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DIVISOR FUNCTION $\tau_3(\omega)$ WEIGHTED BY KLOOSTERMAN SUM

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Abstract: In this article we investige values of $\tau_3(\omega)$ over the Gaussian integer, and obtain estimation sum of $\tau_3(\omega)$ weighted by Kloosterman sum.

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1. Introduction

Let $\mathbb{Z}[i]$ denote ring of the Gaussian integers and for $\omega \in \mathbb{Z}[i]$ function $\tau_3(\omega)$ indicates number of representations of ω as product of 3 the Gaussian integers:

$$\tau_3(\omega) = \sum_{\omega=\omega_1\omega_2\omega_3} 1.$$

Investigation values of $\tau_3(\omega)$ on arithmetic progressions is an important problem of analytic number theory. Like in rational case building of asymptotic formulae for the sum

$$\sum_{\omega\equiv\omega_0(mod\gamma)} \tau_3(\omega)$$

connected with estimation of the sum

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$$T\left(x,\gamma\right) := \sum_{N(\omega)\leq x} \tau_3\left(\omega\right) K\left(1,1,\omega;\gamma\right), \tag{1}$$

where $K\left(\alpha_1,\alpha_2,\alpha_3;\gamma\right)$ is two-dimensional Kloosterman sum over $\mathbb{Z}\left[i\right]$ defined as

$$K\left(\alpha_1,\alpha_2,\alpha_3;\gamma\right) := \sum_{\substack{z_1, z_2, z_3 \in \mathbb{Z}\left[i\right] \\ z_1z_2z_3 \equiv 1\left(mod \gamma\right)}} e^{2\pi i Re\left(\frac{\alpha_1z_1+\alpha_2z_2+\alpha_3z_3}{\gamma}\right)}, \tag{2}$$

with $\alpha_1,\alpha_2,\alpha_3 \in \mathbb{Z}\left[i\right]$.

This sum is a generalization of two-dimensional Kloosterman sum over $\mathbb{Z}$.

The purpose of our article is to get non-trivial asymptotic formulae for $T\left(x\right)$ in case where norm γ tends to infinity with x.

2. Auxiliary Results

In the sequel, we use the following standard notation: $\mathbb{Q}\left(i\right)$ is field of the Gaussian numbers, $\mathbb{Q}\left(i\right) = \{a+bi|a,b\in\mathbb{Q},\ i^2=-1\}$, $G=\mathbb{Z}\left[i\right]=\{a+bi|a,b\in\mathbb{Z},\ i^2=-1\}$ – ring of the Gaussian integers;

$s=\sigma+it$ – complex number, $\sigma = Res, t = Ims$;

Notation G_γ (G_γ^* respectively) states for complete (reduced) system of residues modulo q in G.

By (α,β) we denote GCD α and β over G, that calculated up to $\{\pm1,\pm i\}$;

$N\left(\alpha\right)=a^2+b^2, \alpha = a+bi$ denote the norm of $\alpha\in G$;

$\tilde{\mu},\tilde{\phi}$ are similar Mobius and Euler functions over G;

For $\omega \in G_\gamma^*$ by ω^{-1} we denote multiplicative inverse for ω in G_γ^*;

Notation $\sum_{S(c)}$ means that summaton passes under the condition C that is described separately after corresponding formula;

$$e^{2\pi i x} = e\left(x\right);$$

Notation $\ll_\varepsilon, O_\varepsilon$ means that constant in Vinogradov's symbol $\ll$ or in Landau symbol O may depends on ε.

Lemma 1. *(see Hecke [2], Epstein [1]) The Hecke function $Z(s;\delta_0,\delta_1)$ admits an analytic continuation to the whole complex s-plane, and for $\delta_1 \notin G$ is an entire function, and for $\delta_1 \in G$ $Z\left(s;\delta_0,\delta_1\right)$ is analytic over all s-plane, except point $s=1$, where it has first-order pole with residue π.*

Moreover there is functional equation

$$\pi^{-s}\Gamma(s)\, Z(s;\delta_0,\delta_1) = \pi^{-(1-s)}\Gamma(1-s)\, Z(1-s;-\delta_1,\delta_0)\, e^{-2\pi i Re\left(\bar{\delta}_0\delta\right)}$$

where $\Gamma(z)$ – *Euler gamma-function and as usual by* δ_0 *we denote complex conjugate.*

