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ON PARA KENMOTSU MANIFOLD

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Abstract: A type of para Kenmotsu (briefly p -Kenmotsu) manifold in which $R(\xi, X).C = 0$ has been considered, where C is the conformal curvature tensor of the manifold and R is the curvature transformation. It has been shown that such a manifold is conformally flat and hence is an sp -Kenmotsu manifold.

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Key Words: para contact metric manifold, curvature tensor, Ricci tensor, tangent vector

1. Introduction

Sato [1] defined the notions of an almost para contact Riemannian manifold. After that, T. Adati and K. Matsumoto [2] defined and studied para-Sasakian and SP-Sasakian manifolds which are regarded as a special kind of an almost contact Riemannian manifolds. Before Sato, Kenmotsu [3] defined a class of almost contact Riemannian manifolds. In 1995, B. B. Sinha and K. L. Sai Prasad [4] have defined a class of almost para contact metric manifolds namely

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para Kenmotsu (briefly p-Kenmotsu) and special para Kenmotsu (briefly sp -Kenmotsu) manifolds. In a recent paper, the authors Satyanarayana and Sai Prasad [11] have proved that if in a p-Kenmotsu manifold $(M_n, g)(n > 3)$ the relation $R(X, Y).C = 0$ holds, where C is the conformal curvature tensor of the manifold and R is the Riemannian curvature and $R(X, Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X and Y, then the manifold is conformally flat and hence is an sp-Kenmotsu manifold. In this paper, we have generalized this result by taking the weaker hypothesis $R(\xi, X).C = 0$ instead of $R(X, Y).C = 0$ in a p-Kenmotsu manifold.

Let M_n be an n-dimensional differentiable manifold equipped with structure tensors (Φ, ξ, η) where Φ is a tensor of type (1, 1), ξ is a vector field, η is a 1-form such that

$$\eta(\xi) = 1 \tag{1.1}$$

$$\Phi^2(X) = X - \eta(X)\xi;\ \overline{X} = \Phi X \tag{1.2}$$

Then M_n is called an almost para contact manifold.

Let 'g' be the Riemannian metric satisfying such that, for all vector fields X and Y on M,

$$g(X, \xi) = \eta(X) \tag{1.3}$$

$$\Phi\xi = 0,\ \eta(\Phi X) = 0,\ \text{rank}\ \Phi = n - 1 \tag{1.4}$$

$$g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{1.5}$$

Then the manifold Mn[1] is said to admit an almost para contact Riemannian structure (Φ, ξ, η, g).

A manifold of dimension 'n' with Riemannian metric 'g' admitting a tensor field 'Φ' of type (1, 1), a vector field 'ξ' and a 1-form 'η' satisfying (1.1), (1.3) along with

$$(\nabla_X \eta)Y - (\nabla_Y \eta)X = 0 \tag{1.6}$$

$$(\nabla_X \nabla_Y \eta)Z = [-g(X, Z) + \eta(X)\eta(Z)]\eta(Y) + [-g(X, Y) + \eta(X)\eta(Y)]\eta(Z) \tag{1.7}$$

$$\nabla_X \xi = \Phi^2 X = X - \eta(X)\xi \tag{1.8}$$

is called a para Kenmotsu manifold or briefly p-Kenmotsu manifold [4]. This paper deals with a type of p-Kenmotsu manifold in which

$$R(\xi, X).C = 0 \tag{1.9}$$

where C is the conformal curvature tensor of the manifold and R is the Riemannian curvature.

Let (M_n, g) be an n-dimensional Riemannian manifold admitting a tensor field 'Φ' of type (1, 1), a vector field 'ξ' and a 1-form 'η' satisfying

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y) \tag{1.10}$$

$$g(X, \xi) = \eta(X) \text{ and } (\nabla_X \eta)Y = \varphi(\overline{X}, Y), \text{ where } \varphi \text{ is an associate of } \Phi \tag{1.11}$$

is called a special p-Kenmotsu manifold or briefly sp-Kenmotsu manifold [4].

It is known that [4] in a p-Kenmotsu manifold the following relations hold:

$$S(X, \xi) = -(n-1)\eta(X) \tag{1.12}$$

$$g[R(X, Y)Z, \xi] = \eta[R(X, Y, Z)] = g(X, Z)\eta(Y) - g(Y, Z)\eta(X) \tag{1.13}$$

$$R(X, \xi) = -1 \tag{1.14}$$

$$R(X, \xi, \xi) = -X + \eta(X)\xi \tag{1.15}$$

$$R(X, \xi, X) = \xi \tag{1.16}$$

$$R(\xi, X, \xi) = X \tag{1.17}$$

$$R(X, Y, \xi) = \eta(X)Y - \eta(Y)X; \text{ when } X \text{ is orthogonal to } \xi \tag{1.18}$$

where S is the Ricci tensor and R is the Riemannian curvature.

Moreover, it is also known that if a p-Kenmotsu manifold is projectively flat then it is an Einstein manifold and the scalar curvature has a negative constant value $-n(n-1)$. Especially, if a p-Kenmotsu manifold is of constant curvature, the scalar curvature has a negative constant value $-n(n-1)$ [4]. In this case,

$$S(Y, Z) = -(n-1)g(Y, Z) \tag{1.19}$$

and also

$$'R(X, Y, Z, P) = \frac{1}{(n-1)}[S(Y, Z)g(X, P) - S(X, Z)g(Y, P)] \tag{1.20}$$

The above results will be used in the next section.

