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DISTRIBUTED SOFTWARE SYSTEM FOR TESTING NEAR-RINGS HYPOTHESES AND NEW CONSTRUCTIONS FOR NEAR-RINGS ON FINITE CYCLIC GROUPS

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Abstract: In this paper we describe a software system for testing hypotheses about near-rings on finite cyclic groups. The system allows researchers to define parameters and test π functions. By using this software we make new hypotheses, and formulate and prove statements for constructions of new classes of near-rings. We compute all near-rings on finite cyclic group of order 33, 34 and 35.

AMS Subject Classification: 16Y30
Key Words: near-ring, finite cyclic group, distributed software, grid computing

1. Introduction

An algebraic system $(G, +, *)$ is a *(left) near-ring* on $(G, +)$ if $(G, +)$ is a group, $(G, *)$ is a semigroup and $a * (b + c) = a * b + a * c$ for $a, b, c \in G$. The left

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distributive law yields $x * 0 = 0$ for $x \in G$. A near-ring $(G, +, *)$ is called *zero-symmetric*, if $0 * x = 0$ holds for $x \in G$.

J.R. Clay initiated the study of near-rings whose additive groups are finite cyclic ones in 1964 [1]. Some sufficient conditions for the construction of near-rings on any finite cyclic groups were obtained.

We will assume G coincides with the set $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$, $2 \leq n < \infty$ since every cyclic group of order n is isomorphic to the group of the remainders of modulo n. We will denote the functions mapping $\mathbb{Z}_n$ into itself by π, and the addition and the multiplication modulo n we will denote by $+$ and $\cdot$, respectively. The equality $c = a \cdot b$ will be equivalent to the congruence $ab \equiv c$ (mod n).

It is known [1] that there exists a bijective correspondence between the left distributive binary operations $*$ defined on $\mathbb{Z}_n$ and the n^n functions π mapping $\mathbb{Z}_n$ into itself. If $r * 1 = b$ defines the function $\pi(r) = b$, then according to [1, Theorem II], the binary operation $*$ is left distributive exactly when, for any $x, y \in \mathbb{Z}_n$, the equality

$$\pi(x) \cdot \pi(y) = \pi(x \cdot \pi(y)) \tag{1}$$

holds.

According to the above result, obtaining the near-rings on $\mathbb{Z}_n$ is equivalent to obtaining functions π such that equation (1) holds.

2. Software System for Near-Ring Hypotheses Testing

We created a software system for testing hypotheses about near-rings on finite cycling groups. The system allows researchers to define parameters and π functions, which will be tested. It is possible to test a single π function, or define and test a set of π functions. In the latter case, the functions can be tested if they meet condition (1) for all the values of their parameters.

If the test of condition (1) is not met for a π function, we are certain that the parameterized set of π functions does not describe near-rings.

Having all generated functions satisfy condition (1) is not a sufficient proof that the parameterized set of π functions does describe near-rings. However, this lets us make quick preliminary hypotheses check, before doing detailed proof of the hypotheses.

Testing large sets of π functions can be very intensive computationally, and not feasible on common hardware. A standard personal computer can process only a limited number of π functions. A well-known approach to obtaining more

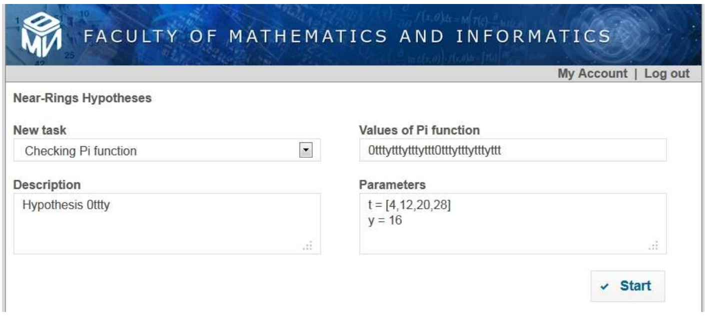

Figure 1

computational power is to use grid computing. Grid computing links disparate computers to form one large infrastructure, harnessing resources. The potential for massive parallel CPU capacity is one of the most attractive features of a grid. The common attribute among such uses is that the applications have been written to use algorithms that can be partitioned into independently running parts. A CPU intensive grid application can be thought of as many smaller "subjobs", each executing on a different machine in the grid [2, 3].

