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SOME COMPARISOM THEOREMS ON TIME SCALES

Chen-Huang Hong[1 §], Huei-Lin Hong[2]
General Education Center
National Taipei University of Technology
Taipei, 10608, Taiwan, REPUBLIC OF CHINA

Abstract: Under the distinct conditions of the function f, some comparison theorems of differential equations are established on time scales.

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1. Introduction

To unify the theory of continuous and discrete dynamic systems, in 1990, Hilger [3] proposed the study of dynamic systems on a time scale and developed necessary calculus for functions on a time scale. In this paper, we give some interesting comparison theorems on time scales. In the last twenty years, some authors discussed some interesting comparison theorems on a differential and integral inequalities, see, for example [1, 2, 4, 5, 7].

We first briefly introduce the time scales calculus as follows:

By a times scale $\mathbb{T}$, we mean any closed subset of $\mathbb{R}$ with order and topological structure in a canonical way. Since a time scale $\mathbb{T}$ may or may not be connected, we need the concept of jump operators.

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[§]Correspondence author

Definition 1.1. Let $t \in \mathbb{T}$, where $\mathbb{T}$ is a time scale. Then two mappings

$$\sigma,\ \rho : \mathbb{T} \to \mathbb{R}$$

satisfying

$$\sigma(t) = \inf\{s \in \mathbb{T} | s > t\},\ \rho(t) = \sup\{s \in \mathbb{T} | s < t\}$$

are called the *jump operators.*

For $t \in \mathbb{T}$, we say that:

(i) t is a *right-scattered point* if $\sigma(t) > t$.

(ii) t is a *left-scattered point* if $\rho(t) < t$.

(iii) t is a *right-dense point* if $\sigma(t) = t$.

(iv) t is a *left-dense point* if $\rho(t) = t$.

Definition 1.2. A mapping $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if it satisfies

(a) f is continuous at each right-dense point or maximal point of $\mathbb{T}$.

(b) $\lim_{s \to t^-} g(s) = g(t^-)$ exists for each left-dense point $t \in \mathbb{T}$.

The set of all rd-continuous functions from $\mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd}[\mathbb{T}, \mathbb{R}]$.

Definition 1.3. The function $p : \mathbb{T} \to \mathbb{R}$ is called *regressive* provided

$$1 + \mu(t)p(t) \neq 0 \quad \text{for each} \quad t \in \mathbb{T},$$

where $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t.$$

Let

$$\mathbb{T}^k := \begin{cases} \mathbb{T} - \{m\},\ \text{if } \mathbb{T} \text{ has a left-scattered maximal point } m. \\ \mathbb{T},\ \text{otherwise.} \end{cases}$$

Definition 1.4. Assume that $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, then we define $f^{\Delta}(t)$ to be the number (if it exists) with property that for any given $\epsilon > 0$, there exists a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s] \right| \leq \epsilon |\sigma(t) - s|$$

for all $s \in U$. In this case, $f^{\Delta}(t)$ is called the Δ-*derivative* of $f(t)$ at t. If f is differentiable at each $t \in \mathbb{T}$, then f is called Δ-*differentiable* on $\mathbb{T}$.

Definition 1.5. A function $g : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ if $g^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^k$, and in this case, we define the integral of f by

$$\int_s^t f(u)\Delta u = g(t) - g(s)$$

for all $s, t \in \mathbb{T}$, and we say that f is *integrable* on $\mathbb{T}$.

Note. Throughout this paper, we suppose that

(a) $\mathbb{R} = (-\infty, \infty)$;

(b) $\mathbb{T}$ is a time scale with t_0 as minimal element;

(c) $\Re^+ = \{p : \mathbb{T} \to \mathbb{R} |\ 1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}\}$, that is, the set $\Re^+$ is all positively regressive functions;

(d) an interval means the intersection of a real interval with the given time scale.

For further information concerning time scales, see [1] and [6].

2. Main Result

Theorem 2.1. *Suppose that $\mathbb{T}$ is a time scale and has the minimal element t_0 but has no maximal element. Let $f \in C[\mathbb{T} \times \mathbb{R}, \mathbb{R}]$ and $v,\ w \in C[\mathbb{T}, \mathbb{R}]$ be differentiable for each $t \in \mathbb{T}$. If one of the following conditions holds:*

(a) $f(t, u)$ is increasing in u (i.e., $u \le v \ \Rightarrow\ f(t, u) \le f(t, v)$ for $t \in \mathbb{T}$) and

$$v^{\Delta}(t) - f(t, v(t)) < w^{\Delta}(t) - f(t, w(t)) \quad \textit{for} \quad t \in \mathbb{T}, \tag{1}$$

or

(b) $f(t, u)$ is decreasing in u (i.e., $-f(t, u)$ is increasing in u) and

$$v^{\Delta}(t) - f(t, v(\sigma(t))) < w^{\Delta}(t) - f(t, w(\sigma(t))) \quad \textit{for} \quad t \in \mathbb{T}. \tag{2}$$

Then $v(t_0) < w(t_0)$ implies $v(t) < w(t)$ for $t \in \mathbb{T}$.

