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AN IMPROVED BINOMIAL DISTRIBUTION TO APPROXIMATE THE NEGATIVE HYPERGEOMETRIC DISTRIBUTION

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Abstract: The aim of this paper is to give an improved binomial distribution with parameters S and p for approximating the negative hypergeometric distribution with parameters R, S and r, where $p = 1 - q = \frac{r}{R+1}$. The improved approximation is more accurate than the binomial approximation when $\frac{S}{R}$ is sufficiently small.

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1. Introduction

Let X be the negative hypergeometric random variable with parameters R, S and r, and its probability function is of the form

$$\mathbf{nh}_{R,S,r}(x) = \frac{\binom{r+x-1}{x}\binom{R-r+S-x}{S-x}}{\binom{R+S}{S}},\ x = 0, 1, ..., S, \tag{1.1}$$

where $R, S \in \mathbb{N}$ and $r \in \{1, ..., R\}$. The mean and variance of X are $E(X) =$

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$\frac{rS}{R+1}$ and $Var(X) = \frac{rS(R+S+1)(R-r+1)}{(R+1)^2(R+2)}$, respectively. This distribution is obtained from a finite sample analogy to the negative binomial distribution, which arises in a scheme of sampling with replacement. Following Teerapabolarn [2], if $R, r \to \infty$ while $\frac{r}{R+1}$ remains a constant, then $\mathbf{nh}_{R,S,r}(x) \to \mathbf{b}_{S,p}(x) = \binom{S}{x} p^x q^{S-x}$ for every $x \in \{0, ..., S\}$, where $p = 1 - q = \frac{r}{R+1}$. Therefore, the binomial probability function can be used as an approximation of the negative hypergeometric probability function if R is large. In this case, Teerapabolarn and Wongkasem [3] gave a bound on $|\mathbf{nh}_{R,S,r}(x) - \mathbf{b}_{S,p}(x)|$ for $x \in \{0, ..., S\}$.

In this paper, we give an improved binomial probability function, $\widehat{\mathbf{b}}_{S,p}(x)$, for approximating the negative hypergeometric probability function, and the accuracy of the approximation is measured in the form of $\left|\mathbf{nh}_{R,S,r}(x) - \widehat{\mathbf{b}}_{S,p}(x)\right|$ for $x \in \{0, ..., S\}$, which is in Section 2. In Section 3, some numerical examples have been given to illustrate the improved approximation and the conclusion of this study is presented in the last section.

2. Result

Applying the property in [1], the following lemma can also be obtained.

Lemma 2.1. *For $x, N \in \mathbb{N}$, then*

$$\prod_{i=0}^{x-1}\left(p + \frac{i}{N}\right) = p^x\left[1 + \frac{x(x-1)}{2Np}\right] + O\left(\frac{1}{N^2}\right), \tag{2.1}$$

$$\frac{1}{\prod_{i=0}^{x-1}\left(1 + \frac{i}{N}\right)} = 1 - \frac{x(x-1)}{2N} + O\left(\frac{1}{N^2}\right). \tag{2.2}$$

Theorem 2.1. *Let $x \in \{0, ..., S\}$ and $p = \frac{r}{R+1}$. Then we have the following:*

$$\mathbf{nh}_{R,S,r}(x) = \widehat{\mathbf{b}}_{S,p}(x) + O\left(\frac{1}{(R+1)^2}\right) \tag{2.3}$$

and for small $\frac{S}{R}$,

$$\widehat{\mathbf{b}}_{S,p}(x) = \mathbf{nh}_{R,S,r}(x), \tag{2.4}$$

where $\widehat{\mathbf{b}}_{S,p}(x) = \mathbf{b}_{S,p}(x)\left\{1 - \frac{S(S-1)}{2(R+1)} + \frac{(S-x)(S-x-1)}{2(R-r+1)} + \frac{x(x-1)}{2r}\right\}$.

Proof. For $x = 0$, applying Lemma 2.1, it follows that

$$\mathbf{nh}_{R,S,r}(0) = \frac{(R-r+1)\cdots(R-r+S)}{(R+1)\cdots(R+S)}$$

$$= \frac{\prod_{i=0}^{S-1}\left(q+\frac{i}{R+1}\right)}{\prod_{i=0}^{S-1}\left(1+\frac{i}{R+1}\right)}$$

$$= q^S\left\{1+\frac{S(S-1)}{2(R+1)q}-\frac{S(S-1)}{2(R+1)}\right\}+O\left(\frac{1}{(R+1)^2}\right)$$

$$= \widehat{\mathbf{b}}_{S,p}(0)+O\left(\frac{1}{(R+1)^2}\right).$$

Next, we have to show that (2.3) holds for $x \in \{1, ..., S\}$. Following [2] and using Lemma 2.1, we can obtain

$$\mathbf{nh}_{R,S,r}(x) = \binom{S}{x}\frac{[r\cdots(r+x-1)][(R-r+1)\cdots(R-r+S-x)]}{(R+1)\cdots(R+S)}$$

$$= \binom{S}{x}\frac{\prod_{i=0}^{x-1}\left(p+\frac{i}{R+1}\right)\prod_{i=0}^{S-x-1}\left(q+\frac{i}{R+1}\right)}{\prod_{i=0}^{S-1}\left(1+\frac{i}{R+1}\right)}$$

