

**DISCONTINUOUS OBLIQUE DERIVATIVE PROBLEM FOR
NONLINEAR ELLIPTIC SYSTEM OF SECOND ORDER
EQUATIONS IN MULTIPLY CONNECTED DOMAINS**

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Abstract: In this article, we discuss that the discontinuous oblique derivative boundary value problem for nonlinear uniformly elliptic system of second order equations in multiply connected domains. We first propose the discontinuous oblique derivative problem and its new modified well-posedness. Next we give a priori estimates of solutions of the modified discontinuous boundary value problem for corresponding elliptic system of first order complex equations and verify its solvability by the above estimates of solutions and the Leray-Schauder theorem. Finally the solvability results of the original discontinuous oblique derivative problem can be derived. Here we mention the discontinuous boundary value problems possess many applications in mechanics and physics etc.

AMS Subject Classification: 35J57, 35J60, 35B45

Received: October 26, 2013

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1. Formulation of Oblique Derivative Problems for Nonlinear Elliptic System of Second Order Equations

In [1-5], the authors discussed some continuous boundary value problems for elliptic equations and their applications, and in [4,6], some discontinuous boundary value problems for spacial elliptic complex equations were discussed. In this article we shall consider the discontinuous oblique derivative problem for general nonlinear elliptic system of second order equations in multiply connected domains. We first introduce the nonlinear uniformly elliptic systems of second order equations

$$\begin{aligned}
 u_{jz\bar{z}} &= F_j(z, u_1, u_2, u_{1z}, u_{2z}, u_{1zz}, u_{2zz}), \quad F_j = \operatorname{Re} [Q_{j1}u_{1zz} + Q_{j2}u_{2zz} \\
 &+ A_{j1}u_{1z} + A_{j2}u_{2z} + A_{j3}u_1 + A_{j4}u_2 + A_{j5}], \quad Q_{jl} = Q_{jl}(z, u_1, u_2, u_{1z}, u_{2z}, \\
 &u_{1zz}, u_{2zz}), \quad l = 1, 2, \quad A_{jl} = A_{jl}(z, u_1, u_2, u_{1z}, u_{2z}), \quad l = 1, \dots, 5, \quad j = 1, 2,
 \end{aligned}
 \tag{1.1}$$

in an $N + 1$ -connected domain G . Denote by $\Gamma = \cup_{j=0}^N \Gamma_j$ the boundary contours of the domain G and let $\Gamma \in C_\mu^2$ ($0 < \mu < 1$). Without loss of generality, we assume that G is a circular domain in the unit disc $|z| < 1$, bounded by the $(N+1)$ -circles $\Gamma_k : |z - z_k| = r_k, k = 0, 1, \dots, N$ and $\Gamma_0 = \Gamma_{N+1} : |z| = 1, z = 0 \in G$. In this article, the notations are the same as in References [3-12]. Suppose that the system (1.1) satisfies

Condition C. 1) $Q_{jl}(z, u_1, u_2, u_{1z}u_{2z}, U, V)$ ($l=1, 2$) and $A_{jl}(z, u_1, u_2, u_{1z}, u_{2z})$ ($l = 1, \dots, 5, j = 1, 2$) are continuous in $u_1, u_2 \in \mathbb{R}$ (the real axis) and $u_{1z}, u_{2z} \in \mathbb{C}$ (the complex plane) for almost every $z \in G, U, V \in \mathbb{C}$. In addition, let

$$\begin{aligned}
 Q_{jl}(z, u_1, u_2, u_{1z}, u_{2z}, U, V) &= 0, \quad j, l = 1, 2, \\
 A_{jl}(z, u_1, u_2, u_{1z}, u_{2z}) &= 0, \quad j = 1, 2, l = 1, \dots, 5,
 \end{aligned}
 \quad z \in \mathbb{C} \setminus G.
 \tag{1.2}$$

2) The above functions are measurable in $z \in G$ for all continuously differentiable functions $u_1(z), u_2(z)$ in $G^* = \overline{G} \setminus Z$ and any measurable function $U(z), V(z) \in G^*$, and satisfy

$$L_p[A_{jl}, \overline{G}] \leq k_{jl} < k_0, \quad j = 1, 2, l = 1, \dots, 5,
 \tag{1.3}$$

in which $G^* = \overline{G} \setminus Z, Z = \{t_1, \dots, t_m\}, t_j$ ($j = 1, \dots, m$) are distinct points on Γ , and p_0, p ($2 < p_0 < p$), k_{jl} ($j = 1, 2, l = 1, \dots, 5$), k_0 are all positive constants.

3) The system (1.1) satisfies the uniform ellipticity condition, i.e.

$$\begin{aligned}
 |F_j(z, u_1, u_2, u_{1z}u_{2z}, U_1, V_1) - F_j(z, u_1, u_2, u_{1z}u_{2z}, U_2, V_2)| \\
 \leq q_{j0}|U_1 - U_2| + q'_{j0}|V_1 - V_2|,
 \end{aligned}
 \tag{1.4}$$

for arbitrary continuously differentiable functions $u_1(z), u_2(z)$ in G^* , almost every point $z \in G$ and $U_1, U_2 \in \mathbb{C}$, where

$$q_{j0}, q'_{j0} (q_{j0} + q'_{j0} < 1, j = 1, 2)$$

are non-negative constants.

Problem P. *The discontinuous oblique derivative boundary value problems for the system (1.1) may be formulated as follows: To find a system of continuously differentiable solutions $(u_1(z), u_2(z))$ in G^* satisfying the boundary conditions*

$$\frac{\partial u_j}{\partial v_j} + 2\sigma_{j1}(t)u_1(t) + 2\sigma_{j2}(t)u_2(t) = 2\tau_{j3}(t), t \in \Gamma^* = \Gamma \setminus Z, j = 1, 2, \tag{1.5}$$

in which $v_j (j = 1, 2)$ are the vectors on Γ . Since

$$\frac{\partial u_j}{\partial v_j} = \frac{\partial u_j}{\partial x} \cos \alpha_{j1} + \frac{\partial u_j}{\partial y} \cos \alpha_{j2} = 2\text{Re}[\overline{\lambda_j(t)}u_{jt}], \overline{\lambda_j(t)} = \cos \alpha_{j1} + i \cos \alpha_{j2}, \tag{1.6}$$

where $\cos \alpha_{j1} = \cos(v_j, x), \cos \alpha_{j2} = \cos(v_j, y)$, hence (1.5) can be written in the form

$$\text{Re}[\overline{\lambda_j(t)}u_{jt}] + \sigma_{j1}(t)u_1(t) + \sigma_{j2}(t)u_2(t) = \tau_{j3}(t), t \in \Gamma^*, j = 1, 2. \tag{1.7}$$