Therefore we built the software as a distributed system which can be run in a grid.

If the number of π functions in the defined set is very large and their test is impossible in real time even on large-scale grid or cloud-based system, the system can test the hypotheses by generating random values for the parameters for a predefined number of π functions and performing the test of condition (1) only for them.

2.1. Overview of the Software System

It consists of these components:

- Near-ring computational service:
  - Task Dispatcher
  - Subtask Runner
- User Dashboard:
  - Database
  - Backend

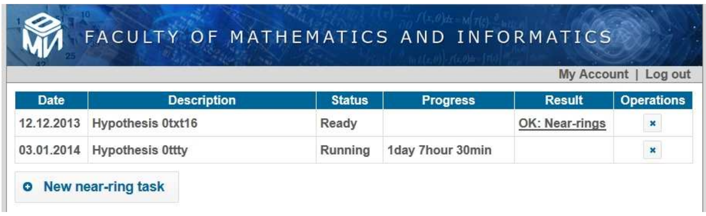

Figure 2

  - Front-end

- Administration

2.2. Task Dispatcher

It performs several functions:

1. Maintains a queue with pending tasks;
2. Splits tasks into sub tasks;
3. Assigns the execution of subtasks across the available computational nodes in the system;
4. Receives the output of the subtasks, and updates the status of the task and stores its result.

The queue with pending tasks is a priority queue which contains the pending tasks. Administrators can change the priority of the pending tasks and influence the execution order. The queue is persisted in a standard file.

Task Dispatcher maintains a list of computational nodes in the system. It actively monitors the status of every node.

2.3. Subtask Runner

It executes a single subtask on a single computational node. When the task is completed, it sends the result of the execution back to the Task Dispatcher.

2.4. User Dashboard

The module provides a user interface for users to:

1. Register into the system;
2. Create new tasks;
3. See the current status of their tasks;
4. Obtain the result of successfully completed tasks;
5. Cancel the execution of tasks;
6. Delete old tasks.

User Dashboard is developed as a standard three-tier web application. It has a database, a backend and a web browser front end. The database is used to store user profiles, tasks and task results. The backend is developed as standalone RESTful services. This gives opportunities for development of other front-ends, for example, mobile ones.

2.5. Administration

This is a special module that extends the User Dashboard with services for administrators. Namely, administrators can:

1. Create, suspend, activate and remove user accounts;
2. Set task priorities;
3. Monitor the status of the Task Dispatcher: number of tasks under execution, number of pending tasks, number of computational nodes, and free computational nodes;
4. Cancel execution of running tasks.

3. New Constructions for Zero-Symmetric Near-Rings on Finite Cyclic Groups

Theorem 1. *Let $n = 2t^2$ and t be an even number. If:*

i) $\pi(0) = \pi(t^2) = 0$;

ii) $\pi(kt) = t^2$, $k = 1, 2, \ldots, 2t-2$, $k \neq t$ and

iii) $\pi(x) \in \{rt\}$, $x \neq it$, $i = 0, 1, \ldots, 2t-2$, $1 \leq r < 2t$ and $(r, t) = 1$,

then equation (1) holds and there are exactly $\varphi(2t)^{n-2t}$ near-rings on $\mathbb{Z}_n$, where $\varphi(m)$ is the number of integers coprime to a positive integer m, between 1 and m.