Lemma 2. *Let* $\delta_0 = \frac{\alpha}{\gamma}$, $\alpha \in G_\gamma^*$. *Then for* $1/2 \le \sigma \le 1+\varepsilon, |t| > t_0 \ge 3$, *we have*

$$\left| Z\left(s;\frac{\alpha}{\gamma},0\right) - \sum_{\omega\in B}\frac{1}{\left(N\left(\omega+\frac{\alpha}{\gamma}\right)\right)^s}\right| \ll_\varepsilon$$
$$\ll_\varepsilon \begin{cases} |t|^{2/3(1-\sigma)+\varepsilon} N(\gamma)^{-(2\sigma-1)}, & if\ t \gg N(\gamma)^{3/2} \\ N(\gamma)^{1-\sigma+\varepsilon}, & if\ t \ll N(\gamma)^{3/2} \end{cases}$$

where B *denotes set* $\{0,\pm 1,\pm i\}$; $\varepsilon > 0$ – *arbitrary positive number.*

Proof. By the definition of $Z(s;\delta_0,\delta_1)$, on line $Res = 1+\varepsilon$ we have

$$\left|\left(Z\left(s;\frac{\alpha}{\gamma},\delta_1\right) - \sum_{\omega\in B}\frac{1}{N\left(\omega+\frac{\alpha}{\gamma}\right)^s}\right)\right| \ll_\varepsilon 1$$

By Kauffman estimate [3], on line $Res = \frac{1}{2}$ we have

$$\left| Z\left(sk\frac{\alpha}{\gamma};0\right) - \sum_{\omega\in B}\frac{1}{N\left(\omega+\frac{\alpha}{\gamma}\right)^s}\right| \le$$
$$\le \left| Z\left(\frac{1}{2}+it;\frac{\alpha}{\gamma},0\right)\right| + \sum_{\omega\in B}\frac{1}{N\left(\omega+\frac{\alpha}{\gamma}\right)^{1/2}} \ll |t|^{\frac{1}{2}+\varepsilon} + N(\gamma)^{1/2}$$

Therefore the principle of Phragmen-Lindelof principle gives

$$\left| Z\left(s;\frac{\alpha}{\gamma},\delta_1\right) - \sum_{\omega\in B}\frac{e(Re(\omega\delta_1))}{N\left(\omega+\frac{\alpha}{\gamma}\right)^s}\right| \ll_\varepsilon$$
$$\ll_\varepsilon \begin{cases} |t|^{2/3(1-\sigma)+\varepsilon} N(\gamma)^{-(2\sigma-1)}, & if\ t \gg N(\gamma)^{3/2}, \\ N(\gamma)^{2-3\sigma}, & if\ t \ll N(\gamma)^{3/2} \end{cases}$$

□

Lemma 3. *Let $\alpha_1, \alpha_2, \alpha_3 \in G, p$ be a Gaussian prime, $m \geq 1$ be integer. Then*

$$|K(\alpha_1, \alpha_2, \alpha_3; p^m)| \leq$$
$$\leq \begin{cases} \bar{\phi}(p^n)^2 \quad if \quad (\alpha_1, \alpha_2, \alpha_3, p^m) = p^m, \\ \bar{\phi}(p) \quad if \quad m = 1 \text{ and } (\alpha_1, p) = 1, \; \alpha_2 = \alpha_3 \equiv 0 \,(mod p), \\ K(1, 1, \alpha; p) \quad if \quad (\alpha_1, \alpha_2, p) = 1, \\ 0 \quad if \quad m \geq 2 \text{ and } (\alpha_1, \alpha_2, \alpha_3; p) = 1, \quad \alpha_1\alpha_2\alpha_3 \equiv 0 \,(mod p). \end{cases}$$

This is a special case of general result on the estimate of n-dimensional Kloosterman sums in [4].