It is clear that $|\lambda_j(t)| = 1 (j = 1, 2)$. In (1.5) and (1.7), $Z = \{t_1, t_2, \dots, t_m\}$ is the set of first kind discontinuous points of $\lambda_j(z) (j = 1, 2)$ on Γ , $\hat{\Gamma}_l$ is an arc from the point t_{l-1} to t_l on $\hat{\Gamma}$, $\hat{\Gamma}_l (l = 1, 2, \dots, m)$ does not include end points. We can assume that $t_l \in \Gamma_0 (l = 1, \dots, m_0), t_l \in \Gamma_1 (l = m_0 + 1, \dots, m_1), \dots, t_l \in \Gamma_N (l = m_{N-1} + 1, \dots, m)$ are all discontinuous points of $\lambda_j(z)$ on Γ . Denote by $\lambda_j(t_l - 0)$ and $\lambda_j(t_l + 0)$ the left limit and right limit of $\lambda_j(z)$ as $z \rightarrow t_l (l \in \Gamma, l = 1, 2, \dots, m, j = 1, 2)$, and

$$e^{i\phi_{jl}} = \frac{\lambda_j(t_l - 0)}{\lambda_j(t_l + 0)}, \gamma_{jl} = \frac{1}{\pi i} \ln \left[\frac{\lambda_j(t_l - 0)}{\lambda_j(t_l + 0)} \right] = \frac{\phi_{jl}}{\pi} - K_{jl}, \tag{1.8}$$

$$K_{jl} = \left[\frac{\phi_{jl}}{\pi} \right] + J_{jl}, J_{jl} = 0 \text{ or } 1, j = 1, 2, l = 1, \dots, m,$$

in which $0 \leq \gamma_{jl} < 1$ when $J_{jl} = 0$, and $-1 < \gamma_{jl} < 0$ when $J_{jl} = 1, j = 1, 2, l = 1, \dots, m$. There is no harm in assuming that the partial indexes $\tilde{K}_{jl} = \sum_{j=m_{l-1}+1}^{m_l} K_{jl}$ are denoted the partial indexes of $\lambda_j(z)$ on $\Gamma_l (l = 0, 1, \dots, N)$, and the partial indexes \tilde{K}_{jl} on $\Gamma_l (l = 1, \dots, N_0 (\leq N))$ are not integers and $\tilde{K}_{jl} = \sum_{j=m_{l-1}+1}^{m_l} K_{jl}$ of $\lambda_j(z)$ on $\Gamma_l (l = N_0 + 1, \dots, N)$ are integers. Set

$$K_j = \frac{1}{2\pi} \Delta_\Gamma \arg \lambda_j(z) = \sum_{l=1}^m \frac{K_{jl}}{2}, j = 1, 2, \tag{1.9}$$

and $K = (K_1, K_2)$ is called the index of Problem P. Moreover, $\lambda_j(z), r_j(z)$ ($j = 1, 2$) satisfies the conditions

$$\begin{aligned}
 C_\alpha[\lambda_j(z), \hat{\Gamma}_l] \leq s_0, \quad C_\alpha[|z - t_l|^{\beta_{jl}} |z - t_{l-1}|^{\beta_{jl-1}} \sigma_{jn}, \hat{\Gamma}_l] \leq s_1, \quad j, n = 1, 2, \\
 C_\alpha[|z - t_l|^{\beta_{jl}} |z - t_{l-1}|^{\beta_{jl-1}} \tau_{j3}(z), \hat{\Gamma}_l] \leq s_0, \quad j = 1, 2, \quad l = 1, \dots, m,
 \end{aligned}
 \tag{1.10}$$

in which s_0, s_1 ($s_1 \leq s_0$), α ($1/2 < \alpha < 1$) are non-negative constants, $\beta_{jl} + |\gamma_{jl}| < 1, j = 1, 2, l = 1, \dots, m$; we require that the solution $[w_1(z), w_2(z)]$ possesses the property

$$\begin{aligned}
 R(z)w_j(z) = C_\delta(\overline{G}), \quad R(z) = \prod_{l=1}^m |z - t_l|^{n_l/\tau^2}, \quad \eta_l = \max(\eta_{1l}, \eta_{2l}), \\
 \eta_{jl} = \begin{cases} \beta_{jl} + \tau, & \text{for } \gamma_{jl} \geq 0, \text{ and } \gamma_{jl} < 0, \beta_{jl} \geq |\gamma_{jl}|, \\ |\gamma_{jl}| + \tau, & \text{for } \gamma_{jl} < 0, \beta_{jl} < |\gamma_{jl}|, \end{cases} \quad j = 1, 2, \quad l = 1, \dots, m,
 \end{aligned}
 \tag{1.11}$$

in the neighborhood ($\subset G$) of t_l ($l = 1, \dots, m$), where δ, τ ($\leq \min(\alpha, 1 - 2/p_0)$) are small positive constants.

We mention that the first boundary value problem, second boundary value problem and third boundary value problem, i.e. regular oblique derivative problem are the special cases of Problem P, because their boundary conditions are the continuous boundary conditions, and their indexes are equal to $K_j = N - 1$ ($j = 1, 2$). Now $2K_j$ ($j = 1, 2$) can be any positive or negative integers, hence Problem P is a vary general discontinuous boundary value problem.

In the next section on we first introduce the modified boundary value problems (Problem Q) for elliptic system of first order complex equations, which corresponds to Problem P for the system (1.1). Moreover we establish then the integral expression and a priori estimates of solutions for Problem Q. By the estimates of solutions and the Leray-Schauder theorem, we can prove the existence of solutions for Problem Q, and so derive the results of the solvability for Problem P for the system (1.1) with some conditions as follows.

Theorem 1.1. (The Main Theorem) *Suppose that the second order non-linear system (1.1) satisfy Condition C. If the constants q'_{10}, k_{1l} ($l = 2, 3, 4$), k_{24} in (1.3), (1.4) and s_1 in (1.10) are all sufficiently small, then Problem P for (1.1) possesses the following results on solvability:*

(1) *When the indexes $K_j = \frac{1}{2\pi} \Delta_\Gamma \arg \lambda_j(t) \geq 0$ ($j = 1, 2$), Problem P for (1.1) has $4N$ solvability conditions, and the solution depends on $2(K_1 + K_2 + 2)$ arbitrarily real constants.*

(2) *When $K_1 < 0, K_2 < 0$, Problem P has $4N - 2K_1 - 2K_2 - 2$ solvability conditions, and the solution depends on two arbitrarily real constants.*

Moreover, we can write down the solvability conditions of Problem P for all other cases.

For Problem P in the simply connected domain, we considered its solvability, the result obtained is a special case of Theorem 1.1, which can be seen from Chapter VI, [3].