Proof. We consider the following cases:

Case 1. Let $x = 0$ or $x = t^2$. Then $\pi(x)\pi(y) = 0$. If $x = 0$ then $\pi(0\pi(y)) = 0$. By assumptions of Theorem all values of π are even numbers and when $x = t^2$ then $\pi(t^2\pi(y)) \equiv \pi(2t^2) \equiv \pi(0) \bmod n$. Equation (1) holds.

Case 2. Assume that $x = kt$, $k = 1, 2, \ldots, 2t-2, k \neq t$. Then $\pi(x)\pi(y) = t^2\pi(x)$ and because all values of π are even numbers the left side of (1) is 0. The right side of (1) is $\pi(kt\pi(y))$. All nonzero values of π have multiplier t. Then $kt\pi(y) \equiv kt^2 \bmod n$. The possible results by modulo n are 0 and p^2. By definition $\pi(0) = \pi(t^2) = 0$. The equation (1) holds.

Case 3. Let $x \neq it$, $i = 0, 1, \ldots, 2t-2$. Then the left side of (1) is equal to $rt\pi(y)$. In the case when $\pi(y) = 0$ the equation (1) holds. In the case when $\pi(y) = t^2$ the left side is equivalent to $2t^2 = 0 \bmod n$. For the right side $x\pi(y) = xt^2$ and the possible values by modulo n are 0 or t^2. We have $\pi(0) = \pi(t^2) = 0$. Equation (1) holds.

Let $\pi(y) = rt$, $(r, t) = 1$. Then the left side by modulo n is equivalent to r^2t^2. By definition $(r, t) = 1$ and t is even number, hence r^2 is odd number and $\pi(x)\pi(y) \equiv t^2 \bmod n$. Consider $x\pi(y)$. By definition $x = it + a$, $1<a<t$ then $x\pi(y) = (it + a)rt = it^2 + art \equiv at^2 \bmod n$. Depending on the value of i, $it^2 \equiv 0 \bmod n$ or $\equiv t^2 \bmod n$. Since $(r, t) = 1$, the expression art will be different from t^2. Therefore $\pi(x\pi(y)) = \pi(kt)$, when $kt \neq 0, t^2$. By definition the right side is equal to t^2. Equation (1) holds.

According to the Theorem, $2t$ elements have constant values 0 or t^2 and the other $2t^2 - 2t$ elements have $\varphi(2t)$ distinct values (the number of coprimes with $2t$). Thus there are exactly $\varphi(2t)^{n-2t}$ functions π such that the equation (1) holds. □

When $n = 32$ we have $4^{24} = 281\,474\,976\,710\,656$ newly described near-rings on Z_{32}, and the values of the function π look like:

$$(0, t, t, t, 16, t, t, t, 16, t, t, t, 16, t, t, t, 0, t, t, t, 16, t, t, t, 16, t, t, t, 16, t, t, t),$$

where $t \in \{4, 12, 20, 28\}$.

Let us denote by M_n the set of all nilpotents of second degree on $(\mathbb{Z}_n, \cdot)$.

Theorem 2. *Let $n = 2t^2$ and t be an even number. If:*

i) $\pi(kt) = 0$, $k = 0, 2, \ldots, 2t{-}2$;

ii) $\pi(kt{+}t) = t^2$, $k = 0, 2, \ldots, 2t{-}2$;

iii) $\pi(e) \in M_n$, e is even number and $e \neq 0, t, 2t, \ldots, n{-}t$ and

iv) $\pi(o) \in \{t, 3t, \ldots, n{-}t\}$, $o = 1, 3, ..., n{-}1$,

then equation (1) holds and there are exactly $t^{t^2}.|M_n|^{t^2-t}$ near-rings on $\mathbb{Z}_n$.

Proof. We use that for $n = 2t^2$ every nilpotent of second degree contains a multiplier $2t$ and will denote $2tm$.

