point* (equilibrium point) and the corresponding strategies which give the *saddle point* are called optimal strategies. The amount of payoff at an equilibrium point is called the crisp game value of the game matrix.

In (1.1), a_{11}, a_{22}, a_{31} are the minimum of rows $1, 2, 3$ and a_{11}, a_{32}, a_{13} are the maximum of columns 1, 2 and 3, therefore a_{11} is the *saddle point* of the game matrix.

Hence $a_{11} = 1$ is the crisp game value of the matrix.

1.3. Triangular Fuzzy Number

It is a fuzzy number representation with three points as follows:

$$A = (a_1, a_2, a_3)$$

This representation is interpreted as membership functions (Figure 1)

$$\mu_A(x) = \begin{cases} 0; & \text{if } x < a_1 \\ \dfrac{x - a_1}{a_2 - a_1}; & \text{if } a_1 \le x \le a_2 \\ \dfrac{a_3 - x}{a_3 - a_2}; & \text{if } a_2 \le x \le a_3 \\ 0; & \text{if } x > a_3 \end{cases}$$

1.4. Fuzzy Game with Payoffs as Triangular Fuzzy Number

Due to uncertainty, the payoffs in a matrix are not fixed numbers. Hence fuzzy games have been studied. Many researchers like Loganathan, Annie Christi, Sakawa and Nishizaki have extensively worked in this field. Loganathan and Annie Christi [6] have explored fuzzy game value of the interval matrix; also Sakawa and Nishizaki [7] have explored max-min solutions for fuzzy multiobjective matrix games. To model such uncertainty we use triangular fuzzy numbers as the entries in the payoff matrix.

Consider a fuzzy game between two players X and Y, the fuzzified payoff matrix is given by A as follows:

$$A = \begin{bmatrix} [1,2,3] & [7,8,9] & [-3,-2,-1] \\ [4,5,6] & [0,4,8] & [-1,1,3] \\ [-7,-5,-3] & [-2,-1,0] & \left[\frac{1}{2}, 1, \frac{3}{2}\right] \end{bmatrix}$$

In this game, if X chooses row one and Y chooses column two then X wins an amount $x \in [7, 8, 9]$, where [7, 8, 9] is a triangular fuzzy number and Y losses the same amount.

2. Operations of Triangular Fuzzy Number

Suppose triangular fuzzy numbers A and B are defined as,

$$A = (a_1, a_2, a_3), \qquad B = (b_1, b_2, b_3)$$

(i) Addition:

$$\begin{aligned} A(+)B &= (a_1, a_2, a_3)(+)(b_1, b_2, b_3) \\ &= (a_1 + b_1, a_2 + b_2, a_3 + b_3) \end{aligned}$$

(ii) Subtraction:

$$\begin{aligned} A(-)B &= (a_1, a_2, a_3)(-)(b_1, b_2, b_3) \\ &= (a_1 - b_3, a_2 - b_2, a_3 - b_1) \end{aligned}$$

(iii) Symmetric image:

$$-(A) = (-a_3, -a_2, -a_1)$$

2.1. Comparison of Triangular Fuzzy Number

Different fuzzy ranking method have been proposed in the literature by Chou et al. [8], Chen [9], Liou and Wang [10], Luu Quoc Dat, Vincent F. Yu and Shuo-Yan Chou [11] and many more.

To compare triangular fuzzy number, we use the following approach,

Consider n fuzzy numbers A_i, $i = 1, 2, \ldots, n$, each with a membership function $f_{A_i}(x)$.

$$f_{A_i}(x) = \begin{cases} f^L_{A_i}(x); & \text{if } a_i \le x < b_i \\ 1; & \text{if } x = b_i \\ f^R_{A_i}(x); & \text{if } b_i < x \le c_i \\ 0; & \text{otherwise.} \end{cases}$$

Let left and right indices, x_L and x_R refer to the intersection of the left and the right membership functions of the fuzzy numbers A_i with different decision levels γ. A larger value of γ ($\gamma = \gamma_2$) indicates a higher-level decision. More specifically, if γ is close to one, the pertaining decision is called a "high level decision", in which case only parts of the two fuzzy numbers, with membership values between γ and "1", will be compared. Likewise, if γ ($\gamma = \gamma_1$) is close to zero, the pertaining decision is referred to as "low level decision", since members with membership values lower than both the fuzzy numbers are involve in the comparison. Figure 2, illustrates the mentioned graphically.

Generally, the left and right indices values are defined as:

$$x_L(A_i) = f_{A_i^L}(x) \wedge \gamma, \quad i = 1, 2, \ldots, n, \ 0 \prec \gamma \prec 1$$
$$x_R(A_i) = f_{A_i^R}(x) \wedge \gamma, \quad i = 1, 2, \ldots, n, \ 0 \prec \gamma \prec 1$$

Let S_i be the support set of A_i, then $S_i = \{x/f_{A_i}(x) \succ 0\}$ and let $S = U_{i=1}^{n} S_i$.

