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ON TAYLOR-SERIES APPROXIMATION OF THE RESPONSE OF ELECTROMAGNETIC FIELD ON A TWO LAYERED EARTH MODEL WITH A CONDUCTIVE OVERBURDEN

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Abstract: We investigate the structure of a two layered earth by presenting a mathematical model of electromagnetic response. The integral expressions provide the electric fields on the ground surface, and Taylor-series expansion are used to find the solution of the electric field. Numerical solutions are computed and plotted to show the behaviors of the electric field while some parameters are assigned approximately. The responses of electric field from the ground surface are influenced by the conductivity of overburden.

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1. Introduction

The electromagnetic method is the most generally used in exploration geophysics

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because it responses best to a good electrical conductors at a shallow depth as deep as kilometers from the ground surface, and it is far less costly than other investigation methods. Lee and Ignetik [7] considered the forward problem of the transient electromagnetic response of a half-space with an exponentially varying conductivity profiles. They indicated out that the conductivity variation of the ground may sometimes be reasonably approximated by an exponential variation since soil salinity profiles frequently show monotonically increasing or decreasing salt conductivity of the ground. Yooyuanyong [9] gave the mathematical model of electromagnetic sounding for a conductive thin disc embedded in an inhomogeneous ground. Ketchanwit [5] used Lee and Ignetik's assumption to model the transient electromagnetic response of a two-layered earth.

In this paper, we present a mathematical model and techniques for studying the structure of the earth's surface layers. We consider the ground containing two layers, the overburden and host medium. The conductivity of overburden given by $\sigma_1(z) = (\sigma_0 + z)e^{-bz/2}$, $0 \leq z \leq d$, where σ_0 is a positive constant, b is constant, d is the thickness of overburden. The conductivity of host medium, $z \geq d$, is a constant given by $\sigma_2(z) = \sigma_0$. The conductivity ground profile in this paper is different from the model used by Lee and Ignetik [7] and Ketchanwit [5].

2. Derivation of Electromagnetic Field

We begin by considering a cylindrical polar coordinates (r, θ, z) which is established with $z > 0$ and is taken vertically positive downwards, and the origin under the center of the horizontal circular loop is used. The azimuthally symmetry gives merely electric field E_θ, and magnetic fields H_r ,and H_z components. Following Morrison et al., [8], these field quantities are found to satisfy Maxwell's equations (Hohmann and Raiche, [3]) in the form of equations

$$i\omega\mu H_r = -\frac{\partial E}{\partial z}, \tag{1}$$

$$i\omega\mu H_z = \frac{1}{r}\frac{\partial (rE)}{\partial r}, \tag{2}$$

and

$$\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = (\sigma(z) - i\omega\varepsilon)E + J_s, \tag{3}$$

where $i = \sqrt{-1}$ is an imaginary number, $J_s = aI(\omega)\delta(r-a)\delta(z+h)/r$ is the source current density, ω is the angular frequency, δ is the delta function, $\sigma(z)$ is the electrical conductivity of medium depending on depth only, ε is the electric permittivity of medium, μ is the magnetic permeability of medium, and $I(\omega)$ is the current in a coil of a small radius a. Eliminating H_r and H_z from above equations lead to get the differential equation for electric fields as.

$$i\omega\mu J_s = -\frac{\partial^2 E}{\partial z^2} - \frac{\partial^2 E}{\partial r^2} - \frac{1}{r}\frac{\partial E}{\partial r} + \frac{E}{r^2} - (i\omega\mu\sigma(z) + \omega^2\mu\varepsilon)E. \quad (4)$$

Taking Hankel transform which is defined as

$$\widetilde{E}(\lambda, z, \omega) = \int_0^\infty rJ_1(\lambda r)E(r, z, \omega)dr,$$

where J_1 is Bessel function of the first kind of order 1, and equation (4) becomes

$$i\omega\mu a I(\omega)\delta(z + h)J_1(\lambda a) = -\frac{\partial^2 \widetilde{E}}{\partial z^2} + (\lambda^2 - (i\omega\mu\sigma(z) + \omega^2\mu\varepsilon))\widetilde{E}. \quad (5)$$

We consider the ground in the form of two layers. An overburden has conductivity denoted by $\sigma_1(z) = (\sigma_0 + z)e^{-bz/2}$, $0 \le z \le d$, where σ_0 is a positive constant, b is constant, d is the thickness of overburden. The conductivity of host medium is a constant, which can be defined as $\sigma_2(z) = \sigma_0$, $z \ge d$. We consider a primary alternating source current carried by a coil of radius a, at $z = -h$ above the surface of the earth, (see figure 1).