Lemma 4. *Let $\gamma = \gamma_1\gamma_2, \; (\gamma_1, \gamma_2) = 1, \; \gamma_1, \gamma_2$ be square-free Gaussian integeres, $(\omega, \gamma) = \delta, \delta = \delta_1\delta_2, \;\; \delta_1|\gamma_1, \;\; \delta_2|\gamma_2$.*

Then

$$K(1, 1, \omega; \gamma) = \begin{cases} \mu(\delta_1) K\left(1, 2, \omega\tilde{\delta}_1^2; \frac{\gamma}{\delta_1}\right), & if \; N(\delta_2) = 1 \\ 0, & else \end{cases}$$

Proof. Taking into account quasi-multiplicativity of $K(1, 1, \omega; \gamma)$ on γ, i.e. by equality

$$K(1, 1, \omega; \gamma_1\gamma_2) = K\left(1, 2, \omega\gamma_2'^2; \gamma_1\right) \cdot K\left(1, 1, \omega\gamma_1'^2; \gamma_2\right), \tag{3}$$

where $\gamma_1\gamma_1' \equiv 1 \,(mod\gamma_2), \gamma_2\gamma_2' \equiv 1 \,(mod\gamma_1)$

it suffices to prove the lemma under the assumption $\gamma = \mathfrak{p}^m, \mathfrak{p}$ is Gaussian prime.

If $(\omega, \gamma_2) = \delta_2, N(\delta_2) > 1$ then from (3) it follows that $\gamma_2 = \mathfrak{p}^m, m > 1, \delta_2 = \mathfrak{p}^l, l \geq 1$. But then from lemma 3 we get $K(1, 1, \omega; \gamma) = K\left(1, 1, \omega\left(\frac{\gamma}{\delta_1}\right)'^2; \delta_1\right) \cdot K\left(1, 1, \omega\delta_1'^2; \frac{\gamma}{\delta_1}\right)$.

Next

$$\begin{aligned} &K\left(1, 1, \omega\left(\frac{\gamma}{\delta_1}\right)'^2; \delta_1\right) = K(1, 1, 0; \delta_1) = \textstyle\prod_{j=1}^l K(1, 1, 0; \mathfrak{p}_j) = \\ &= \textstyle\prod_{j=1}^l \sum_{z_3} \sum_{z_1z_2=1} e\left(Re\left(\frac{z_1z_3+z_2}{\mathfrak{p}}\right)\right) = \\ &= \textstyle\prod_{j=1}^l \sum_{z_2} e\left(Re\frac{z_2}{\mathfrak{p}}\right) \sum_{z_3} e\left(Re\frac{z_3}{\mathfrak{p}}\right) = \\ &= \textstyle\prod_{j=1}^l \mu(\mathfrak{p}_j) \sum_{z_1z_2\equiv 1(\mathfrak{p}_j)} e\left(Re\frac{z_2}{\mathfrak{p}}\right) = \\ &= \textstyle\prod_{j=1}^l \mu(\mathfrak{p}_j) \sum_{z_1 \in C_{\mathfrak{p}}^*} e\left(\frac{z_1}{\mathfrak{p}_j}\right) = 1 \end{aligned}$$

Thus the lemma is proved. □

Corollary 1. *In terms of lemma 4 we have for $Res > 1$*

$$\sum_{0\neq\omega\in G} \frac{\tau_3(\omega)K(1,1,\omega;\gamma)}{N(\omega)^3} =$$

$$= \sum_{\delta|\gamma_1} \bar{\mu}(\delta) \sum_{\substack{\omega \\ (\omega,\gamma)=\delta}} \frac{\tau_3(\omega)}{N(\omega)^3} K\left(1,1,\omega\left(\delta'\right)^2;\frac{\gamma}{\delta}\right)$$

(here $\delta\cdot\delta' \equiv 1 \left(mod\frac{\gamma}{\delta}\right)$

3. Main Theorem

In this section we construct an asymptotic formula for the sum

$$\sum_{N(\omega)\leq x} \tau_3(\omega) K(1,1,\omega).$$

Theorem. *Let $\gamma \in G, N(\gamma) > 1$. Then*

$$\sum\nolimits_{N(\omega)\leq x} \tau_3(\omega) K(1,1,\omega,p) =$$
$$= x\tilde{\mu}(\gamma) \sum\nolimits_{\delta|\gamma} \tau_3\left(\frac{\gamma}{\delta}\right) \prod\nolimits_{p|\delta}\left(1-\frac{1}{N(p)}\right)^2 Q_2\left(\log\frac{x}{N^2(\delta)}\right)$$

where $Q_2(u)$ – polynomial of degree 2 with leading coefficient π^3.