Case 1. Let $x = kt$, $k = 0, 2, \ldots, 2t{-}2$. Then $\pi(x)\pi(y) = 0$. In the right side of (1) we have $kt\pi(y)$. All nonzero values of π contain multiplier t. Hence $kt\pi(y) \equiv kt^2 \equiv 0 \bmod n$. The right side is equal to 0. The equation (1) holds.

Case 2. Let $x = kt{+}t$. Then the left side of (1) is equal to $t^2\pi(y)$. All values of π are even numbers. Then $\pi(y)t^2 \equiv 0 \bmod n$. Since $(k+1)\pi(y)$ is even number we have $(k+1)\pi(y)t \equiv kt \bmod n$ and $\pi(kt) = 0$. Equation (1) holds.

Case 3. Let x is even number and $x \neq 0, t, 2t, \ldots, n{-}t$. The left side is $\pi(x)\pi(y) = 2tm\pi(y)$. All values of π are zero or contain multiplier t. Hence the left side becomes $2t^2.m_2 \equiv 0 \bmod n$. The parameter in the right side of (1) is $z = x.\pi(y)$. All values of x are even and all other values contain multiplier t, hence $z = kt$. From the first assumption of Theorem the right side is also equal to zero.

Case 4. Let x is odd number. Then the left side of (1) is $m_1 t\pi(y)$.

When $\pi(y) = 0$ the equation (1) holds.

Case 4.1. Assume that $\pi(y)$ is equal to t^2 or belongs to M_n. We have multiplier t and multiplier equal to even number. Hence $(even)t^2 = 0 \bmod n$. The right side parameter in π function is equal to $o.\pi(y) \equiv kt \bmod n$. Therefore the right side is 0.

Case 4.2. Let y is odd number. Then the left side is $o_1t.o_2t = o.t^2 \equiv t^2 \bmod n$. The right side is $\pi(o.o_2t)$, the values of o and o_2 are odd and by assumption (ii) the right side of (1) is equal to t^2. Equation (1) holds.

According to the Theorem, $2t$ elements have constant values 0 or t^2, for t^2 elements we have t distinct values (iv) and $t^2 - t$ element belong to M_n (iii). Thus there are exactly $t^{t^2}.|M_n|^{t^2-t}$ functions π such that the equation (1) holds. □

When $n = 32$ we have $4^{16}.4^8 = 4^{24}$ newly described near-rings on Z_{32} and the values of the function π look like:

$$(0, t, x, t, 16, t, x, t, 0, t, x, t, 16, t, x, t, 0, t, x, t, 16, t, x, t, 0, t, x, t, 16, t, x, t),$$

where $t \in \{4, 12, 20, 28\}$ and $x \in \{0, 8, 16, 24\}$.

Theorem 3. *Let $n = 2t^2$ and t be an even number. If:*

i) $\pi(kt) = 0,\ k = 0, 2, \dots, 2t-2$;

ii) $\pi(e) \in M_n$, e is even number and $e \neq 0, t, 2t, \dots, n-t$;

For every $s = 1, 3, 5, \dots, 2t - 1$ we have

iii) $\pi(s) = \pi(2t+s) = \cdots = \pi(n-2t+s) = rt,\ 1 \le r < 2t,\ (r, t)=1$;

iv) $\pi(o) \in M_n$, $o = 1, 3, 5, ..., n-1$, $o \notin \{s, 2t+s, \dots, n-2t+s\}$;
and for every specific value rt in positions "s" we have

v) $\pi(z) = t^2$, $z = s.rt \bmod n$ and

vi) $\pi(kt+t) = 0,\ k = 0, 2, \dots, 2t-2$ and $k \neq z$

then equation (1) holds and there are exactly $t\varphi(2t)|M_n|^{n-3t}$ near-rings on $\mathbb{Z}_n$.