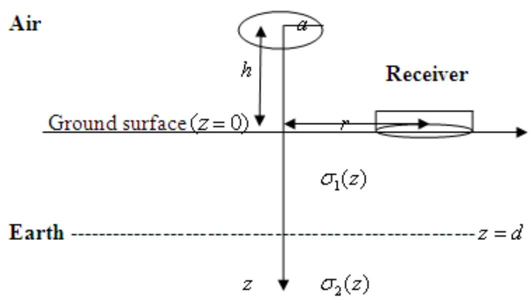

Figure 1: IIllustration of the two layered earth model with a positively skewed bulge conductivity profile.

The electric field in air can be denoted by $\widetilde{E}_a(\lambda, z, \omega)$ and expressed as the sum of two components,

$$\widetilde{E}_a(\lambda, z, \omega) = \widetilde{E}^p(\lambda, z, \omega) + \widetilde{E}^s(\lambda, z, \omega),$$

where $\widetilde{E}^p(\lambda, z, \omega)$ is the primary field and $\widetilde{E}^s(\lambda, z, \omega)$ is the secondary field. Both electric fields can be obtained from equation (5). In air, $\sigma_{air}(z) \cong 0$ and the electric field is given by

$$\widetilde{E}_a(\lambda, z, \omega) = \frac{i\omega\mu_0 aI(\omega)J_1(\lambda a)e^{-\lambda|z+h|}}{2\lambda} + Ae^{\lambda z},\ z \leq 0 \tag{6}$$

which remains bounded as $z \to -\infty$, and A is arbitrary constant to be determined.

From equation (4), we can obtain the partial differential equation for the electric filed in overburden as

$$\frac{\partial^2}{\partial z^2}E_o(r, z, \omega) + \frac{\partial^2}{\partial r^2}E_o(r, z, \omega) + \frac{1}{r}\frac{\partial}{\partial r}E_o(r, z, \omega) - \frac{E_o(r, z, \omega)}{r^2} + k_o^2 E_o(r, z, \omega) = 0. \tag{7}$$

where $E_o(r, z, \omega)$ is the electric field in an overburden, $k_o^2 = i\omega\mu_o\sigma(z) + \omega^2\mu_o\varepsilon_o$, μ_o is the magnetic permeability of overburden, ε_o is the electric permittivity of overburden and $\sigma(z) = (\sigma_0 + z)e^{-bz/2}$ is the conductivity of overburden. Taking Hankel transform to equation (7), we have

$$\frac{d^2}{dz^2}\widetilde{E}_o(\lambda, z, \omega) - (\lambda^2 - k_o^2)\widetilde{E}_o(\lambda, z, \omega) = 0. \tag{8}$$

By Taylor-series approximation, we obtain the solution of equation (8) as

$$\begin{aligned}\widetilde{E}_o(\lambda, z, \omega) &= A_1 e^{-(\lambda^2 - \alpha\sigma_0 + b)z} \\ &\quad + A_2\left[(\lambda^2 - \alpha\sigma_0 + b + 1)z + [\lambda^2 - \alpha\sigma_0 - (\lambda^2 - \alpha\sigma_0 + b)^2]\frac{z^2}{2}\right],\end{aligned} \tag{9}$$

where A_1 and A_2 are arbitrary constants to be determined from the boundary conditions.

We can obtain partial differential equation for the electric filed in host medium from equation (4) as

$$\frac{\partial^2}{\partial z^2}E_h(r, z, \omega) + \frac{\partial^2}{\partial r^2}E_h(r, z, \omega) + \frac{1}{r}\frac{\partial}{\partial r}E_h(r, z, \omega) - \frac{E_h(r, z, \omega)}{r^2} + k_h^2 E_h(r, z, \omega) = 0. \tag{10}$$

where $E_h(r,z,\omega)$ is the electric field in host medium, $k_h^2 = i\omega\mu_h\sigma_0 + \omega^2\mu_h\varepsilon_h$, μ_h is the magnetic permeability of host medium, ε_h is the electric permittivity of overburden and σ_0 is the conductivity of host medium. Taking Hankel transform to equation (10), we have

$$\frac{d^2}{dz^2}\widetilde{E}_h(\lambda,z,\omega) - m_h^2\widetilde{E}_h(\lambda,z,\omega) = 0, \tag{11}$$

where $m_h^2 = \lambda^2 - k_h^2$. The solution of equation (11) is

$$\widetilde{E}_h(\lambda,z,\omega) = A_3 e^{-m_h(z-d)} + A_4 e^{m_h(z-d)}, \tag{12}$$

where A_3, A_4 are arbitrary constants to be determined from the boundary conditions. Under the condition that $z \to \infty$, then $\widetilde{E}_h \to 0$, we require that $A_4 = 0$. Consequently, equation (12) becomes

$$\widetilde{E}_{ho}(\lambda,z,\omega) = A_3 e^{-m_h(z-d)}, \quad z \geq d \tag{13}$$

