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ON THE DIOPHANTINE EQUATION $143^x + 145^y = z^2$

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Abstract: In this paper, we show that $(1, 0, 12)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $143^x + 145^y = z^2$ where x, y and z are non-negative integers.

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1. Introduction

In 2012, Chotchaisthit [1] showed that, for any positive prime p, the Diophantine equation $4^x + p^y = z^2$ has no non-negative integer solution where x, y, z are non-negative integers. In the same year, Sroysang [6] showed that $(1, 0, 2)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $3^x + 5^y = z^2$ where x, y and z are non-negative integers. In 2013, Sroysang [8] showed that the Diophantine equation $5^x + 7^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers. For similar equations, we refer to [4, 5, 7, 9, 10, 11, 12, 13, 14]. Recently, Rabago [3] showed that $(1, 0, 3)$, $(1, 1, 5)$, $(2, 1, 9)$ and $(3, 1, 23)$ are only four solutions (x, y, z) for the Diophantine equation $8^x + 17^y = z^2$ where x, y and z are non-negative integers.

In this paper, we show that $(1, 0, 12)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $143^x + 145^y = z^2$ where x, y and z

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are non-negative integers.

2. Preliminaries

Proposition 2.1. [2] (**Catalan's conjecture**) $(3, 2, 2, 3)$ *is a unique solution* (a, b, x, y) *for the Diophantine equation* $a^x - b^y = 1$ *where* a, b, x *and* y *are integers such that* $\min\{a, b, x, y\} > 1$.

Lemma 2.2. $(1, 12)$ *is a unique solution* (x, z) *for the Diophantine equation* $143^x + 1 = z^2$ *where* x *and* z *are non-negative integers.*

Proof. Let x and z be non-negative integers such that $143^x + 1 = z^2$. If $x = 0$, then $z^2 = 2$ which is impossible. Then $x \geq 1$. It follows that $z^2 = 143^y + 1 \geq 143^1 + 1 = 144$. Then $z \geq 12$. Now, we consider on the equation $z^2 - 143^x = 1$. By Proposition 2.1, we have $x = 1$. We obtain that $z^2 = 144$ and then $z = 12$. □

Lemma 2.3. *The Diophantine equation* $1 + 145^y = z^2$ *has no non-negative integer solution where* y *and* z *are non-negative integers.*

Proof. Suppose that there are non-negative integers y and z such that $1 + 145^y = z^2$. If $y = 0$, then $z^2 = 2$ which is impossible. Then $y \geq 1$. It follows that $z^2 = 1 + 145^y \geq 1 + 145^1 = 146$. Then $z \geq 13$. Now, we consider on the equation $z^2 - 145^y = 1$. By Proposition 2.1, we have $y = 1$. We obtain that $z^2 = 146$. This is a contradiction. □

3. Main Results

Theorem 3.1. $(1, 0, 12)$ *is a unique non-negative integer solution* (x, y, z) *for the Diophantine equation* $143^x + 145^y = z^2$ *where* x, y *and* z *are non-negative integers.*

Proof. Let x, y and z be non-negative integers such that $143^x + 145^y = z^2$. By Lemma 2.3, it follows that $x \geq 1$. Note that z is even. This implies that $z^2 \equiv 0 \pmod 4$. Since $145^y \equiv 1 \pmod 4$, we obtain that $143^x \equiv 3 \pmod 4$. Then x is odd. Now, we will divide the number y into two cases.

Case $y = 0$. By Lemma 2.2, we have $x = 1$ and $z = 12$.

Case $y \geq 1$. Note that $145^{\mathsf{y}} \equiv 0 \pmod 5$. Since $143^{\mathsf{y}} \equiv 2 \pmod 5$ or $143^{\mathsf{y}} \equiv 3 \pmod 5$, it follows that $z^2 \equiv 2 \pmod 5$ or $z^2 \equiv 3 \pmod 5$. This implies that z is odd. This is a contradiction.