2. Formulation of Modified Problem for Nonlinear Elliptic System of First Order Equations

Now we introduce the new well posedness of discontinuous oblique derivative problem for the elliptic system (1.1), namely

Problem Q. To find a system of continuous solutions $(u_1(z), u_2(z), w_1(z), w_2(z))$ in G for the nonlinear system of first order equations

$$\begin{aligned} w_{j\bar{z}} &= f_j(z, u_1, u_2, w_1, w_2, w_{1z}, w_{2z}), \quad f_j = \operatorname{Re} [Q_{j1}w_{1z} + Q_{j2}w_{2z} \\ &+ A_{j1}w_1 + A_{j2}w_2 + A_{j3}u_1 + A_{j4}u_2 + A_{j5}], \quad Q_{jl} = Q_{jl}(z, u_1, u_2, w_1, w_2, \\ &w_{1z}, w_{2z}), \quad j, l = 1, 2, \quad A_{jl} = A_{jl}(z, u_1, u_2, w_1, w_2), \quad j = 1, 2, \quad l = 1, \dots, 5, \end{aligned} \quad (2.1)$$

with the boundary conditions

$$\begin{aligned} \operatorname{Re}[\overline{\lambda_j(t)}w_j(t)] + \sigma_{j1}(t)u_1(t) + \sigma_{j2}(t)u_2(t) &= \tau_{j3}(t) + h_j(t)\overline{\lambda_j(z)}X_j(z), \quad t \in \Gamma^*, \\ j &= 1, 2. \end{aligned} \quad (2.2)$$

in which

$$h_j(z) = \begin{cases} \left. \begin{aligned} &h_{jl} \quad z \in \Gamma_l, \quad l = 1, \dots, N, \\ &0, \quad z \in \Gamma_0, \quad z \in \Gamma_0, \end{aligned} \right\} \text{if } K_j \geq 0, \\ \left. \begin{aligned} &h_{jl}, \quad z \in \Gamma_l, \quad l = 1, \dots, N \\ &[1 + (-1)^{2K_j}]h_0 + \operatorname{Re} \sum_{m=1}^{\lfloor K_j + 1/2 \rfloor - 1} (h_{jm}^+ + ih_{jm}^-)z^m, \quad z \in \Gamma_0 \end{aligned} \right\} \text{if } K_j < 0, \\ j &= 1, 2, \end{cases}$$

in which h_{jl} ($l = [1 - (-1)^{2K_j}]/2, \dots, N$) and h_{jm}^\pm ($m = 1, \dots, [|K_j| + 1/2] - 1, j = 1, 2$) are unknown real constants to be determined appropriately, and

$$\begin{aligned}
 Y_j(z) = & \prod_{l=1}^{m_0} (z - t_l)^{\gamma_{jl}} \prod_{s=1}^N (z - z_s)^{-[\tilde{K}_{js}]} \prod_{l=m_0+1}^{m_1} \left(\frac{z - t_l}{z - z_1} \right)^{\gamma_{jl}} \cdots \prod_{l=m_{N_0-1}+1}^{m_{N_0}} \left(\frac{z - t_l}{z - z_{N_0}} \right)^{\gamma_{jl}} \\
 & \times \prod_{l=m_{N_0}+1}^{m_{N_0+1}} \left(\frac{z - t_l}{z - z_{N_0+1}} \right)^{\gamma_{jl}} \left(\frac{z - t'_{N_0+1}}{z - z_{N_0+1}} \right) \cdots \prod_{l=m_{N-1}+1}^m \left(\frac{z - t_l}{z - z_N} \right)^{\gamma_{jl}} \left(\frac{z - t'_N}{z - z_N} \right), \\
 & j = 1, 2,
 \end{aligned}$$

where $\tilde{K}_{js} = \sum_{l=m_{s-1}+1}^{m_s} K_{jl}$ ($j = 1, 2$) is denoted the partial indexes on Γ_s ($s = 1, \dots, N$); there is in no harm assuming that the partial indexes K_{js} of $\lambda_j(z)$ on Γ_s ($s = 0, \dots, N_0$ ($\leq N$), $j = 1, 2$) are integers, and the partial indexes K_{js} of $\lambda_j(z)$ on Γ_s ($s = N_0 + 1, \dots, N, j = 1, 2$) are no integers, and t'_l ($\in \Gamma_l, l = N_0 + 1, \dots, N$) are fixed points, which are not the discontinuous points at Z ; we must give the attention that the boundary circles Γ_l ($l = 0, 1, \dots, N$) of the domain G are moved round the positive direct. Similarly to (1.7)-(1.12), Chapter V, [4], we see that

$$\frac{\lambda_j(t_l - 0)}{\lambda_j(t_l + 0)} \overline{\left[\frac{Y_j(t_l - 0)}{Y_j(t_l + 0)} \right]} = \frac{\lambda_j(t_l - 0)}{\lambda_j(t_l + 0)} e^{-i\pi\gamma_{jl}} = \pm 1, \quad j = 1, 2, \quad l = 1, \dots, m,$$

it only needs to charge the symbol on some arcs on Γ , then $\lambda_j(z)\overline{Y_j(z)}/|Y_j(z)|$ ($j = 1, 2$) on Γ are continuous. In this case, its index

$$\kappa_j = \frac{1}{2\pi} \Delta_\Gamma [\lambda_j(z)\overline{Y_j(z)}] = K_j - \frac{N - N_0}{2}, \quad j = 1, 2$$

are integers; and

$$\begin{aligned}
 X_j(z) = & \begin{cases} iz^{[\kappa_j]} e^{iS_j(z)} Y_j(z), & z \in \Gamma_0, \\ ie^{i\theta_{jl}} e^{iS_j(z)} Y_j(z), & z \in \Gamma_l, l = 1, \dots, N, \end{cases} \quad \text{Im}[\overline{\lambda_j(z)} X_j(z)] = 0, \quad z \in \Gamma, \\
 \text{Re} S_j(z) = & \begin{cases} \arg \lambda_j(z) - [\kappa_j] \arg z - \arg Y_j(z), & z \in \Gamma_0, \\ \arg \lambda_j(z) - \arg Y_j(z) - \theta_{jl}, & z \in \Gamma_l, l = 1, \dots, N, \end{cases} \quad \text{Im}[S_j(1)] = 0, \\
 & j = 1, 2,
 \end{aligned} \tag{2.3}$$

in which $S_j(z)$ ($j = 1, 2$) are solutions of the modified Dirichlet problem with the above boundary conditions for analytic functions.