The constants A from equation (6), A_1, A_2 from equation (9), and A_3 from equation (13) can be solved by imposing the continuity of $\widetilde{E}$ and $\dfrac{\partial \widetilde{E}}{\partial z}$ at the interface. At air-Earth interface, $z = 0$. Therefore, we have

$$\widetilde{E}_a(\lambda,0,\omega) = \widetilde{E}_o(\lambda,0,\omega), \qquad \frac{\partial}{\partial z}\widetilde{E}_a(\lambda,0,\omega) = \frac{\partial}{\partial z}\widetilde{E}_o(\lambda,0,\omega)$$

At $z = d$, we have

$$\widetilde{E}_o(\lambda,d,\omega) = \widetilde{E}_h(\lambda,d,\omega), \qquad \frac{\partial}{\partial z}\widetilde{E}_o(\lambda,d,\omega) = \frac{\partial}{\partial z}\widetilde{E}_h(\lambda,d,\omega)$$

We obtain the electric fields in air as

$$\begin{aligned} &E_a(r,z,\omega) \\ &= i\omega\mu_0 a I(\omega) \int_0^\infty \left[\frac{\lambda e^{-\lambda|z+h|}}{(\lambda + \lambda^2 - \alpha\sigma_0 + b)} + \frac{\lambda m_h e^{-\lambda|h| + (\lambda^2 - \alpha\sigma_0 + b)d + \lambda z}}{(\lambda^2 - \alpha\sigma_0 + b - m_h)} \right] \\ &\times J_1(\lambda a) J_1(\lambda r) d\lambda. \end{aligned} \tag{14}$$

The electric fields in overburden as

$$E_o(r,z,\omega)$$

$$= i\omega\mu_0 aI(\omega) \int_0^\infty \lambda \left[\frac{1}{(\lambda + \lambda^2 - \alpha\sigma_0 + b)} + \frac{m_h e^{(\lambda^2 - \alpha\sigma_0 + b)d}}{(\lambda^2 - \alpha\sigma_0 + b - m_h)} \right]$$
$$\times J_1(\lambda a) J_1(\lambda r) e^{-\lambda|h| + \lambda z} d\lambda. \quad (15)$$

and the electric fields in host medium as

$$E_h(r, z, \omega)$$
$$= i\omega\mu_0 aI(\omega) \int_0^\infty \lambda \left[\frac{(\lambda + m_h + b) e^{-(\lambda^2 - \alpha\sigma_0 + b)z - m_h(z-d) - \lambda|h|}}{(\lambda^2 - \alpha\sigma_0 + b - m_h)} \right]$$
$$\times J_1(\lambda a) J_1(\lambda r) d\lambda. \quad (16)$$

The electric field on the ground surface can be determined from equations (14) or equation (15) by taking $z = 0$. Thus, we have

$$E_{sur}(r, 0, \omega)$$
$$= i\omega\mu_0 aI(\omega) \int_0^\infty \lambda \left[\frac{1}{(\lambda + \lambda^2 - \alpha\sigma_0 + b)} + \frac{m_h e^{(\lambda^2 - \alpha\sigma_0 + b)d}}{(\lambda^2 - \alpha\sigma_0 + b - m_h)} \right]$$
$$\times J_1(\lambda a) J_1(\lambda r) e^{-\lambda|h|} d\lambda. \quad (17)$$

Chave's algorithm [2] is used for numerical calculating the inverse Hankel transform of the electric field solutions.