Proof. First, we suppose that case (a) holds. We apply the induction principle to the statement

$$A(t) : v(t) < w(t),\ \ t \in \mathbb{T}.$$

(I) Clearly, $A(t_0)$ is satisfied since $v(t_0) < w(t_0)$.

(II) Let t be right-scattered and $A(t)$ be true. We show that $A(\sigma(t))$ is true. It follows from $f(t, u)$ is increasing in $u \in \mathbb{R}$ and $v(t) \leq w(t)$ that

$$f(t, v(t)) \leq f(t, w(t)).$$

Using the definition of derivative of v and w on the right scattered point,

$$v(\sigma(t)) - w(\sigma(t)) = [v^{\Delta}(t) - w^{\Delta}(t)](\sigma(t) - t) + v(t) - w(t),$$

which implies, by (1)

$$v(\sigma(t)) - w(\sigma(t)) \leq [f(t, v(t)) - f(t, w(t))](\sigma(t) - t) + v(t) - w(t) < 0.$$

Hence, $A(\sigma(t))$ is true.

(III) Let t be right-dense and N be a neighborhood of t. In this case, $v^{\Delta}(t) = v'(t)$. Assume that $A(t)$ is true. We show that $A(s)$ is true for $s \geq t$, $s \in N$. Suppose not, then there exists $s_0 \in N$ with $s_0 > t$ such that

$$v(s_0) \leq w(s_0) \quad \text{and} \quad v'(s_0) \geq w'(s_0).$$

Since $f(t, u)$ is increasing in u,

$$f(s_0, v(s_0)) \leq f(s_0, w(s_0)).$$

Thus

$$v'(s_0) - f(s_0, v(s_0)) \geq w'(s_0) - f(s_0, w(s_0)),$$

which ia a contradiction to (1).

(IV) Let t be a left-dense point such that $A(s)$ is true for $s < t$. We show that $A(t)$ is true. It follows from the continuity of v and w that

$$v(t) = \lim_{s \to t^-} v(s) \leq \lim_{s \to t^-} w(s) = w(t).$$

It remains to show that $v(t) = w(t)$ is impossible. Assume, on the contrary that $v(t) = w(t)$, then, by (1),

$$v^{\Delta}(t) - w^{\Delta}(t) < f(t, v(t)) - f(t, w(t)) = 0. \tag{3}$$

Using the definition of the derivative, there exists $\epsilon > 0$ such that

$$v(\sigma(t)) - v(s) - \frac{\epsilon}{2}(\sigma(t) - s) \leq v^{\Delta}(t)(\sigma(t) - s)$$

and

$$-w(\sigma(t)) + w(s) + \frac{\epsilon}{2}(\sigma(t) - s) \leq -w^{\Delta}(t)(\sigma(t) - s),$$

which yield

$$[v^{\Delta}(t) - w^{\Delta}(t)](\sigma(t) - s) \geq [v(\sigma(t)) - w(\sigma(t))] - (v(s) - w(s)).$$

It follows from $\sigma(t) - s \geq t - s > 0$ that A(s) is true for $s < t$, $v(t) = w(t)$, and $v^{\Delta}(t) - w^{\Delta}(t) > 0$, which contradicts (3). Hence, by the induction principle, we conclude that $v(t) < w(t),\ t \in \mathbb{T}$. This completes the proof.

The proof of the theorem with case (b) is similar. We only show that $A(\sigma(t))$ is true for t is a right-scattered point. Assume that $A(\sigma(t))$ is not true, i.e.,

$$v(\sigma(t)) > w(\sigma(t)).$$

Since $f(t, u)$ is decreasing in u,

$$f(t, v(\sigma(t))) \leq f(t, w(\sigma(t))).$$

It follows from (2) that

$$\begin{aligned} v(\sigma(t)) - w(\sigma(t)) &= (v^{\Delta}(t) - w^{\Delta}(t))(\sigma(t) - t) + v(t) - w(t) \\ &\leq [f(t, v(\sigma(t))) - f(t, w(\sigma(t)))](\sigma(t) - t) + v(t) - w(t) \\ &\leq 0, \end{aligned}$$

which is a contradiction with assumption

$$v(\sigma(t)) > w(\sigma(t)).$$

Thus, the proof is complete. □

Remark. The case (a) of Theorem 2.1 is similar to that of the inequatities in [6, p. 47].