Moreover we suppose that the solution satisfies the point conditions:

$$\operatorname{Im}[\overline{\lambda_j(a_l)}w_j(a_l)] = b_{jl}, \quad l \in J_j, \quad J_j = \{1, \dots, 2K_j + 1\}, \quad K_j \geq 0, \quad j = 1, 2, \quad (2.4)$$

where $a_l (\in \Gamma_0, l \in J_j, K_j \geq 0, j = 1, 2)$ are distinct points; and $b_{jl} (l \in J_j, j = 1, 2)$ are all real constants satisfying the conditions

$$|b_{jl}| \leq s_0 < \infty, \quad l \in J_j, \quad j = 1, 2. \quad (2.5)$$

Besides, we require that the solution $(u_1(z), u_2(z), w_1(z), w_2(z))$ satisfies $R'(z)u_j(z) \in C_\alpha^1(\overline{G})$, $RSw_j(z) \in W_{p_0}^1(G)$ ($\alpha = 1 - 2/p_0$, $2 < p_0 < p$, $j = 1, 2$, $S(z) = \prod_{l=1}^m |z - t_l|^{l/\tau^2}$) and the relation

$$u_j(z) = b_{j0} + \operatorname{Re} \int_{a_0}^z [w_j(z) + \sum_{k=1}^N \frac{id_{jk}}{z - z_k} dz], \quad (2.6)$$

where $a_0 = 1$, $|b_{j0}| \leq s_0$, $d_{jk} (j = 1, 2, k = 1, \dots, N)$ are appropriate real constants such that the functions $u_j(z) (j = 1, 2)$ determined by the integral in (2.6) is single-valued in G .

Theorem 2.1. (The Equivalence Theorem) *Let $(u_1(z), u_2(z))$ be a solution of Problem P for the second order system (1.1). Then $(u_1(z), u_2(z), w_1(z), w_2(z))$ is a solution of Problem Q for the system (2.1) of first order, which satisfies the boundary conditions*

$$\operatorname{Re}[\overline{\lambda_j(t)}w_j(t)] + \sigma_{j1}(t)u_1(t) + \sigma_{j2}(t)u_2(t) = \tau_{j3}(t), \quad t \in \Gamma, \quad j = 1, 2, \quad (2.7)$$

and the relation (2.6). Conversely, let $(u_1(z), u_2(z), w_1(z), w_2(z))$ be a solution of Problem Q for the system (2.1), which satisfy the boundary conditions (2.2) – (2.5) and the relation (2.6) with $h_j(t) = 0$ and $d_{jk} = 0 (j = 1, 2, k = 1, \dots, N)$. Then $(u_1(z), u_2(z))$ is a solution of Problem P for (1.1).

The proof of the theorem is immediate.

3. Estimates of Solutions of Modified Problem for Nonlinear Elliptic System of First Order Equations

In this section, we shall give a priori estimates of solutions for Problem Q.

Theorem 3.1. *Let $(u_1(z), u_2(z), w_1(z), w_2(z))$ be a solution of Problem Q for the first order system (2.1) satisfying the conditions*

$$\begin{aligned} C[R'(z)u_j(z), \overline{G}] &\leq M_1, \quad C[R(z)w_j(z), \overline{G}], \\ \|RSw_{j\bar{z}}\|_{L_{p_0}(\overline{G})} &= \|RS\rho_j(z)\|_{L_{p_0}(\overline{G})} \leq M_2, \quad j = 1, 2, \end{aligned} \quad (3.1)$$

where $S(z) = \prod_{l=1}^m |z - t_l|^{l/\tau^2}$, M_1, M_2 are unknown real constants to be determined appropriately. Then when the constants q'_{10}, k_{1l} ($l = 2, 3, 4$), k_{24} in (1.3), (1.4) and s_1 in (1.10) are sufficiently small, the solution $(u_1(z), u_2(z), w_1(z), w_2(z))$ certainly satisfies the estimates

$$\begin{aligned} C_\delta[R'(z)u_j(z), \overline{G}] < M_3, \quad C_\delta[R(z)w_j(z), \overline{G}] < M_4, \\ \|\!|RS(|w_{j\bar{z}}| + |w_{jz}|)\!\|_{L_{p_0}(\overline{G})} < M_5, \quad j = 1, 2, \end{aligned} \tag{3.2}$$

in which $M_k = M_k(q_0, p_0, k_0, \delta, s_0)$ ($k = 3, 4, 5$), $q_0 = \max(q_{10}, q'_{20})$, δ is a small positive constant, and we may be choose that M_1 and M_2 are equal to M_3 and M_4 respectively.

Proof. We first consider the case $K_j \geq 0$ ($j = 1, 2$). Let the stated solution $u_j(z), w_j(z)$ ($j = 1, 2$) in the theorem substitute the corresponding functions in the system (2.1), the boundary conditions (2.2)–(2.5) and the formula (2.6). It is clear that $w_2(z)$ is a solution of the modified Riemann-Hilbert boundary value problem for the complex equation $w_{2\bar{z}} = \rho_2(z)$ with the boundary condition (2.2)–(2.6) ($j = 2$). By virtue of the condition (3.1) and using the method analogous to that in Theorem 3.1, Chapter V, [3], we can prove that $w_2(z)$ satisfies the estimations

$$C_\delta[R(z)w_2(z), \overline{G}] < M_6, \quad \|\!|RSw_{2z}\!\|_{L_0(\overline{G})} \leq M_7, \tag{3.3}$$

where δ and p_0 are similar to those mentioned above and $M_l = M_l(q_0, p_0, k_0, \delta, s_0, M_1, M_2)$, $l = 6, 7$. Moreover, (2.1) having $j = 1$ can be written in the form

$$\begin{aligned} w_{1\bar{z}} - \text{Re}[Q_{11}w_{1z} + A_{11}w_1] &= A_1, \\ A_1 &= \text{Re}[Q_{12}w_{2z} + A_{12}w_2 + A_{13}u_1 + A_{14}u_2 + A_{15}], \end{aligned} \tag{3.4}$$

and it is not difficult to see that no matter how large the constants M_1 and M_2 in (3.1) are, provided that the constants q'_{10}, k_{1l} ($l = 2, 3, 4$) in Condition C and s_1 in (1.10) are sufficiently small, we must have

$$\|\!|RSA_1\!\|_{L_{p_0}(\overline{G})} \leq s_2 = s_2(q_0, p_0, k_0, \delta, s_0), \tag{3.5}$$

$$\begin{aligned} C_\alpha[|z - t_l|^{\beta_{jl}} |z - t_{l-1}|^{\beta_{j,l-1}} (\sigma_{11}(t)u_1(t) + \sigma_{12}(t)u_2(t)), \hat{\Gamma}_l] \\ \leq s_3 = s_3(q_0, p_0, k_0, \delta, s_0), \quad l = 1, \dots, m. \end{aligned} \tag{3.6}$$