Then, the subtraction of left relative values from right relative values of each fuzzy number A_i with index of optimism α is then defined as follows:

$$D_\alpha^\gamma(A_i) = \alpha[x_L(A_i) - x_{\min}] - (1-\alpha)[x_{\max} - x_R(A_i)]$$

where $x_{\min} = \inf S$, $x_{\max} = \sup S$.

Obviously, the fuzzy number A_i is larger if the right relative values, $x_{\max} - x_R(A_i)$, is smaller and the left relative values, $x_L(A_i) - x_{\min}$, is larger. Therefore, for any two fuzzy numbers A_i and A_j, if, $D_\alpha^\gamma(A_i) < D_\alpha^\gamma(A_j)$, then $A_i \prec A_j$ if $D_\alpha^\gamma(A_i) > D_\alpha^\gamma(A_j)$, then $A_i \succ A_j$. Finally, if $D_\alpha^\gamma(A_i) = D_\alpha^\gamma(A_j)$, then $A_i \sim A_j$. It is clear that if $\gamma \geq 1$, then $D_\alpha^\gamma(A_i) = 0$.

Consider the fuzzy numbers $A_1 = (3, 6, 9)$ and $A_2 = (5, 6, 7)$. Using the corresponding membership functions are:

$$\mu_{A_1}(x) = \begin{cases} 0; & \text{if } x < 3 \\ \dfrac{x-3}{3}; & \text{if } 3 \leq x \leq 6 \\ \dfrac{9-x}{3}; & \text{if } 6 \leq x \leq 9 \\ 0; & \text{if } x > 9 \end{cases}$$

and

$$\mu_{A_2}(x) = \begin{cases} 0; & \text{if } x < 5 \\ x-5; & \text{if } 5 \leq x \leq 6 \\ 7-x; & \text{if } 6 \leq x \leq 7 \\ 0; & \text{if } x > 7 \end{cases}$$

In Figure 3, Using the above approach, the difference between left relative values and right relative values of fuzzy number A_1 and A_2 with index of optimism α

can be obtained as $D^{\gamma}_{\alpha}(A_1) = \alpha(3\gamma) - (1-\alpha)3\gamma$ and $D^{\gamma}_{\alpha}(A_2) = \alpha(\gamma+2) - (1-\alpha)(2+\gamma)$, respectively. It is observed that for a pessimistic decision maker i.e. $\alpha = 0$, we have $A_1 \succ A_2$ for every $\gamma \in (0,1)$; for an optimistic decision maker, i.e. $\alpha = 1$, we have $A_1 \prec A_2$ for every $\gamma \in (0,1)$; and for a moderate decision maker, i.e. $\alpha = 0.5$, we have $A_1 \sim A_2$, for every $\gamma \in (0,1)$. Obviously, decision maker's attitudes towards risks effect on the ranking order of fuzzy numbers.

3. Problem in Consideration

The problem that we are aiming to solve is a two player zero sum fuzzy game in which the entries in the payoff matrix A are triangular fuzzy number i.e. the payoff matrix is as follows.

$$A = \begin{bmatrix} \widetilde{a}_{11} & \widetilde{a}_{12} & \widetilde{a}_{13} & \cdots & \widetilde{a}_{1n} \\ \widetilde{a}_{21} & \widetilde{a}_{22} & \widetilde{a}_{23} & \cdots & \widetilde{a}_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \widetilde{a}_{m1} & \widetilde{a}_{m2} & \widetilde{a}_{m3} & \cdots & \widetilde{a}_{mn} \end{bmatrix}_{m\times n}$$

In the above game player I has m strategies and player II has n strategies. If player I chooses the ith strategies and player II chooses jth strategies, then player I win an amount $x \in \widetilde{a}_{ij}$.

4. Strategy Involved to Solve the Problem

Since the entries in the payoff matrix are all triangular fuzzy numbers, so by using the above fuzzy ranking method, we first try to find an entry $\widetilde{a}_{ij}$ of the matrix A, which has the properties

(i) $\widetilde{a}_{ij}$ is minimum of the ith row

(ii) $\widetilde{a}_{ij}$ is maximum of the jth column

Then we say $\widetilde{a}_{ij}$ is a fuzzy *saddle* point. Since, the ranking method is dependent upon the parameter α so we find game value for different values of α. If we get the same game value of a given matrix for all values of $\alpha \in [0,1]$, then we call that particular game value as the fixed game value of the given game matrix.

Let us, for instance, consider the game between two players X and Y, the fuzzifieds payoff matrix is given by A as follows:

$$A = \begin{bmatrix} [1,2,3] & [7,8,9] & [-3,-2,-1] \\ [4,5,6] & [0,4,8] & [-1,1,3] \\ [-7,-5,-3] & [-2,-1,0] & \left[\frac{1}{2},1,\frac{3}{2}\right] \end{bmatrix}$$

For $\alpha = 0$, $\widetilde{a}_{13}, \widetilde{a}_{23}, \widetilde{a}_{32}$ are the minimum of rows 1, 2 and 3 respectively and $\widetilde{a}_{21}, \widetilde{a}_{12}, \widetilde{a}_{23}$ are the maximum of columns 1, 2 and 3 respectively. Therefore $\widetilde{a}_{23}$ is the *saddle* fuzzy game value of matrix A.