Proof. We consider the following cases:

Case 1. Let $x = kt$, $k = 0, 2, \dots, 2t-2$. Then $\pi(x)\pi(y)=0$. From assumptions in Theorem $\pi(y)$ is equal to 0, rt, t^2 or $2tm$. Since k is even, the value of the right side of (1) is $\pi(kt.tx \bmod 2t^2) = \pi(0)$. The (1) holds.

Case 2. Let x is even number and $x \neq 0, t, 2t, \dots, n-t$. Then the left side is $2tm\pi(y)$ and as the previous case the value is 0. Since x is even and the $\pi(y)$ contains multiplier t we have $\pi(2tr) = 0$. The equation (1) is true.

Case 3. Let's have fixed $s = 1, 3, 5, \dots, 2t-1$ and $x \in s, 2t+s, ..., (2t-2)t+s$, i.e. $x = 2it + s$.

Case 3.1. Let $y = 2jt + s$. The left side is $r_1 t r_2 t$, (r_1, t)=(r_2, t)=1. The t is even number and r_1 and r_2 are coprimes with t. Then the left side is et^2, where e is odd number and $et^2 \bmod 2t^2 = t^2$. The parameter in the right side $z = (2it + s)\pi(y) = (2it + s)rt = ir2t^2 + srt \bmod n = srt$. By assumptions of Theorem we have $\pi(z) = t^2$. The equation (1) holds.

Case 3.2. Let $y \neq 2jt + s$. The left side is $rt\pi(y)$. When $\pi(y)$=0 the (1) holds. In the other case $\pi(y)$ have values t^2 or $2tm$, i.e. $2tq$. Hence the left side is 0 by modulo n. The parameter $z = (2it+s)2tq = 2iq2t^2+2tsq \bmod n = 2tsq$. Then $\pi(2tsq) = 0$ by the first assumption of the Theorem. The equation (1) is true.

Case 4. Let's have fixed $s = 1, 3, 5, \ldots, 2t - 1$, x is odd and $x \notin s, 2t+s, ..., (2t-2)t+s$, i.e. $x = 2it+s$. The left side of (1) is $2tm\pi(y)$. The nonzero values of $\pi(y)$ have multiplier t. Hence the left side is equal to 0. The parameter in the right side is equal to $x\pi(y) = it$. By the first and last assumptions of Theorem $\pi(kt) = \pi(kt+t) = 0$, excluding $z = srt$ from the case 3. The equation (1) holds.

According to the terms of Theorem, we have t^2-2t elements (ii) and t^2-t elements (iv) with values in M_n, for every s (t times) we have $\varphi(2t)$ constant values for t elements (iii) and the other elements have constant values. Thus there are exactly $t\varphi(2t)|M_n|^{n-3t}$ functions π such that the equation (1) holds. □

When $n = 32$ we have $4.4.4^{20}$ newly described near-rings on Z_{32} and for $s = 1$ the values of the function π look like:

$$(0, 4, x, x, 16, x, x, x, 0, 4, x, x, 0, x, x, x, \\ 0, 4, x, x, 0, x, x, x, 0, 4, x, x, 0, x, x, x),$$

$$(0, 12, x, x, 0, x, x, x, 0, 12, x, x, 16, x, x, x, \\ 0, 12, x, x, 0, x, x, x, 0, 12, x, x, 0, x, x, x),$$

$$(0, 20, x, x, 0, x, x, x, 0, 20, x, x, 0, x, x, x, \\ 0, 20, x, x, 16, x, x, x, 0, 20, x, x, 0, x, x, x),$$

$$(0, 28, x, x, 0, x, x, x, 0, 28, x, x, 0, x, x, x, \\ 0, 28, x, x, 0, x, x, x, 0, 28, x, x, 16, x, x, x)$$

where $x \in \{0, 8, 16, 24\}$.

When $n = 72$ we have $6.4.6^{54} = 4.6^{55}$ near-rings on Z_{72} from the described class.