Noting that the complex equation (3.4) satisfies Condition C and $w_1(z)$ is a solution of the complex equation (3.4) with the boundary conditions (2.2)–(2.6) ($j = 1$), we conclude that $w_1(z)$ satisfies the estimates:

$$C_\alpha[R(z)w_1(z), \overline{G}] \leq M_8, \quad \|\!|RS(|w_{1\bar{z}}| + |w_{1z}|)\!\|_{L_{p_0}(\overline{G})} \leq M_9, \tag{3.7}$$

where $M_l = M_l(q_0, p_0, k_0, \delta, s_0)$, $l = 8, 9$. Substituting the function $w_1(z)$ into (2.6) with $j = 1$, we have

$$\begin{aligned} C_\delta[R'(z)u_1(z), \overline{G}] &\leq M_{10}, \quad C_\delta[R(z)w_1, \overline{G}], \\ \|RS(|u_{1\overline{z}\overline{z}}| + |u_{1zz}|)\|_{L_{p_0}(\overline{G})} &\leq M_{11}, \end{aligned} \quad (3.8)$$

where $M_l = M_l(q_0, p_0, k_0, \delta, s_0)$, $l = 10, 11$. Afterwards, we discuss the complex equation (2.1) with $j = 2$, and rewrite it into the form

$$\begin{aligned} w_{2\overline{z}} - \operatorname{Re}[Q_{22}w_{2z} + A_{22}w_2] &= A_2, \\ A_2 = \operatorname{Re}[Q_{21}w_{1z} + A_{21}w_1 + A_{23}u_1 + A_{24}u_2 + A_{25}] &, \end{aligned} \quad (3.9)$$

On account of (3.1), (3.7), (3.8) and that k_{24} and s_1 are sufficiently small, we have then

$$\|RSA_2\|_{L_{p_0}(\overline{G})} \leq s_4 = s_4(q_0, p_0, k_0, \delta, s_0), \quad (3.10)$$

$$\begin{aligned} C_\alpha[|z - t_l|^{\beta_{ji}} |z - t_{l-1}|^{\beta_{ji-1}} (\sigma_{21}(t)u_1(t) + \sigma_{22}(t)u_2(t)), \Gamma_l] \\ \leq s_5 = s_5(q_0, p_0, k_0, \delta, s_0), \quad l = 1, \dots, m. \end{aligned} \quad (3.11)$$

Because $u_2(z)$ and $w_2(z)$ satisfy (2.2)–(2.6) with $j = 2$, we can obtain the estimations

$$C_\delta[R(z)w_2(z), \overline{G}] \leq M_{12}, \quad \|RS(|w_{2\overline{z}}| + |w_{2z}|)\|_{L_{p_0}(\overline{G})} \leq M_{13}, \quad (3.12)$$

$$C_\delta[R'(z)u_2(z), \overline{G}] \leq M_{14}, \quad \|RS(|u_{2z\overline{z}}| + |u_{2zz}|)\|_{L_{p_0}(\overline{G})} \leq M_{15}, \quad (3.13)$$

where $M_l = M_l(q_0, p_0, k_0, \delta, s_0)$, $l = 12, \dots, 15$. From (3.7), (3.8), (3.12) and (3.13), we can derive (3.2). Moreover we may select $M_1 = M_3$ and $M_2 = M_4$ in (3.1).

Next, we consider the case $K_1 < 0$ or $K_2 < 0$. When $K_2 \geq 0$, by the same reason as above, we have the estimates in (3.3). When $K_2 < 0$, in order to obtain (3.3), we use the estimates of the constants h_{2l} ($l = [1 - (-1)^{2K_2}/2, \dots, N)$, h_{2m}^+ , h_{2m}^- ($m = 1, \dots, [K_2 + 1/2] - 1$) determined by $w_2(z)$ satisfying

$$|h_{2l}|, |h_{2m}^+|, |h_{2m}^-| \leq s_6 = s_6(q_0, p_0, k_0, \delta, s_0, M_1, M_2), \quad (3.14)$$

and hence (3.3) is derived. When $K_1 < 0$, since $w_1(z)$ is a solution of the modified Riemann-Hilbert boundary value problem for (3.4) with the boundary condition (2.2)–(2.6) ($j = 1$), then similarly to the above, we can also prove that the constants h_{1l} ($l = [1 - (-1)^{2K_1}/2, \dots, N)$, h_{+1m} , h_{-1m}^- ($m = 1, \dots, [K_1 + 1/2] - 1$) satisfy the inequality (3.14) with $j = 1$. It follows that (3.7) holds. The remanent part of the proof is the same as the case $K_j \geq 0$ ($j = 1, 2$). This completes the proof.

4. Solvability of Modified Problem and Original Problem for Nonlinear Systems of Elliptic Equations

Suppose that the system (2.1) satisfies the following conditions, i.e. for any real functions $u_j^n(z)$ ($R'(z)u_j^n(z) \in C_\delta(\overline{G})$, $j, n = 1, 2$), any complex functions $w_j^n(z)$ ($R(z)w_j^n(z) \in C_\delta(\overline{G})$, $j, n = 1, 2$), $U_j(z)$ ($RSU_j \in L_{p_0}(G)$, $j = 1, 2$) satisfy

$$\begin{aligned}
 &F_j(z, u_1^1, u_2^1, w_1^1, w_2^1, U_1, U_2) - F_j(z, u_1^2, u_2^2, w_1^2, w_2^2, U_1, U_2) = \text{Re}[\tilde{Q}_{j1}U_1 \\
 &+ \tilde{Q}_{j2}U_2 + \tilde{A}_{j1}(w_1^1 - w_1^2) + \tilde{A}_{j2}(w_2^1 - w_2^2) + \tilde{A}_{j3}(u_1^1 - u_1^2) + \tilde{A}_{j4}(u_2^1 - u_2^2)], \quad (4.1) \\
 & \qquad \qquad \qquad j = 1, 2,
 \end{aligned}$$

where $\tilde{Q}_{jl} = \tilde{Q}_{jl}(z, u_1^1, u_2^1, u_2^2, w_1^1, w_1^2, w_2^1, w_2^2, U_1, U_2)$ ($j, l = 1, 2$) $\tilde{A}_{jl} = \tilde{A}_{jl}(z, u_1^1, u_2^1, u_2^2, w_1^1, w_1^2, w_2^1, w_2^2, U_1, U_2)$ ($j = 1, 2, l = 1, \dots, 4$) satisfy the conditions

$$\begin{aligned}
 &|\tilde{Q}_{j1}| \leq q_{j0}, \quad |\tilde{Q}_{j2}| \leq q'_{j0} \quad (q_{j0} + q'_{j0} < 1), \\
 &\|A_{jl}\|_{L_p(\overline{G})} \leq k_{jl} < k_0, \quad j = 1, 2, l = 1, \dots, 4.
 \end{aligned}$$