Let γ_1 - square-free part of γ and let $\sigma|\gamma_1$. Its clear that $\left(\delta,\frac{\gamma}{\delta}\right)=1$. For Re$s>1$ lets take a look at sum

$$F(s,\delta) = \sum_{S(c_1)}\sum_{S(c_2)}\prod_{j=1}^{3}\sum_{S}(c_{3j}) e\left(\mathrm{Re}\frac{\alpha_j\omega_j\delta_j\delta'}{\gamma/\delta}\right)(N(\omega_j)N(\delta_j))^{-s},$$

where

$$c_1 := \left\{\alpha_1,\alpha_2,\alpha_3 \in G_{\gamma/\delta} | \alpha_1\alpha_2\alpha_3 \equiv 1 \left(mod\frac{\gamma}{\delta}\right)\right\},$$
$$c_2 := \left\{\delta_1,\delta_2,\delta_3 \in G | \delta_1\delta_2\delta_3 = \delta\right\},$$
$$c_3 := \left\{\omega_j \in G | \left(\omega_j,\gamma\delta^{-1}\right)=1\right\}, j=1,2,3$$

Proof. We have

$$F(s,\delta) =$$
$$= \sum_{\substack{\omega \\ (\omega,\gamma)=\delta}} N(\omega)^{-s} \sum_{S(c_2)}\sum_{S(c_4)}\sum_{S(c_1)} e\left(\mathrm{Re}\frac{\left(\alpha_1\delta_1\omega_1+\alpha_2\delta_2\omega_2+\alpha_3\delta_3\omega_3\right)\delta'}{\gamma/\delta}\right), \quad , \tag{5}$$

where $c_4 := \{\omega_1\omega_2\omega_3 \in G | \delta_1\omega_1 \cdot \delta_2\omega_2 \cdot \delta_3\omega_3 = \omega\}$.

For the other hand

$$F(s,\delta) = \sum_{S(c_1)} \sum_{S(c_2)} \prod_{j=1}^{3} \left(\sum_{\omega_j \in G} e\left(\mathrm{Re}\frac{\alpha_j\omega_j\delta_j\delta'}{\omega/\delta}\right) N(\omega_j)^{-s} \sum_{\beta_j|(\omega_j,\gamma/\delta)} \tilde{\mu}(\beta_j)\right) = \sum_{\substack{\beta_j|\gamma/\delta \\ j=1,2,3}} \frac{\tilde{\mu}(\beta_1)\tilde{\mu}(\beta_2)\tilde{\mu}(\beta_3)}{N(\beta_1\beta_2\beta_3)^3} \sum_{S(c_1)} \sum_{S(c_3)} \prod_{j=1}^{3} Z\left(s; 0, \frac{\alpha_j\delta_j\beta_j\delta'}{\gamma/\delta}\right) \tag{6}$$

From (5) and (6) for $Res > 1$ we obtain

$$F(s) = \sum_{\omega} \frac{\tau_3(\omega) K(1,1,\omega;\gamma)}{N(\omega)^3} = \sum_{\delta|\gamma_1} \frac{\tilde{\mu}(\delta) F(s,\delta)}{N(\delta)^3}, \tag{7}$$

where

$$F(s,\delta) = \sum_{\substack{\beta_j|\gamma/\delta \\ j=1,2,3}} \frac{\tilde{\mu}(\beta_1)\tilde{\mu}(\beta_2)\tilde{\mu}(\beta_3)}{N(\beta_1\beta_2\beta_3)^3} \sum_{S(c_1)} \sum_{S(c_2)} \prod_{j=1}^{3} Z\left(s; 0, \frac{\alpha_j\beta_j\delta_j\delta'}{\gamma/\delta}\right) \tag{8}$$

From (7), (8) we conclude that $F(s)$ is analytically continuous to all complex s-plane except maybe $s = 1$, where $F(s)$ can has 3rd-order pole.

Now from (6), (7) we can see that we can by Perron's formulae, build asymptotic formulae for summa (4). But for this we need to research $F(s)$ in stripe $-\varepsilon \le Res \le 1+\varepsilon$, where ε is arbitrary positive number, $0 < \varepsilon < 1$.

