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ON THE DIOPHANTINE EQUATION $3^x + 45^y = z^2$

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Abstract: In this paper, we show that the Diophantine equation $3^x + 45^y = z^2$ has a unique non-negative integer solution where x, y and z are non-negative integers. The solution (x, y, z) is $(1, 0, 2)$.

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1. Introduction

In 2012, Sroysang [9] proved that the Diophantine equation $3^x + 5^y = z^2$ has a unique non-negative integer solution where x, y and z are non-negative integers. The solution (x, y, z) is $(1, 0, 2)$.

In 2013, Sroysang [10] proved that the Diophantine equation $3^x + 17^y = z^2$ has a unique non-negative integer solution where x, y and z are non-negative integers. The solution (x, y, z) is $(1, 0, 2)$.

In the same year, Rabago [4] proved that the two Diophantine equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$ have exactly two solutions (x, y, z) where x, y and z are non-negative integers. The solutions are in $\{(1, 0, 2), (4, 1, 10)\}$ and $\{(1, 0, 2), (2, 1, 10)\}$, respectively.

Recently, Chotchaisthit [1] proved that $(7, 0, 1, 3)$ and $(3, 2, 2, 5)$ are only two solutions (p, x, y, z) for the Diophantine equation $p^x + (p + 1)^y = z^2$ where x, y, z are non-negative integers and p is a Mersenne prime. For related papers,

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we refer to [3, 5, 6, 7, 8, 11, 12, 13, 14, 15, 16, 17, 18].

In this paper, we show that $(1, 0, 2)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $3^x + 45^y = z^2$ where x, y and z are non-negative integers.

2. Preliminaries

Proposition 2.1. [2] (**Catalan's conjecture**) $(3, 2, 2, 3)$ *is a unique solution* (a, b, x, y) *for the Diophantine equation* $a^x - b^y = 1$ *where* a, b, x *and* y *are integers such that* $\min\{a, b, x, y\} > 1$.

Lemma 2.2. [9] $(1, 2)$ *is a unique solution* (x, z) *for the Diophantine equation* $3^x + 1 = z^2$ *where* x *and* z *are non-negative integers.*

Lemma 2.3. *The Diophantine equation* $1 + 45^y = z^2$ *has no non-negative integer solution where* y *and* z *are non-negative integers.*

Proof. Suppose that there are non-negative integers y and z such that $1 + 45^y = z^2$. If $y = 0$, then $z^2 = 2$ which is impossible. Then $y \geq 1$. Then $z^2 = 1 + 45^y \geq 1 + 45^1 = 46$. Thus, $z \geq 7$. Now, we consider on the equation $z^2 - 45^y = 1$. By Proposition 2.1, we have $y = 1$. Thus, $z^2 = 46$. This is a contradiction. Hence, the Diophantine equation $1 + 45^y = z^2$ has no non-negative integer solution. □

3. Main Results

Theorem 3.1. $(1, 0, 2)$ *is a unique non-negative integer solution* (x, y, z) *for the Diophantine equation* $3^x + 45^y = z^2$ *where* x, y *and* z *are non-negative integers.*

Proof. Let x, y and z be non-negative integers such that $3^x + 45^y = z^2$. By Lemma 2.3, we have $x \geq 1$. Note that z is even. Then $z^2 \equiv 0 \pmod 4$. Since $45^y \equiv 1 \pmod 4$, it follows that $3^x \equiv 3 \pmod 4$. We obtain that x is odd. Now, we will divide the number y into two cases.

Case $y = 0$. By Lemma 2.2, we obtain that $x = 1$ and $z = 2$.

Case $y \geq 1$. Then $45^y \equiv 0 \pmod 5$. Note that $3^y \equiv 2 \pmod 5$ or $3^y \equiv 3 \pmod 5$. Then $z^2 \equiv 2 \pmod 5$ or $z^2 \equiv 3 \pmod 5$. In fact, $z^2 \equiv 0 \pmod 5$ or $z^2 \equiv 1 \pmod 5$ or $z^2 \equiv 4 \pmod 5$. This is a contradiction.

Hence, $(1, 0, 2)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $3^x + 45^y = z^2$ where x, y and z are non-negative integers. □

Corollary 3.2. *The Diophantine equation $3^x + 45^y = w^4$ has no non-negative integer solution where x, y and w are non-negative integers.*

Proof. Suppose that there are non-negative integers x, y and w such that $3^x + 45^y = w^4$. Let $z = w^2$. Then $3^x + 45^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (1, 0, 2)$. Then $w^2 = z = 2$. This is a contradiction. Hence, the Diophantine equation $3^x + 45^y = w^4$ has no non-negative integer solution where x, y and w are non-negative integers. □

References

[1] S. Chotchaisthit, On the Diophantine equation $p^x + (p+1)^y = z^2$ where p is a Mersenne prime, *Int. J. Pure Appl. Math.*, **88** (2013), 169–172.

[2] P. Mihailescu, Primary cycolotomic units and a proof of Catalan's conjecture, *J. Reine Angew. Math.*, **27** (2004), 167–195.

[3] J. F. T. Rabago, More on the Diophantine equation of type $p^x + q^y = z^2$, *Int. J. Math. Sci. Comp.*, **3** (2013), 15–16.

[4] J. F. T. Rabago, On two Diophantine equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$, *Int. J. Math. Sci. Comp.*, **3** (2013), 28–29.

[5] B. Sroysang, More on the Diophantine equation $2^x + 3^y = z^2$, *Int. J. Pure Appl. Math.*, **84** (2013), 133–137.

[6] B. Sroysang, More on the Diophantine equation $2^x + 19^y = z^2$, *Int. J. Pure Appl. Math.*, **88** (2013), 157–160.

[7] B. Sroysang, More on the Diophantine equation $2^x + 37^y = z^2$, *Int. J. Pure Appl. Math.*, **89** (2013), 275–278.

[8] B. Sroysang, More on the Diophantine equation $8^x + 19^y = z^2$, *Int. J. Pure Appl. Math.*, **81** (2012), 601–604.

[9] B. Sroysang, On the Diophantine equation $3^x + 5^y = z^2$, *Int. J. Pure Appl. Math.*, **81** (2012), 605–608.

[10] B. Sroysang, On the Diophantine equation $3^{\mathsf{x}} + 17^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **89** (2013), 111–114.

[11] B. Sroysang, On the Diophantine equation $5^{\mathsf{x}} + 7^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **89** (2013), 115–118.

[12] B. Sroysang, On the Diophantine equation $5^{\mathsf{x}} + 23^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **89** (2013), 119–122.

[13] B. Sroysang, On the Diophantine equation $7^{\mathsf{x}} + 8^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **84** (2013), 111–114.

[14] B. Sroysang, On the Diophantine equation $23^{\mathsf{x}} + 32^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **84** (2013), 231–234.

[15] B. Sroysang, On the Diophantine equation $31^{\mathsf{x}} + 32^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **81** (2012), 609–612.

[16] B. Sroysang, On the Diophantine equation $47^{\mathsf{x}} + 49^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **89** (2013), 279–282.

[17] B. Sroysang, On the Diophantine equation $89^{\mathsf{x}} + 91^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **89** (2013), 283–286.

[18] A. Suvarnamani, A. Singta, S. Chotchaisthit, On two Diophantine equations $4^{\mathsf{x}} + 7^{\mathsf{y}} = z^2$ and $4^{\mathsf{x}} + 11^{\mathsf{y}} = z^2$, *Sci. Technol. RMUTT J.*, **1** (2011), 25–28.