3. Conclusion

For this paper, we present the method to study the structure of the earth's surface layers. The integral expressions are used to produce the forward modeling. We started off by formulating the problem to get the electric fields, which could be used to find the electric fields on the ground surface. We consider two layered earth models having a positively skewed bulge conductivity profile on the first layer which is denoted by $\sigma_1(z) = (\sigma_0 + z)e^{-bz/2}$, $0 \leq z \leq d$, where σ_0 is a positive constant, b is constant, d is the thickness of overburden. The conductivity of host medium, $z \geq d$, is a constant given by $\sigma_2(z) = \sigma_0$ on the second layer. In our experiments, we fix the value of b through the process. The values of σ_0 and d are varied. We firstly set up frequency =150 Hz, $d = 1.5\,m$, $\sigma_0 = 0.001\,Sm^{-1}$, $\sigma_0 = 0.002\,Sm^{-1}$ and $\sigma_0 = 0.003\,Sm^{-1}$. Similarly, when $d = 2.5\,m$. Next, we set up $\sigma_0 = 0.001\,Sm^{-1}$, $d = 1.5\,m$, $d = 2.5\,m$

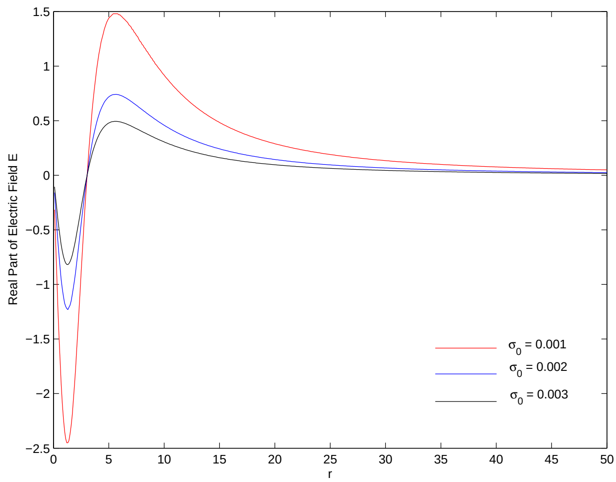

Figure 2: Graph of real part of electric field E versus r for a two layered earth model with a positively skewed bulge conductivity profile, $\sigma_0 = 0.001\,S\,m^{-1}, \sigma_0 = 0.002\,S\,m^{-1}, \sigma_0 = 0.003\,S\,m^{-1}, d = 1.5\,m$, frequency $= 150\,$Hz.

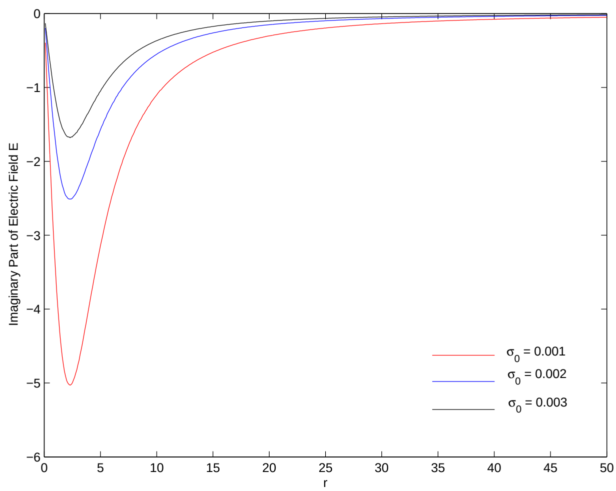

Figure 3: Graph of imaginary part of electric field E versus r for a two layered earth model with a positively skewed bulge conductivity profile, $\sigma_0 = 0.001\,S\,m^{-1}, \sigma_0 = 0.002\,S\,m^{-1}, \sigma_0 = 0.003\,S\,m^{-1}, d = 1.5\,m$, frequency $= 150\,$Hz.

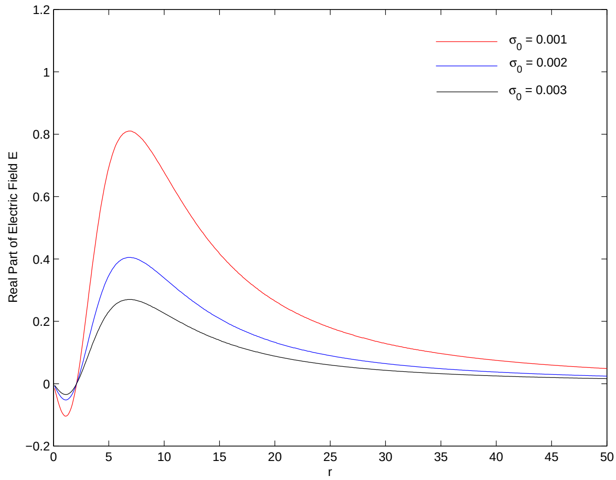

Figure 4: Graph of real part of electric field E versus r for a two layered earth model with a positively skewed bulge conductivity profile, $\sigma_0 = 0.001\,S\,m^{-1}, \sigma_0 = 0.002\,S\,m^{-1}, \sigma_0 = 0.003\,S\,m^{-1}, d = 2.5\,m$, frequency $= 150\,\mathrm{Hz}$.