We first assume that the coefficients Q_{jl} , A_{jl} of the first order system (2.1) equal zero in the neighborhood $\overline{G} \setminus G_m$ of the boundary Γ , where $G_m = \{z \mid |z - Z| \geq 1/m, z \in \overline{G}, Z = \{t_1, \dots, t_m\}\}$, m is a sufficiently large positive integer, and write this system in the form

$$w_{j\bar{z}} = f_j^{(m)}(z, u_1, u_2, w_1, w_2, w_{1z}, w_{2z}), \quad j = 1, 2. \quad (4.2)$$

Theorem 4.1. *Suppose that the system (4.2) of first order complex equations satisfies Condition C and (4.1). If the constants q'_{10} , k_{1l} ($l = 2, 3, 4$), k_{24} in (1.3), (1.4) and s_1 in the (1.10) are sufficiently small, then Problem Q for the system (4.2) is solvable.*

Proof. Let us introduce the Banach space $B = C^1(G_m) \times C^1(G_m) \times L_{p_0}(G_m) \times L_{p_0}(G_m)$ and denote by B_M the totality of the systems of the functions: $\omega = (u_1(z), u_2(z), \rho_1(z), \rho_2(z))$ satisfying the inequalities

$$C^1[u_j(z), G_m] \leq M_3, \quad \|\rho_j(z)\|_{L_0(G_m)} \leq M_4, \quad j = 1, 2, \quad (4.3)$$

where M_3 and M_4 are the constants similar to (3.2). Evidently, B_M is a bounded open set in B . We arbitrarily select $\omega = (u_1(z), u_2(z), \rho_1(z), \rho_2(z)) \in \overline{B_M}$ and make the double integrals

$$H\rho_j = \frac{2}{\pi} \iint_{G_m} \ln \left| 1 - \frac{z}{\zeta} \right| \cdot \rho_j(\zeta) d\sigma_\zeta, \quad j = 1, 2,$$

and so

$$(H\rho_j)_z = -\frac{1}{\pi} \iint_{G_m} \frac{\rho_j(\zeta)}{\zeta - z} d\sigma_\zeta = T\rho_j, \quad j = 1, 2.$$

Substitute $u_j(z)$ and $T\rho_j$ ($j = 1, 2$) into suitable positions of the boundary conditions (2.2)–(2.6), and similarly to [3],[10], we can prove the existence and uniqueness of the analytic function $\Phi_j(z)$ in G satisfying the boundary conditions

$$\begin{aligned} \operatorname{Re}[\overline{\lambda_j(t)}(\Phi_j(t) + T\rho_j)] + \sigma_{j1}(t)u_1(t) + \sigma_{j2}(t)u_2(t) \\ = \tau_{j3}(t) + h_j(t)\overline{\lambda_j(z)}X_j(z), \quad t \in \Gamma, \end{aligned} \quad (4.4)$$

$$\operatorname{Im}[\overline{\lambda_j(a_l)}(\Phi_j(t) + T\rho_j)]_{t=a_l} = b_{jl}, \quad l = J_j, \quad K_j \geq 0, \quad j = 1, 2, \quad (4.5)$$

where $\Phi_j(z)$, $w_j(z) = \Phi_j(z) + T\rho_j$ ($j = 1, 2$) satisfy the estimations

$$C_\delta[\Phi'_j(z), G_m] \leq M_{16}, \quad C_\delta[w_j(z), G_m] \leq M_{17}, \quad j = 1, 2, \quad (4.6)$$

in which $M_l = M_l(q_0, p_0, k_0, \delta, s_0, G_m, M_3, M_4)$, $l = 16, 17$. Now we substitute $u_j(z)$, $w_j(z)$, $\Phi'_j(z)$ into the appropriate places of the system (4.2) and consider the system of the integral equations with the parameter t ($0 \leq t \leq 1$):

$$\rho_j^*(z) = t f_j^{(m)}(z, u_1, u_2, w_1, w_2, \Phi'_1(t) + \Pi\rho_1^*, \Phi'_2(t) + \Pi\rho_2^*), \quad j = 1, 2, \quad (4.7)$$

where

$$\Pi\rho_j^* = -\frac{1}{\pi} \iint_{G_m} \frac{\rho_j^*(\zeta)}{(\zeta - z)^2} d\sigma_\zeta, \quad j = 1, 2.$$

By Condition C and the principle of contraction, the system (4.7) has the unique solutions $\rho_j^*(z) \in L_{p_0}(G_m)$, $j = 1, 2$. Afterwards we seek the analytic functions $\Phi_j^*(z)$ ($j = 1, 2$) in G , which satisfy the boundary conditions:

$$\begin{aligned} \operatorname{Re}[\overline{\lambda_j(t)}(\Phi_j^*(t) + T\rho_j^*)] + \sigma_{j1}(t)u_1(t) + \sigma_{j2}(t)u_2(t) \\ = \tau_{j3}(t) + h_j(t)\overline{\lambda_j(z)}X_j(z), \quad t \in \Gamma, \end{aligned} \quad (4.8)$$

$$\begin{aligned} \operatorname{Im}[\overline{\lambda_j(t)}(\Phi_j^*(t) + T\rho_j^*)]_{t=a_l} = b_{jl}, \\ l \in J_j, \quad K_j \geq 0, \quad j = 1, 2, \end{aligned} \quad (4.9)$$

and then find single-value functions

$$u_j^*(z) = b_{j0} + \operatorname{Re} \int_{a_0}^z [w_j^*(z) + \sum_{k=1}^N \frac{id_{jk}}{z - z_k} dz], \quad j = 1, 2, \quad (4.10)$$

where $w_j^*(z) = \Phi_j^*(z) + T\rho_j^*$, $j = 1, 2$. Denote by $\omega^* = T[\omega, t]$ ($0 \leq t \leq 1$) a mapping from $\omega = \omega(z)$ to $\omega^* = [u_1^*(z), u_2^*(z), \rho_1^*(z), \rho_2^*(z)]$. Moreover, if the constants q'_{10}, k_{1l} ($l = 2, 3, 4$), k_{24} in (1.3), (1.4) and s_1 in the boundary condition (1.10) are sufficiently small, then similarly to the proof of Theorem 3.1, we can derive that the solution $\omega = (u_1(z), u_2(z), \rho_1(z), \rho_2(z)) \in \overline{B_M}$ of Problem Q for the system

$$\begin{aligned} \rho_j(z) &= t f_j^{(m)}(z, u_1, u_2, w_1, w_2, \Phi'_1(t) + \Pi\rho_1, \Phi'_2(t) + \Pi\rho_2), \\ 0 \leq t \leq 1, \quad j &= 1, 2, \end{aligned} \tag{4.11}$$

satisfies (4.2), where α, p_0 are stated before and $M_k = M_k(q_0, p_0, k_0, G_m, \nu, s_0)$ ($k = 3, 4$), and hence $\omega = (u_1(z), u_2(z), \rho_1(z), \rho_2(z)) \in \overline{B_M}$. Further, we can verify that the mapping $\omega^* = T[\omega, t]$ ($0 \leq t \leq 1$) satisfies three conditions of the Leray-Schauder theorem, i.e.