Hence, $(1, 0, 12)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $143^{\mathsf{x}} + 145^{\mathsf{y}} = z^2$ where x, y and z are non-negative integers. □

Corollary 3.2. *The Diophantine equation* $143^{\mathsf{x}} + 145^{\mathsf{y}} = w^4$ *has no non-negative integer solution where* x, y *and* w *are non-negative integers.*

Proof. Suppose that there are non-negative integers x, y and w such that $143^{\mathsf{x}} + 145^{\mathsf{y}} = w^4$. Let $z = w^2$. This implies that $143^{\mathsf{x}} + 145^{\mathsf{y}} = z^2$. By Theorem 3.1, we have $(x, y, z) = (1, 0, 12)$. Then $w^2 = z = 12$. This is a contradiction. □

Corollary 3.3. $(1, 0, 2)$ *is a unique non-negative integer solution* (x, y, z) *for the Diophantine equation* $143^{\mathsf{x}} + 145^{\mathsf{y}} = 9u^4$ *where* x, y *and* u *are non-negative integers.*

Proof. Let x, y and u be non-negative integers such that $143^{\mathsf{x}} + 145^{\mathsf{y}} = 9u^4$. Let $z = 3u^2$. This implies that $143^{\mathsf{x}} + 145^{\mathsf{y}} = z^2$. By Theorem 3.1, we have $(x, y, z) = (1, 0, 12)$. Then $3u^2 = z = 12$. It follows that $u = 2$. □

Corollary 3.4. $(1, 0, 3)$ *is a unique non-negative integer solution* (x, y, z) *for the Diophantine equation* $143^{\mathsf{x}} + 145^{\mathsf{y}} = 16v^2$ *where* x, y *and* v *are non-negative integers.*

Proof. Let x, y and v be non-negative integers such that $143^{\mathsf{x}} + 145^{\mathsf{y}} = 16v^2$. Let $z = 4v$. This implies that $143^{\mathsf{x}} + 145^{\mathsf{y}} = z^2$. By Theorem 3.1, we have $(x, y, z) = (1, 0, 12)$. Then $4v = z = 12$. It follows that $v = 3$. □

References

[1] S. Chotchaisthit, On the Diophantine equation $4^{\mathsf{x}} + p^{\mathsf{y}} = z^2$ where p is a prime number, *Amer. J. Math. Sci.*, **1** (2012), 191–193.

[2] P. Mihailescu, Primary cycolotomic units and a proof of Catalan's conjecture, *J. Reine Angew. Math.*, **27** (2004), 167–195.

[3] J. F. T. Rabago, On an open problem by B. Sroysang, *Konuralp J. Math.*, **1** (2013), 30–32.

[4] B. Sroysang, More on the Diophantine equation $2^{\mathsf{x}} + 3^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **84** (2013), 133–137.

[5] B. Sroysang, More on the Diophantine equation $8^{\mathsf{x}} + 19^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **81** (2012), 601–604.

[6] B. Sroysang, On the Diophantine equation $3^{\mathsf{x}} + 5^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **81** (2012), 605–608.

[7] B. Sroysang, On the Diophantine equation $3^{\mathsf{x}} + 17^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **89** (2013), 111–114.

[8] B. Sroysang, On the Diophantine equation $5^{\mathsf{x}} + 7^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **89** (2013), 115–118.

[9] B. Sroysang, On the Diophantine equation $5^{\mathsf{x}} + 23^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **89** (2013), 119–122.

[10] B. Sroysang, On the Diophantine equation $7^{\mathsf{x}} + 8^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **84** (2013), 111–114.

[11] B. Sroysang, On the Diophantine equation $23^{\mathsf{x}} + 32^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **84** (2013), 231–234.

[12] B. Sroysang, On the Diophantine equation $31^{\mathsf{x}} + 32^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **81** (2012), 609–612.

[13] B. Sroysang, On the Diophantine equation $47^{\mathsf{x}} + 49^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **89** (2013), 279–282.

[14] B. Sroysang, On the Diophantine equation $89^{\mathsf{x}} + 91^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **89** (2013), 283–286.