By lemma 3 and corolary from lemma 4, we have

$$F(1+\varepsilon+it) \ll \sum_{\omega} \frac{\tau_3(\omega)|K(1,1,\omega;\gamma)|}{N(\omega)^{1+\varepsilon}} \ll \tau(\gamma) N(\gamma) \cdot \frac{1}{\varepsilon} \tag{9}$$

From functional equation for $Z(s;\delta_0,\delta_1)$ we conclude

$$F(s) = \sum_{\delta|\gamma_1} \tilde{\mu}(\delta) \sum_{\substack{\beta_j|\gamma/\delta \\ j=1,2,3}} \frac{\tilde{\mu}(\beta_1)\tilde{\mu}(\beta_2)\tilde{\mu}(\beta_3)}{N(\beta_1\beta_2\beta_3)^s} \cdot F_1(s),$$

where

$$F_1(s) = \sum_{S(c_2)} \sum_{S(c_1)} \frac{\pi^{6s-3}\Gamma(1-s)}{\Gamma^3(s)} \prod_{j=1}^{3} Z\left(1-s; \frac{\alpha_j\delta_j\beta_j(\delta')^2}{\gamma/\delta}, 0\right)$$

Therefore on line$Res = \frac{1}{2}$, using lemma 2, we have

$$F(s) \ll \sum_{\delta|\gamma_1} \sum_{\substack{\beta_j | \gamma/\delta \\ j=1,2,3}} \frac{|\tilde{\mu}(\beta_1)\tilde{\mu}(\beta_2)\tilde{\mu}(\beta_3)|}{N(\beta_1\beta_2\beta_3)^{\frac{1}{2}}} \sum_{S(c_1)} \prod_{j=1}^{3} \left| Z\left(\frac{1}{2} - it, \frac{\alpha_j\delta_j\beta_j(\delta')^2}{\gamma/\delta}\right)\right| \qquad (10)$$

By the inequality

$$\begin{aligned} &\left|Z\left(\tfrac{1}{2} - it, \tfrac{\alpha_j\delta_j\beta_j(\delta')^2}{\gamma/\delta}\right)\right| \le \\ &\le \left|Z\left(\tfrac{1}{2} - it, \tfrac{\alpha_j\delta_j\beta_j(\delta')^2}{\gamma/\delta}\right) - N\left(\tfrac{\alpha_j\delta_j\beta_j(\delta')^2}{\gamma/\delta}\right)^{\frac{1}{2}+it}\right| + \\ &+ \left(N\left(\tfrac{\alpha_j\delta_j\beta_j(\delta')^2}{\gamma/\delta}\right)\right)^{1/2} \end{aligned}$$

we obtain from (10)

$$F(s) \ll \sum_{\delta|\gamma_1} \sum_{\substack{\beta_j | \gamma/\delta \\ j=1,2,3}} \frac{1}{N(\beta_1\beta_2\beta_3)^{1/2}} \sum_{S(c_1)} (\Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4), \qquad (11)$$

where

$$\begin{cases} \Sigma_1 := \displaystyle\sum_{\substack{i_1,i_2,i_3 \\ i_1 \ne i_2, i_3 \\ i_2 \ne i_3}} \left| z\left(\frac{1}{2} - it, A_{i_1}\right) - N(A_{i_i})^{\frac{1}{2}-it}\right| \cdot \\ \qquad \cdot \left| Z(-it, A_{i_2}) - N(A_{i_2})^{-\frac{1}{2}+it}\right| N(A_{i_3})^{\frac{1}{2}}, \\ \Sigma_2 := \displaystyle\sum_{\substack{i_1 \ne i_2, i_3 \\ i_2 \ne i_3}} \left| Z\left(\frac{1}{2} - it, A_{i_1}\right) - N(A_{i_2})^{\frac{1}{2}-it}\right| N(A_{i_2} \cdot A_{i_3})^{-\frac{1}{2}}, \\ \Sigma_3 := \prod_{j=1}^{3} \left| Z\left(\tfrac{1}{2} - it, A_j\right) - N(A_j)^{\frac{1}{2}-it}\right|, \\ \Sigma_4 := \left(\frac{N(\gamma)^3}{N(A_1A_2A_3)}\right)^{1/2}, \\ A_j := \frac{\alpha_j\delta_j\beta_j(\delta')^2}{\gamma/\delta}, \quad j = 1,2,3 \end{cases} \qquad (12)$$

Using lemma 2, we infer

$$\Sigma_i \ll \begin{cases} N(\gamma)^{3/2}\tau^2(\gamma)\log^3 N(\gamma), & if \ \ |t| \le N(\gamma)^{3/2}, \\ |t|\,\tau^2(\gamma)\log^2 N(\gamma), & if \ \ |t| > N(\gamma)^{3/2} \end{cases} . \tag{13}$$
$$i = 1,2,3,4$$