1) In order to prove the complete continuity of $\omega^* = T[\omega, t]$ ($0 \leq t \leq 1$) in $\overline{B_m}$, we chose arbitrarily $\omega_n \in \overline{B_M}$, $n = 1, 2, \dots$, and so $\omega_n^* = T[\omega_n, t] = (u_{1n}^*(z), u_{2n}^*(z), \rho_{1n}^*(z), \rho_{2n}^*(z)) \in B$, i.e.

$$\rho_{jn}^*(z) = t f_j^{(m)}(z, u_{1n}, u_{2n}, w_{1n}, w_{2n}, \Phi'_{1n}(t) + \Pi\rho_{1n}^*, \Phi'_{2n}(t) + \Pi\rho_{2n}^*), \quad j = 1, 2. \tag{4.12}$$

We can prove as above that $w_{jn}(z)$ and $\Phi'_{jn}(z)$ satisfy the estimation (4.6). Thus we may select subsequences of $\{u_{jn}(z)\}$, $\{w_{jn}(z)\}$, $\{\Phi'_{jn}(z)\}$, which uniformly converge to $u_{j0}(z)$, $w_{j0}(z)$, $\Phi'_{j0}(z)$ in G_m respectively, and denote by the same symbols the subsequences. Note that the right side of the system

$$\rho_{j0}^*(z) = t f_j^{(m)}(z, u_{10}, u_{20}, w_{10}, w_{20}, \Phi'_{10}(t) + \Pi\rho_{10}^*, \Phi'_{20}(t) + \Pi\rho_{20}^*), \quad j = 1, 2. \tag{4.13}$$

equals zero for $z \notin G_m$, and apply the principle of contraction, then the system (4.13) has a unique solution $(\rho_{10}^*(z), \rho_{20}^*(z))$ and $\rho_{j0}^*(z) \in L_{p_0}(G_m)$, $j = 1, 2$. Let (4.2) be minus (4.13), we have

$$\begin{aligned} \rho_{jn}^*(z) - \rho_{j0}^*(z) &= t [f_j^{(m)}(z, u_{1n}, u_{2n}, w_{1n}, w_{2n}, \Phi'_{1n}(t) + \Pi\rho_{1n}^*, \Phi'_{2n}(t) + \Pi\rho_{2n}^*) \\ &\quad - f_j^{(m)}(z, u_{1n}, u_{2n}, w_{1n}, w_{2n}, \Phi'_{1n}(t) + \Pi\rho_{10}^*, \Phi'_{2n}(t) + \Pi\rho_{20}^*) + c_{jn}(z)], \\ c_{jn}(z) &= f_j^{(m)}(z, u_{1n}, u_{2n}, w_{1n}, w_{2n}, \Phi'_{1n}(t) + \Pi\rho_{10}^*, \Phi'_{2n}(t) + \Pi\rho_{20}^*) \\ &\quad - f_j^{(m)}(z, u_{10}, u_{20}, w_{10}, w_{20}, \Phi'_{10}(t) + \Pi\rho_{10}^*, \Phi'_{20}(t) + \Pi\rho_{20}^*), \quad j = 1, 2. \end{aligned} \tag{4.14}$$

By virtue of Condition C, we can derive $\|c_{jn}(z)\|_{L_{p_0}(G_m)} \rightarrow 0$, $j = 1, 2$, when $n \rightarrow \infty$, and from (4.14), it follows that

$$\begin{aligned} & \sum_{j=1}^2 \|c_{jn}^*(z) - c_{j0}^*(z)\|_{L_{p_0}(G_m)} \\ & \leq \frac{1}{1 - 2 \max_{j=1,2} (q_{j0} + q'_{j0}) \|\Pi\|_{L_{p_0}}} \sum_{j=1}^2 \|c_{jn}\|_{L_{p_0}(G_m)}. \end{aligned} \quad (4.15)$$

Therefore, $\sum_{j=1}^2 \|c_{jn}^*(z) - c_{j0}^*(z)\|_{L_{p_0}(G_m)} \rightarrow 0$ for $n \rightarrow \infty$. Besides, according to $C[w_{jn}^*(z) - w_{j0}^*(z), G_m] \rightarrow 0$ for $n \rightarrow \infty$, it is clear that $C^1[u *_{jn}(z) - u_{j0}^*(z), G_m] \rightarrow 0$ ($j = 1, 2$) as $n \rightarrow \infty$. This shows the complete continuity of $\omega^* = T[\omega, t]$ ($0 \leq t \leq 1$) in $\overline{B_m}$. By the same reason, we can also prove that $\omega^* = T[\omega, t]$ ($0 \leq t \leq 1$) maps continuously $\overline{B_m}$ onto B , and $T[\omega, t]$ is uniformly continuous with respect to $t \in [0, 1]$ for $\omega \in \overline{B_M}$.

2) When $t = 0$, it is easy to see that $\omega^* = T[\omega, 0] = (u_1^*(z), u_2^*(z), 0, 0) \in B_M$ from (4.7) and (4.10).

3) From (3.2), we find that $\omega = T[\omega, t]$ ($0 \leq t \leq 1$) has not a solution $\omega(z)$ on the boundary $\partial B_M = \overline{B_M} \setminus B_M$.

Hence by the Leray-Schauder theorem, we can know that the system (4.11) with $t = 1$ has a solution $\omega = (u_1(z), u_2(z), \rho_1(z), \rho_2(z)) \in B_M$, and thus $(u_1(z), u_2(z), w_1(z), w_2(z))$ is a solution of Problem Q for the first order system (4.1), where $w_j(z) = \Phi_j(z) + T\rho_j$ ($j = 1, 2$).

Theorem 4.2. *Let the first order system (2.1) of the complex equations satisfy Condition C and (4.1). If the constants q'_{10}, k_{1l} ($l = 2, 3, 4$), k_{24} in (1.3), (1.4) and s_1 in (1.10) are sufficiently small, then Problem Q for the system (2.1) is solvable.*

Proof. On the basis of Theorems 4.1 and 3.1, it is easy to see that Problem Q for the system (4.2) ($m = 1, 2, \dots$) have solutions $u_{jm}(z), w_{jm}(z) = \Phi_{jm}(z) + T\rho_{jm}$ ($j = 1, 2, m = 1, 2, \dots$), and these solutions satisfy the estimation (3.2). Hence we can choose subsequences $\{u_{jm}^*(z)\}, \{w_{jm}^*(z)\}$ of $\{u_{jm}(z)\}, \{w_{jm}(z)\}$, which converge to $u_{j0}(z), w_{j0}(z)$ ($j = 1, 2$) in G^* respectively, and $u_{j0}(z), w_{j0}(z)$ ($j = 1, 2$) satisfy the boundary conditions (2.2)–(2.5) and the relation (2.6).