2. p-Kenmotsu Manifold Satisfying $R(\xi, X).C = 0$

The conformal curvature tensor C is given by

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \quad (2.1)$$

where S is the Ricci tensor, r is the scalar curvature and Q is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S [5] i.e.,

$$g(QX,Y) = S(X,Y). \quad (2.2)$$

Then

$$\begin{aligned}\eta(C(X,Y)Z) &= g(C(X,Y)Z,\xi) \\ &= \frac{1}{n-2}\Big[\Big(\frac{r}{n-1}+1\Big)(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)) \\ &\qquad - (S(Y,Z)\eta(X) - S(X,Z)\eta(Y))\Big].\end{aligned} \quad (2.3)$$

Putting $Z = \xi$ in (2.3), we get

$$\eta(C(X,Y)\xi) = 0. \quad (2.4)$$

Again putting $X = \xi$ in (2.3), we get

$$\eta(C(\xi,Y)Z) = \frac{1}{n-2}\Big[\Big(\frac{r}{n-1}+1\Big)g(Y,Z) - S(Y,Z) - \Big(\frac{r}{n-1}+n\Big)\eta(Y)\eta(Z)\Big]. \quad (2.5)$$

Now

$$\begin{aligned}(R(\xi,X)\cdot C)(U,V)W &= R(\xi,X)C(U,V)W - C(R(\xi,X)U,V)W \\ &\quad - C(U,R(\xi,X)V)W - C(U,V)R(\xi,X)W.\end{aligned}$$

Using (1.9), we get

$$\begin{aligned}R(\xi,X)C(U,V)W - C(R(\xi,X)U,V)W - C(U,R(\xi,X)V)W \\ - C(U,V)R(\xi,X)W = 0.\end{aligned} \quad (2.6)$$

Therefore

$$\begin{aligned}g(R(\xi,X)C(U,V)W,\xi) - g(C(R(\xi,X)U,V)W,\xi) \\ - g(C(U,R(\xi,X)V)W,\xi) - g(C(U,V)R(\xi,X)W,\xi) = 0.\end{aligned} \quad (2.7)$$

From this, it follows that

$$\begin{aligned}&{}'C(U,V,W,X)-\eta(X)\eta(C(U,V)W)+\eta(U)\eta(C(X,V)W)+\eta(V)\eta(C(U,X)W)\\&\quad+\eta(W)\eta(C(U,V)X)-g(X,U)\eta(C(\xi,V)W)-g(X,V)\eta(C(U,\xi)W)\\&\quad-g(X,W)\eta(C(U,V)\xi)=0\end{aligned} \tag{2.8}$$

where

$${}'C(U,V,W,X)=g(C(U,V)W,X).$$

Putting $X=U$ in (2.8), we get

$$\begin{aligned}&{}'C(U,V,W,U)+\eta(V)\eta(C(U,U)W)+\eta(W)\eta(C(U,V)U)-g(U,U)\eta(C(\xi,V)W)\\&\qquad-g(U,V)\eta(C(U,\xi)W)-g(U,W)\eta(C(U,V)\xi)=0.\end{aligned} \tag{2.9}$$

Let $\{e_i\}, i=1,2,\cdots n$ be an orthogonal basis of the tangent space at any point. Then the sum $1\leq i\leq n$ of the relation (2.9) for $U=e_i$ gives

$$\eta(C(\xi,V)W)=0. \tag{2.10}$$

By using (2.4), we have from (2.8)

$$\begin{aligned}&{}'C(U,V,W,X)-\eta(X)\eta(C(U,V)W)+\eta(U)\eta(C(X,V)W)+\eta(V)\eta(C(U,X)W)\\&\quad+\eta(W)\eta(C(U,V)X)-g(X,U)\eta(C(\xi,V)W)-g(X,V)\eta(C(U,\xi)W)=0.\end{aligned} \tag{2.11}$$

In virtue of (2.5) and (2.10), we have

$$S(Y,Z)=\Big(\frac{r}{n-1}+1\Big)g(Y,Z)-\Big(\frac{r}{n-1}+n\Big)\eta(Y)\eta(Z). \tag{2.12}$$

Using (2.3), (2.4), (2.10) and (2.12), the relation (2.11) reduces to

$${}'C(U,V,W,X)=0. \tag{2.13}$$

From (2.13) it follows that

$$C(U,V)W=0. \tag{2.14}$$

which proves that the p-Kenmotsu manifold is conformally flat and hence it is of constant curvature.

Then from (1.19) and (1.20) we have

$${}'R(X,Y,Z,P)=g(X,Z)g(Y,P)-g(Y,Z)g(X,P) \tag{2.15}$$

Also from (1.19) and (1.5) we have

$$S(\Phi X, \Phi Y) = -(n-1)[g(X,Y) - \eta(X)\eta(Y)] \quad (2.16)$$

On contraction (2.16) with covariant tensor $\varphi(X,Y) = g(\overline{X}, Y)$ and hence we find

$$\varphi(\overline{X}, Y) = g(X,Y) - \eta(X)\eta(Y)$$

that is, the manifold is an *sp*-Kemotsu one. Thus, we can state the following theorem.

Theorem: If a *p*-Kenmotsu manifold (M_n, g) is conformally flat, the manifold is an *sp*-Kenmotsu one.

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