4. Generating of Near-Rings on Finite Cyclic Groups

Using the constructions describing classes of near-rings from theorems in Section 3, constructions from other theorems and constructions describing only near-rings on $\mathbb{Z}_{32}$ we generated all other near-rings that do not belong to above-mentioned classes and we calculated the number of near-rings on Z_{32} which is exactly 72 651 402 778 958 353 [4].

We also computed all near-rings on finite cyclic groups of order 33, 34 and 35.

In our research since 2009 [5, 4] we generated all near-ring on $\mathbb{Z}_n$, $16 \leq n \leq 35$, $n \neq 25, 27$ and 32. We received the exact number of near-rings on $\mathbb{Z}_{25}$ and $\mathbb{Z}_{27}$ through the same technique.

5. Conclusion

We created a distributed software system for testing hypotheses about near-rings on finite cycling groups.

We use the software system to test various constructions of π functions on $\mathbb{Z}_8$, $\mathbb{Z}_{32}$ and $\mathbb{Z}_{72}$.

We make our findings based on the dependences for the structure of near-rings on $\mathbb{Z}_{32}$ [4]. The hypotheses made on the grounds of $\mathbb{Z}_{32}$ are tested and validated for other $\mathbb{Z}_n$. In Section 3 we formulate and prove the statements about this constructions of near-rings on $\mathbb{Z}_{2t^2}$, where t is positive even number.

In future, the software system can be extended with other features to analyze the properties and generate near-rings.

	Zero-symmetric	Non-zero-symmetric	Total number
$\mathbb{Z}_3$	6	1	7
$\mathbb{Z}_4$	16	1	17
$\mathbb{Z}_5$	28	1	29
$\mathbb{Z}_6$	65	33	98
$\mathbb{Z}_7$	111	1	112
$\mathbb{Z}_8$	349	1	350
$\mathbb{Z}_9$	1169	1	1170
$\mathbb{Z}_{10}$	807	393	1200
$\mathbb{Z}_{11}$	1311	1	1312
$\mathbb{Z}_{12}$	4467	1055	5522
$\mathbb{Z}_{13}$	5263	1	5264
$\mathbb{Z}_{14}$	10505	5256	15761
$\mathbb{Z}_{15}$	21783	6215	27998
$\mathbb{Z}_{16}$	16834653	1	16834654
$\mathbb{Z}_{17}$	72816	1	72817
$\mathbb{Z}_{18}$	15032215	610684	15642899
$\mathbb{Z}_{19}$	286380	1	286381
$\mathbb{Z}_{20}$	876919	109847	986766
$\mathbb{Z}_{21}$	1164023	304834	1468857
$\mathbb{Z}_{22}$	2225545	1111088	3336633
$\mathbb{Z}_{23}$	4371615	1	4371616
$\mathbb{Z}_{24}$	15 821 973	2 619 758	18 441 731
$\mathbb{Z}_{25}$	95 367 449 527 555	1	95 367 449 527 556
$\mathbb{Z}_{26}$	34 749 177	17 400 576	52 149 753
$\mathbb{Z}_{27}$	286 174 087 734	1	286 174 087 735
$\mathbb{Z}_{28}$	207 919 830	19 570 310	227 490 140
$\mathbb{Z}_{29}$	273 300 895	1	273 300 896
$\mathbb{Z}_{30}$	552 602 256	461 986 240	1 014 588 496
$\mathbb{Z}_{31}$	1 089 204 381	1	1 089 204 382
$\mathbb{Z}_{32}$	72 651 402 778 958 352	1	72 651 402 778 958 353
$\mathbb{Z}_{33}$	4 364 742 735	1 092 510 166	5 457 252 901
$\mathbb{Z}_{34}$	8 677 365 263	4 338 542 561	13 015 907 824
$\mathbb{Z}_{35}$	17 373 338 997	1 362 452 660	18 735 791 657

Table 1: The number of near-rings on $\mathbb{Z}_n,\ n \leq 35$

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