For $\alpha = 0.5$, $\widetilde{a}_{13}, \widetilde{a}_{23}, \widetilde{a}_{31}$ are the minimum of rows 1, 2 and 3 respectively and $\widetilde{a}_{21}, \widetilde{a}_{12}, \widetilde{a}_{23}$ are the maximum of columns 1, 2 and 3 respectively. Therefore $\widetilde{a}_{23}$ is the *saddle* fuzzy game value of matrix A.

Similarly, for $\alpha = 1$, $\widetilde{a}_{23}$ is the fuzzy game value.

Thus in this case $\widetilde{a}_{23}$ is the fixed game value of the above game matrix.

5. Remarks

In the above example we are getting the same fuzzy game value for different values of α but we might get different fuzzy game value for different values of α.

Let us consider a game between two players X and Y, the fuzzified payoff matrix is given by A as follows:

$$A = \begin{bmatrix} [3,6,9] & [3,9,12] & [5,6,7] \\ [3,4,5] & [1,2,3] & [5,6,7] \\ [3,5,6] & [2,5,7] & [5,6,7] \end{bmatrix}$$

For $\alpha = 0$, $\widetilde{a}_{13}, \widetilde{a}_{22}, \widetilde{a}_{31}$ are the minimum of rows 1, 2 and 3 respectively and $\widetilde{a}_{11}, \widetilde{a}_{12}, \widetilde{a}_{31}$ are the maximum of columns 1, 2 and 3 respectively. Therefore, $\widetilde{a}_{31}$ is the *saddle* fuzzy game value of matrix A.

For $\alpha = 1$, $\widetilde{a}_{11}, \widetilde{a}_{22}, \widetilde{a}_{32}$ are the minimum of rows 1, 2 and 3 respectively and $\widetilde{a}_{11}, \widetilde{a}_{12}, \widetilde{a}_{13}$ are the maximum of columns 1, 2 and 3 respectively. Therefore $\widetilde{a}_{11}$ is the *saddle* fuzzy game value of matrix A.

For $\alpha = 0.5$, $\widetilde{a}_{11}$ and $\widetilde{a}_{13}$ are both fuzzy game value of the matrix A.

6. Conclusions

The paper considers a two person zero sum game with imprecise values having triangular possibility distribution. The proposed method is an interactive method that integrates the concept of fuzzy ranking and the minimax principle to get an imprecise game value. The parameter α can be modified suitably by the decision maker to get the desired result. We may get different fuzzy game value for different values of α for the same fuzzy game. The proposed method can be applied in the behavioural science of the animals, biological aspect of the cell of the creatures, the study of micro and macro level economic aspects, all the branches of engineering with aspect of control theory of the system and many other fields. The method can similarly be applied to a three person zero sum game.

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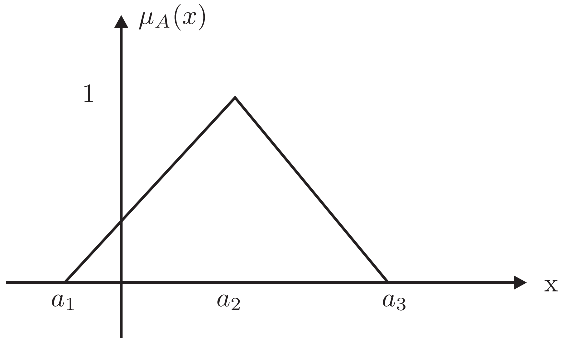

Figure 1: Triangular fuzzy number $A = (a_1, a_2, a_3)$

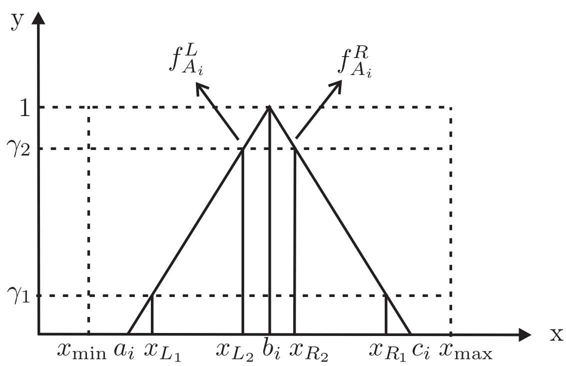

Figure 2: The left and right indices for fuzzy number A_i

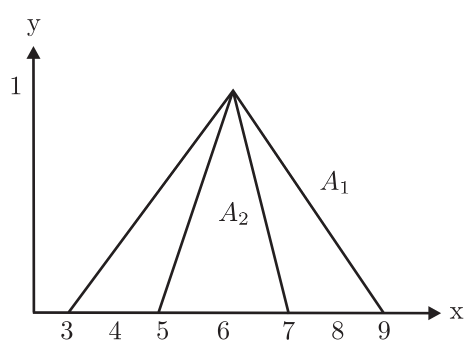

Figure 3: Fuzzy numbers A_1 and A_2