Therefore using Phragmen-Lindelof principe, (9) and (13), we have for $1/2 \le \sigma \le 1+\varepsilon, |Ims| = |t| \ge 3$

$$F(\sigma+it) \ll_{\varepsilon} \begin{cases} N(\gamma)^{2-\sigma+\varepsilon}, & |t| \le N(\gamma)^{3/2}, \\ |t|^{2-2\sigma+\varepsilon} N(\gamma)^{2\sigma-1+\varepsilon}, & |t| > N(\gamma)^{3/2} \end{cases} \tag{14}$$

Bearing in mind that generating series for $\tau(\omega)K(1,1,\omega\gamma)$ defines by (7), we, using Perron's formulat, can write

$$\sum_{N(\omega)\le x} \tau_3(\omega)\, K(1,1,\omega;\gamma) =$$
$$= \frac{1}{2\pi i}\int\limits_{1+\varepsilon-iT}^{1+\varepsilon+iT} F(s)\frac{x^s}{s}ds + O\left(\frac{x^{1+\varepsilon}N(\gamma)\,\tau(\gamma)}{T\varepsilon^2}\right)$$

Move integrating contour on line $Res = 1/2$ and count that integrating over horizontal parts have estimate

$$\ll \max\left(|F(s)|\,\frac{x^{\sigma}}{T}\right).$$

So after simple calculations, we conclude

$$\sum_{N(\omega)\le x} \tau_3(\omega)\, K(1,1,\omega;\gamma) =$$
$$= \underset{s=1}{res}\left(F(s)\,\tfrac{x^s}{s}\right) + \frac{1}{2\pi i}\int\limits_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} F(s)\,\tfrac{x^s}{s}ds + O\left(\frac{x^{1+\varepsilon}N(\gamma)^{1/2}\tau(\gamma)}{T\varepsilon^2}\right) + \tag{15}$$
$$+O\left(\max\left(\frac{N(\gamma)^{\frac{3}{2}+\varepsilon}}{T}, T^{1+\varepsilon}N(\gamma)^{\varepsilon}\right)\right)$$

To estimate integral at the right of last equation we use inequality (11),

(12) and Lemma 6. We deduce

$$\int\limits_{Res=\frac{1}{2}}^{T} |\Sigma_1| \frac{x^{1/2}}{t} dt \ll$$

$$\ll \sum_{\substack{i_1 \neq i_2, i_3 \\ i_2 \neq i_3}} x^{1/2} N(A_{i_3})^{1/2} \left(\int\limits_{1}^{T} \left| Z\left(\frac{1}{2} - it, A_{i_1}, 0\right) - N(A_{i_1})^{\frac{1}{2}} \right|^2 \frac{dt}{t} \cdot \right.$$

$$\left. \cdot \int\limits_{1}^{T} \left| Z\left(\tfrac{1}{2} - it, A_{i_2}, 0\right) - N(A_{i_2})^{\frac{1}{2} - it} \right|^2 \frac{dt}{t} \right)^{1/2} \ll \tag{16}$$

$$\ll \sum_{\substack{i_1 \neq i_2, i_3 \\ i_2 \neq i_3}} x^{1/2} N(A_{i_3})^{1/2} T^{\varepsilon} N(\gamma)^{-\varepsilon} \log^2 T$$

Similar estimates we have for integrals Σ_2 and Σ_4.

Currently we have not appropriate estimate for 4th moment of the Hecke Zeta-function on the half line. Therefore use the following inequality

$$\int\limits_{1}^{T} |\Sigma_3| \frac{x^{\frac{1}{2}}}{t} dt \ll \int\limits_{1}^{T} |\Sigma_1| \frac{x^{\frac{1}{2}}}{t} dt \max_{1 \leq |t| \leq T} \left| Z\left(\frac{1}{2} - it, A_{i_3}, 0\right) - N(A_{i_2})^{\frac{1}{2} - it} \right|, \tag{17}$$

Since we have ability to estimate 4th moment Z-function Hecke on the half line.