$$= \binom{S}{x}p^x q^{S-x}\left\{1-\frac{S(S-1)}{2(R+1)}+\frac{(S-x)(S-x-1)}{2(R+1)q}\right\}$$

$$+\frac{x(x-1)}{2(R+1)p}+O\left(\frac{1}{(R+1)^2}\right)$$

$$= \mathbf{b}_{S,p}(x)\left\{1-\frac{S(S-1)}{2(R+1)}+\frac{(S-x)(S-x-1)}{2(R-r+1)}+\frac{x(x-1)}{2r}\right\}$$

$$+O\left(\frac{1}{(R+1)^2}\right)$$

$$= \widehat{\mathbf{b}}_{S,p}(x)+O\left(\frac{1}{(R+1)^2}\right).$$

Also, if $\frac{S}{R}$ is small, then $O\left(\frac{1}{(R+1)^2}\right) = 0$. Hence $\widehat{\mathbf{b}}_{S,p}(x) = \mathbf{nh}_{R,S,r}(x)$. □

3. Numerical Examples

The following examples have been given to illustrate how well the improved binomial distribution approximates the negative hypergeometric distribution (when R is sufficiently large or $\frac{S}{R}$ is small).

3.1. Let $R = 100$, $S = 10$ and $r = 10$, then $p = \frac{10}{101}$ and the numerical results are as follows:

x	$\mathbf{nh}_{R,S,r}(x)$	$\widehat{\mathbf{b}}_{S,p}(x)$	$\mathbf{b}_{S,p}(x)$	$\lvert\mathbf{nh}_{R,S,r}(x)-\widehat{\mathbf{b}}_{S,p}(x)\rvert$	$\lvert\mathbf{nh}_{R,S,r}(x)-\mathbf{b}_{S,p}(x)\rvert$
0	0.36910835	0.36979369	0.35253333	0.00068534	0.01657502
1	0.36910835	0.36805248	0.38739927	0.00105587	0.01829091
2	0.18455418	0.18431967	0.19157107	0.00023451	0.00701689
3	0.06026259	0.06092239	0.05613804	0.00065980	0.00412455
4	0.01413375	0.01424277	0.01079578	0.00010902	0.00333797
5	0.00247341	0.00236939	0.00142362	0.00010401	0.00104979
6	0.00032545	0.00027643	0.00013037	0.00004902	0.00019508
7	0.00003165	0.00002200	0.00000819	0.00000965	0.00002347
8	0.00000217	0.00000114	0.00000034	0.00000103	0.00000183
9	0.00000009	0.00000003	0.00000001	0.00000006	0.00000009

3.2. Let $R = 500$, $S = 30$ and $r = 50$, then $p = \frac{50}{501}$ and the numerical results are as follows:

x	$\mathbf{nh}_{R,S,r}(x)$	$\widehat{\mathbf{b}}_{S,p}(x)$	$\mathbf{b}_{S,p}(x)$	$\lvert\mathbf{nh}_{R,S,r}(x)-\widehat{\mathbf{b}}_{S,p}(x)\rvert$	$\lvert\mathbf{nh}_{R,S,r}(x)-\mathbf{b}_{S,p}(x)\rvert$
0	0.04680740	0.04678191	0.04267411	0.00002549	0.00413329
1	0.14627313	0.14646752	0.14193163	0.00019439	0.00434149
2	0.22582250	0.22585025	0.22816061	0.00002775	0.00233811
3	0.22928700	0.22900543	0.23608636	0.00028157	0.00679936
4	0.17196525	0.17178848	0.17667217	0.00017677	0.00470692
5	0.10144505	0.10153772	0.10185092	0.00009268	0.00040587
6	0.04894279	0.04910514	0.04704865	0.00016236	0.00189414
7	0.01982493	0.01989925	0.01788355	0.00007432	0.00194137
8	0.00686852	0.00686257	0.00570013	0.00000595	0.00116838
9	0.00206314	0.00203501	0.00154475	0.00002814	0.00051839
10	0.00054272	0.00052257	0.00035964	0.00002016	0.00018308
11	0.00012597	0.00011678	0.00007249	0.00000919	0.00005348
12	0.00002594	0.00002279	0.00001273	0.00000315	0.00001322
13	0.00000476	0.00000389	0.00000195	0.00000086	0.00000281
14	0.00000078	0.00000058	0.00000026	0.00000020	0.00000052
15	0.00000011	0.00000008	0.00000003	0.00000004	0.00000008
16	0.00000001	0.00000001	0.00000000	0.00000001	0.00000001

For approximating the negative hypergeometric distribution in the examples 3.1 and 3.2, it can be seen that the improved binomial distribution is more appropriate than the binomial distribution .

4. Conclusion

The result of this study is an improved binomial distribution with parameters S and $p = \frac{r}{R+1}$. It is more accurate for approximating the negative hypergeometric distribution, that is, the improved binomial distribution can be used as an approximation of the negative hypergeometric distribution when $\frac{S}{R}$ is sufficiently small. In addition, the improvement of the approximation is more appropriate than the binomial approximation.

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