Theorem 2.2. *Suppose that $\mathbb{T}$ is a time scale and has the minimal element t_0 but has no maximal element. Let $F \in C[\mathbb{T} \times \mathbb{T} \times \mathbb{R}, \mathbb{R}]$ satisfy*

$$u \leq v \quad \Rightarrow \quad F(t, s, u) \leq F(t, s, v)$$

for all $(t,s) \in \mathbb{T} \times \mathbb{T}$*. If there are three functions* v*,* w *and* $g \in C[\mathbb{T}, \mathbb{R}]$ *satisfy the following two inequalities:*

$$v(t) \le g(t) + \int_{t_0}^{t} F(t,s,v(s))\Delta s, \quad w(t) \ge g(t) + \int_{t_0}^{t} F(t,s,w(s))\Delta s, \tag{4}$$

where the equality holds in at most one place for each $t \in \mathbb{T}$*, then*

$$v(t) < w(t) \quad \text{for} \quad t \in \mathbb{T}.$$

Proof. It follows from (4) that $v(t_0) \le g(t_0))$ and $w(t_0) \ge g(t_0)$, where the equality holds in at most one place. Thus

$$v(t_0) < w(t_0).$$

If the conclusion were false, then there exists $t \in \mathbb{T}$ such that

$$v(t) = w(t)$$

and

$$v(s) < w(s) \quad \text{for} \quad s \in [t_0, t) \cap \mathbb{T}.$$

By the increasing property of F,

$$F(t,s,v(s)) \le F(t,s,w(s)) \quad \text{for} \quad s \in [t_0, t) \cap \mathbb{T}.$$

This and (4) imply

$$v(t) \le g(t) + \int_{t_0}^{t} F(t,s,v(s))\Delta s \le g(t) + \int_{t_0}^{t} F(t,s,w(s))\Delta s \le w(t),$$

where there is strictly inequality in at least one place, hence

$$v(t) < w(t) \text{ for } t \in \mathbb{T}.$$

This contradiction proves our theorem. □

It follows from Theorem 2.2 that we have the following

Corollary 2.1. *Suppose that* $\mathbb{T}$ *is a time scale and has the minimal element* t_0 *but has no maximal element. Let* F*,* $v(t)$ *and* $w(t)$ *be defined as in Theorem 2.2 for* $t \in \mathbb{T}$*. If*

$$v(t) - \int_{t_0}^{t} F(t,s,v(s))\Delta s < w(t) - \int_{t_0}^{t} F(t,s,w(s))\Delta s \quad \text{for} \quad t \in \mathbb{T},$$

then $v(t) < w(t)$ *for all* $t \in \mathbb{T}$.

Finally, we consider the case that "$<$" is replaced by "$\leq$" in Corollary 2.1

Theorem 2.3. *Suppose that* $\mathbb{T}$ *is a time scale and has the minimal element* t_0 *but has no maximal element. Let* F, v, w *and* g *be defined as in Theorem 2.2 and for each* $g(t)$, *the following equation*

$$w(t) = g(t) + \int_{t_0}^{t} F(t, s, w(s))\Delta s$$

has a unique solution $w(t, g)$. *If for any function sequence* $\{h_k(s)\}_{k=1}^{\infty}$ *with* $h_k : \mathbb{T} \to \mathbb{R}$,

$$\max_{0 \leq s \leq t} |h_k(s) - g(s)| \to 0 \;\Rightarrow\; |w(h_k(t)) - w(g(t))| \to 0 \text{ as } k \to \infty, \tag{5}$$

and

$$v(t) - \int_{t_0}^{t} F(t, s, v(s))\Delta s < w(t) - \int_{t_0}^{t} F(t, s, w(s))\Delta s \quad \text{for} \quad t \in \mathbb{T},$$

then $v(t) \leq w(t)$ *for all* $t \in \mathbb{T}$.

Proof. Let

$$g(t) = w(t) - \int_{t_0}^{t} F(t, s, w(s))\Delta s.$$

For any given $\epsilon > 0$, let $w^{\epsilon}(t)$ be the solution of

$$g(t) + \epsilon = w(t) - \int_{t_0}^{t} F(t, s, w(s))\Delta s.$$

Thus

$$v(t) - \int_{t_0}^{t} F(t, s, v(s))\Delta s \leq g(t) < g(t) + \epsilon = w^{\epsilon}(t) - \int_{t_0}^{t} F(t, s, w^{\epsilon}(s))\Delta s.$$

It follows from Corollary 2.1 that

$$v(t) < w^{\epsilon}(t) \quad \text{for} \quad t \in \mathbb{T}.$$

But, by (5),

$$\epsilon \to 0 \quad \text{implies} \quad w^{\epsilon}(t) \to w^{0}(t) = w(t).$$

Thus $v(t) \leq w(t)$ for $t \in \mathbb{T}$. □

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