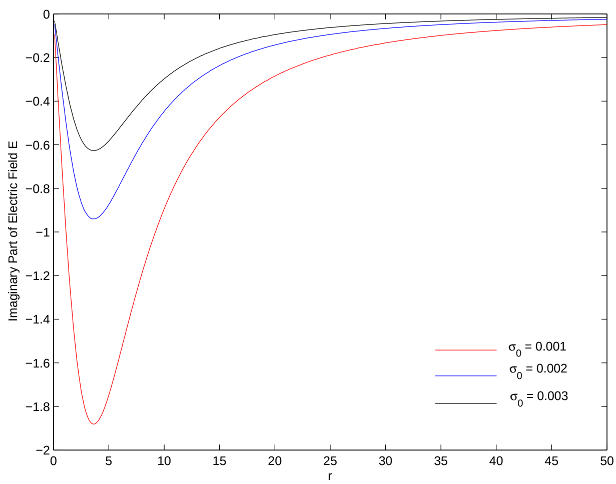

Figure 5: Graph of imaginary part of electric field E versus r for a two layered earth model with a positively skewed bulge conductivity profile, $\sigma_0 = 0.001\,S\,m^{-1}, \sigma_0 = 0.002\,S\,m^{-1}, \sigma_0 = 0.003\,S\,m^{-1}, d = 2.5\,m$, frequency $= 150\,\mathrm{Hz}$.

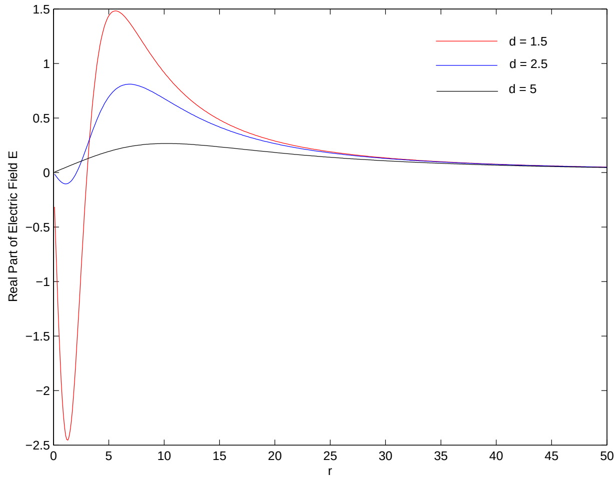

Figure 6: Graph of real part of electric field E versus r for a two layered earth model with a positively skewed bulge conductivity profile, $\sigma_0 = 0.001\,S\,m^{-1}, d = 1.5\,m, d = 2.5\,m, d = 5\,m$, frequency $= 150\,$Hz.

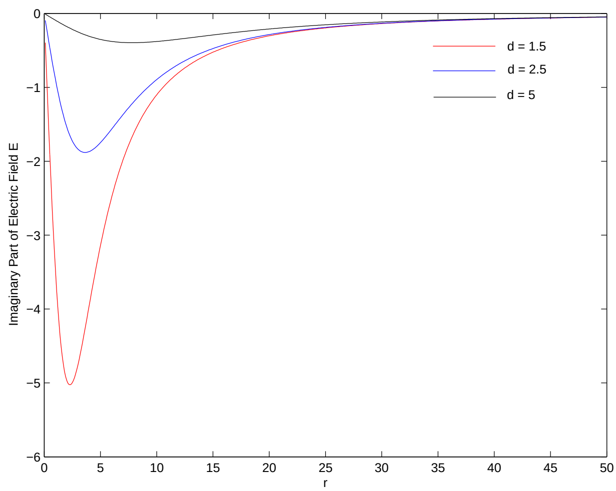

Figure 7: Graph of imaginary part of electric field E versus r for a two layered earth model with a positively skewed bulge conductivity profile, $\sigma_0 = 0.001\,S\,m^{-1}, d = 1.5\,m, d = 2.5\,m, d = 5\,m$, frequency $= 150\,$Hz.

and $d = 5\,m$. The graphs are shown the behavior of the electric field against source-receiver spacing r at different depths. The curves of electric fields for the real part and imaginary part are oscillated and decreasing to zero. The curves of the electric fields are similar to the figure of a positively skewed bulge conductivity ground profile as shown in figures. These implied that the curves of electric fields are dominated by the ground conductivity profile. For the additional information, the inversion process will be studied in the near future work to find out the optimal values for all the parameters.

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