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ON THE DIOPHANTINE EQUATION $131^x + 133^y = z^2$

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Abstract: In this paper, we show that the Diophantine equation $131^{\mathsf{x}}+133^{\mathsf{y}} = z^2$ has no non-negative integer solution where x, y and z are non-negative integers.

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1. Introduction

In 2007, Acu [1] found all solutions of the Diophantine equation $2^{\mathsf{x}} + 5^{\mathsf{y}} = z^2$ where x, y and z are non-negative integers.

In 2011, Suvarnamani, Singta and Chotchaisthit [12] showed that the two Diophantine equations $4^{\mathsf{x}} + 7^{\mathsf{y}} = z^2$ and $4^{\mathsf{x}} + 11^{\mathsf{y}} = z^2$ have no non-negative integer solution where x, y and z are non-negative integers.

In 2012, Chotchaisthit [3] found all non-negative integer solutions for the Diophantine equation of type $4^{\mathsf{x}} + p^{\mathsf{y}} = z^2$ with p is a positive prime number where x, y and z are non-negative integers.

In 2013, Chotchaisthit [2] found all solutions of the Diophantine equation $2^{\mathsf{x}} + 11^{\mathsf{y}} = z^2$ where x, y and z are non-negative integers.

Recently, we [5, 6, 7, 8, 9, 10, 11] solved the Diophantine equations $32^{\mathsf{x}} + 49^{\mathsf{y}} = z^2$, $3^{\mathsf{x}}+5^{\mathsf{y}} = z^2$, $8^{\mathsf{x}}+19^{\mathsf{y}} = z^2$, $31^{\mathsf{x}}+32^{\mathsf{y}} = z^2$, $7^{\mathsf{x}}+8^{\mathsf{y}} = z^2$, $2^{\mathsf{x}}+3^{\mathsf{y}} = z^2$, and $23^{\mathsf{x}} + 32^{\mathsf{y}} = z^2$, where x, y and z are non-negative integers.

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In this paper, we show that the Diophantine equation $131^x + 133^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers.

2. Preliminaries

Proposition 2.1. [4] $(3, 2, 2, 3)$ *is a unique solution* (a, b, x, y) *for the Diophantine equation* $a^x - b^y = 1$ *where* a, b, x *and* y *are integers with* $\min\{a, b, x, y\} > 1$.

Lemma 2.2. *The Diophantine equation* $131^x + 1 = z^2$ *has no non-negative integer solution where* x *and* z *are non-negative integers.*

Proof. Suppose that $131^x + 1 = z^2$ for some non-negative integers x and z. If $x = 0$, then $z^2 = 2$ which is impossible. This implies that $x \geq 1$. Then $z^2 = 131^x + 1 \geq 131^1 + 1 = 132$. Thus, $z \geq 12$. Now, we consider on the equation $z^2 - 131^x = 1$. By Proposition 2.1, we have $x = 1$. We obtain that $z^2 = 132$. This is a contradiction. Hence, the equation $131^x + 1 = z^2$ has no non-negative integer solution. □

Lemma 2.3. *The Diophantine equation* $1 + 133^y = z^2$ *has no non-negative integer solution where* y *and* z *are non-negative integers.*

Proof. Suppose that $1 + 133^y = z^2$ for some non-negative integers y and z. If $y = 0$, then $z^2 = 2$ which is impossible. This implies that $y \geq 1$. Then $z^2 = 1 + 133^y \geq 1 + 133^1 = 134$. Thus, $z \geq 12$. Now, we consider on the equation $z^2 - 133^y = 1$. By Proposition 2.1, we have $y = 1$. We obtain that $z^2 = 134$. This is a contradiction. Hence, the equation $1 + 133^y = z^2$ has no non-negative integer solution. □

3. Results

Theorem 3.1. *The Diophantine equation* $131^x + 133^y = z^2$ *has no non-negative integer solution where* x, y *and* z *are non-negative integers.*

Proof. Suppose that $131^x + 133^y = z^2$ for some non-negative integers x, y and z. By Lemma 2.2 and 2.3, we have $x \geq 1$ and $y \geq 1$. Note that z is even. This implies that $z^2 \equiv 0, 4 \pmod 6$. However, $133^y \equiv 1 \pmod 6$. Thus, $131^x \equiv 5, 3 \pmod 6$. Note that $131^x \equiv 5^x \pmod 6$ and $5^x \equiv 1, 5 \pmod 6$. We obtain that $131^x \equiv 5^x \equiv 5 \pmod 6$. It follows that x is odd. Hence

,$131^{\mathsf{x}} \equiv 3, 5, 6 \pmod 7$. Note that $133^{\mathsf{x}} \equiv 0 \pmod 7$. We obtain that $z^2 \equiv 3, 5, 6 \pmod 7$. This is impossible since $z^2 \equiv 0, 1, 2, 4 \pmod 7$. □

Corollary 3.2. *The Diophantine equation* $131^{\mathsf{x}} + 133^{\mathsf{y}} = w^4$ *has no non-negative integer solution where* x, y *and* w *are non-negative integers.*

Proof. Suppose that $131^{\mathsf{x}} + 133^{\mathsf{y}} = w^4$ for some non-negative integers x, y and w. Let $z = w^2$. Then $131^{\mathsf{x}} + 133^{\mathsf{y}} = z^2$. By Theorem 3.1, the equation $131^{\mathsf{x}} + 133^{\mathsf{y}} = z^2$ has no non-negative integer solution. We obtain that the equation $131^{\mathsf{x}} + 133^{\mathsf{y}} = w^4$ has no non-negative integer solution. This is a contradiction. □

4. Open Problem

Let p be a positive odd prime number. We ask for the set of all solutions (x, y, z) for the Diophantine equation $p^{\mathsf{x}} + (p + 2)^{\mathsf{y}} = z^2$ where x, y and z are non-negative integers.

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