So gathering (15)-(17) and summarizing by $\delta_j \beta_j \alpha_j$, we obtain

$$\sum_{N(\omega) \leq x} \tau_3(\omega) K(1, 1, \omega; \gamma) =$$

$$= \underset{s=1}{res} \left(F(s) \tfrac{x^s}{s} \right) + O\left(\tfrac{x^{1+\varepsilon} N(\gamma)^{1/2} \tau(\gamma)}{T \varepsilon^2} \right) +$$

$$+ \left(\max\left(\tfrac{N(\gamma)^{\frac{1}{2}+\varepsilon}}{T}, T^{1+\varepsilon} N(\gamma)^{\varepsilon} \right) \right) + O\left(x^{\frac{1}{2}} T^{\frac{1}{3}} N(\gamma) \tau(\gamma) T^{\varepsilon} \log^2 T \right)$$

Taking $T = x^{3/8} N(\gamma)^{-3/8}$, $\varepsilon = (\log x)^{-1}$, we get

$$\sum_{N(\omega) \leq x} \tau_3(\omega) K(1, 1, \omega; \gamma) =$$

$$= \underset{s=1}{res} \left(F(s) \tfrac{x^s}{s} \right) + O\left(x^{\frac{5}{8}} N(\gamma)^{\frac{7}{8}} \tau(\gamma) \log^2 x \right) \tag{18}$$

It left to find residue of function $F(s) \frac{x^s}{s}$ in point $s = 1$.

We have

$$\underset{s=1}{res}\left(F(s)\tfrac{x^s}{s}\right)=\sum_{\delta|\gamma}\mu(\delta)\sum_{\substack{\beta_j|\gamma/\delta\\ j=1,2,3}}\mu(\beta_1)\mu(\beta_2)\mu(\beta_3)\times$$

$$\times\sum_{S(c_1)}\sum_{S(c_2)}\underset{s=1}{res}\left\{\prod_{j=1}^{3}Z\left(s;0,\frac{\alpha_j\delta_j\beta_j(\delta')^2}{\gamma/\delta}\right)\cdot\frac{x^s}{N(\beta_1\beta_2)^s}\right\}.$$

Let us remark here that $Z\left(s;0,\frac{\delta_j\alpha_j\beta_j(\delta')^2}{\gamma/\delta}\right)$ has singular point $s=1$ only in case $\frac{\delta_j\alpha_j\beta_j(\delta')^2}{\gamma/\delta}\in\mathbb{Z}$.

But if $\gamma_2\neq 1$ then residue can be not equal zero only if $\beta_1=\gamma_2$ or $\beta_2=\gamma_2$, (but then $\mu(\gamma_2)=0$). This means non-zero residue possible if $\gamma_2=1,\gamma_1=\gamma$. Largest contribution in gives that collection of residue consist of such values $\beta_j=\gamma/\delta, j=1,2,3$.

We obtain a pole of 3rd order for the summand.

In that case

$$\sum_{\delta|\gamma}\tilde{\mu}(\delta)\tilde{\mu}^3\left(\frac{\gamma}{\delta}\right)\sum_{S(c_1)}\sum_{S(c_2)}Z^3(s;0,0)\cdot\frac{x^s}{s\left(N\left(\frac{\gamma}{\delta}\right)\right)^{2s}}$$

and its residue in point $s=1$ equals

$$x\cdot\pi^3\tilde{\mu}(\gamma)\sum_{\delta|\gamma}\tau_3\left(\frac{\gamma}{\delta}\right)\tilde{\phi}^2(\delta)N^{-2}(\delta)P_2\left(\log\frac{x}{N^2(\delta)}\right),$$

where $P_2(u)$ is a polynomial of power 2 with lead coefficient equals 1.

From that we conclude

$$\underset{s=1}{res}\left(F(s)\frac{x^s}{s}\right)=x\tilde{\mu}(\gamma)\sum_{\delta|\gamma}\tau_s\left(\frac{\gamma}{\delta}\right)\frac{\tilde{\phi}^2(\delta)}{N^2(\delta)}Q_2\left(\log\frac{x}{N^2(\delta)}\right),\qquad(19)$$

where $Q_2(u)$ – polynomial of power 2 with lead coefficient equals π^3.

Now from (18), (19) follows that theorem proved. □

We established asymptotic formulae is non-trivial for $N(\gamma)\ll_\varepsilon x^{\frac{3}{7}-\varepsilon}$, where ϵ is a arbitrary small positive.

The method of proof of the main theorem can be applied for building asymptotic formulae of sums $\tau_3(\omega)K(1,1,\omega;\gamma)$ in sectors

$$S=\left\{\omega\in G|0\le\phi_1<\arg\omega\le\phi_2\le\frac{\pi}{2},N(\omega)\le x\right\}.$$

For this requires to apply the Hecke Zeta-function $Z_m(s;\delta_0,\delta_1)$ with the character $\lambda_m(\omega) = \exp(4mi\arg w)$.

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