In the following, we shall prove that $(u_{10}(z), u_{20}(z), w_{10}(z), w_{20}(z))$ is a solution of the first order system (2.1). Because $u_{jm}(z), w_{jm}(z) = \Phi_{jm}(z) + T\rho_{jm}$ ($j = 1, 2$) satisfy (3.1), we may choose the subsequences $\{\rho_{jm}^*(z)\}$ of $\{\rho_{jm}(z)\} = \{w_{jm\bar{z}}\}$ ($j = 1, 2$), which weakly converges to $\rho_{j0}(z) \in L_{p_0}(G_n)$ ($j =$

1, 2) for any fixed large positive integer n . Therefore $\{\Phi_{jm}^*(z)\}$ and $\{T\rho_{jm}^*\}$ uniformly converge to $\Phi_{j0}(z)$, $T\rho_{j0}$ ($j = 1, 2$) in G_n respectively, where $\Phi_{j0}(z)$ ($j = 1, 2$) are analytic functions, and $w_{j0}(z) = \Phi_{j0}(z) + T\rho_{j0} \in W_{p_0}^1(G_n)$ ($j = 1, 2$), it is clear that G_n is any compact subset of G^* . For convenience, we denote still $\{u_{jm}^*(z)\}$, $\{w_{jm}^*(z)\}$, $\{\rho_{jm}^*(z)\}$ by $\{u_{jm}(z)\}$, $\{w_{jm}(z)\}$, $\{\rho_{jm}(z)\}$ respectively. In particular, let $S = \{z \mid |z - z_*| \leq R, \{|z - z_*| = R\} \cap Z = \emptyset\}$ ($z_* \in G^*, R > 0$) be a set in G^* , and $w_{jm}(z)$ ($m = 1, 2, \dots$) admit the expression in S :

$$w_{jm}(z) = \Psi_{jm}(z) + P\rho_{jm}, P\rho_{jm} = Q\rho_{jm} - \text{Im}Q\rho_{jm}|_{z=z_*},$$

$$Q\rho_{jm} = -\frac{1}{\pi} \iint_S \left[\frac{\rho_{jm}(\zeta)}{\zeta - z} + \frac{(z - z_*)\rho_{jm}(\zeta)}{R^2 - (\bar{\zeta} - \bar{z})(z - z_*)} \right] d\sigma_\zeta, \tag{4.16}$$

$$j = 1, 2, m = 1, 2, \dots$$

We note that $\{\rho_{jm}(z)\}$ weakly converge to $\rho_{j0}(z)$ ($j = 1, 2$) in S , $\{\Psi'_{jm}(z)\}$ uniformly converge to $\Psi'_{j0}(z)$ in S and $\Psi'_{jm}(z)$ satisfy

$$\|\Psi'_{jm}(z)\|_{L_{p_0}(S)} \leq M_{18} = M_{18}(q_0, p_0, k_0, \delta, S, s_0), j = 1, 2, m = 1, 2, \dots, \tag{4.17}$$

and apply the principle of contraction that the system of the integral equations

$$\rho_j^*(z) = f_j(z, u_{10}, u_{20}, w_{10}, w_{20}, \Psi'_{10} + Q\rho_{10}^*, \Psi'_{20}(t) + Q\rho_{20}^*), j = 1, 2, \tag{4.18}$$

has a unique solution $\rho_{j0}^*(z) \in L_{p_0}(S)$, where $Q\rho_{j0}^* = (P\rho_{j0}^*)_z$, $j = 1, 2$. Let the system

$$\rho_{jm}^*(z) = f_j^{(m)}(z, u_{1m}, u_{2m}, w_{1m}, w_{2m}, \Psi'_{1m} + Q\rho_{1m}^*, \Psi'_{2m}(t) + Q\rho_{2m}^*), \tag{4.19}$$

$$j = 1, 2, m = 1, 2, \dots$$

be minus (4.13), and notice that $f_j = f_j^{(m)}$ in S , $j = 1, 2$, when m is large enough. We can prove that $\sum_{j=1}^2 \|\rho_{jm}^*(z) - \rho_{j0}^*(z)\|_{L_{p_0}(S)} \rightarrow 0$ for $m \rightarrow \infty$. Hence $(u_{10}(z), u_{20}(z), w_{10}(z), w_{20}(z))$ is a solution of (2.1) in S , where $w_{j0}(z) = \Psi_{j0}(z) + P\rho_{j0}^* = \Psi_{j0}(z) + P\rho_{j0}$ ($j = 1, 2$). Owing to the arbitrariness of S , $(u_{10}(z), u_{20}(z), w_{10}(z), w_{20}(z))$ is therefore a solution of the system (2.1) in G . This completes the proof.

In the following, we shall derive the results of solvability on Problem P of the second order system (1.1).

Proof of Theorem 1.1. From Theorem 4.2, we know that Problem Q of the first order system (2.1) has a solution $(u_1(z), u_2(z), w_1(z), w_2(z))$. Let us substitute the solution into the boundary conditions (2.2)–(2.4). The functions

$h_j(t)$ ($j = 1, 2$) and the constants d_{jk} ($j = 1, 2, k = 1, \dots, N$) are then determined. If the functions and the constants are equal zero, i.e.

$$h_j(t) = \begin{cases} h_{jl} = 0, l = 1, \dots, N, & \text{when } K_j \geq 0, \\ h_{jl} = 0, l = [1 - (-1)^{2K_j}]/2, \dots, N, \\ \eta_{jm}^\pm = 0, m = 1, \dots, [|K_j| + 1/2] - 1, \end{cases} \text{ when } K_j < 0, j = 1, 2, \quad (4.20)$$

$$d_{jk} = 0, j = 1, 2, k = 1, \dots, N, \quad (4.21)$$

then the solution $(u_1(z), u_2(z), w_1(z), w_2(z))$ of Problem Q for the first order system (2.1) is also a solution of Problem P for the second order system (1.1), and $u_{jz} = w_j(z), j = 1, 2$. From (4.20) and (4.21), it is not difficult to see the total number of solvability conditions of Problem P for the system (1.1). For example, when the indices $K_j \geq 0$ ($j = 1, 2$), Problem P for the system (1.1) has $4N$ solvability conditions. Besides the solution of Problem P depends on some arbitrarily real constants. For instance, when $K_j \geq 0$ ($j = 1, 2$), the solution of Problem P depends on $2(K_1 + K_2 + 2)$ arbitrarily real constants. As for the other cases, we use the same way to obtain also the results as stated in Theorem 1.1.

Acknowledgments

The research was supported by PHR (IHLB 201106206), and College Students Scientific Research and Undertaking Starting Action Project (SJ201301016).

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